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Forbidden subgraphs for constant domination number

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In this paper, we characterize the sets $\mathcal{H}$ of connected graphs such that there exists a constant $c = c(\mathcal{H})$ satisfying $\gamma(G) \leq c$ for every connected $\mathcal{H}$-free graph G, where $\gamma(G)$ is the domination number of G.

Keywords: Domination number, Forbidden induced subgraph, Ramsey number

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph. Let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For a vertex $x \in V(G)$, let $N_G(x)$ and $N_G[x]$ denote the *open neighborhood* and the *closed neighborhood*, respectively; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$. For a set $X \subseteq V(G)$, let $N_G[X] = \bigcup_{x \in X} N_G[x]$. For a vertex $x \in V(G)$ and a non-negative integer i, let $N_G^i(x) = \{y \in V(G) :$ the distance between x and y in G is $i\}$. Note that $N_G^0(x) = \{x\}$ and $N_G^1(x) = N_G(x)$. Let K_n and P_n denote the *complete graph* and the *path* of order n, respectively. For terms and symbols not defined in this paper, we refer the reader to [3].

Let G be a graph. For two sets $X, Y \subseteq V(G)$, we say that X *dominates* Y if $Y \subseteq N_G[X]$. A subset of $V(G)$ which dominates $V(G)$ is called a *dominating set* of G. The minimum cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. Since the determining problem of the value $\gamma(G)$ is NP-complete (see [7]), many researchers have tried to find good bounds for the domination number (see [9]). One of the most famous results is due to Ore [11] who proved that every connected graph G of order at least two satisfies $\gamma(G) \leq |V(G)|/2$. Here one problem naturally arises: What additional conditions allow better upper bounds on the domination number? In this paper, we focus on forbidden induced subgraph conditions.

For a graph G and a set $\mathcal{H}$ of connected graphs, G is said to be *$\mathcal{H}$-free* if G contains no graph in $\mathcal{H}$ as an induced subgraph. In this context, members of $\mathcal{H}$ are called *forbidden subgraphs*. If G is $\{H\}$-free, then G is simply said to be *H-free*. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \leq \mathcal{H}_2$ if for every $H_2 \in \mathcal{H}_2$, there exists $H_1 \in \mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. The relation

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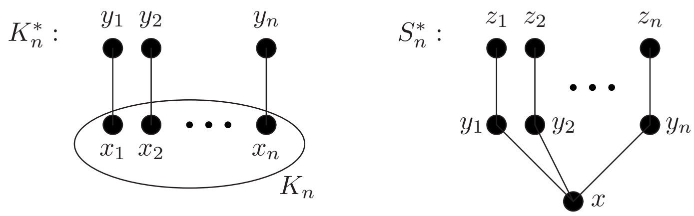

Fig. 1: Graphs K_n^* and S_n^*

"$\leq$" between two sets of forbidden subgraphs was introduced in [6]. Note that if $\mathcal{H}_1 \leq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

Let $K_{1,3}$ and K_3^* denote the two unique graphs having degree sequence $(3,1,1,1)$ and $(3,3,3,1,1,1)$, respectively. Cockayne, Ko and Shepherd [1] (see also Theorem 2.9 in [9]) proved that every connected $\{K_{1,3}, K_3^*\}$-free graph G satisfies $\gamma(G) \leq \lceil |V(G)|/3 \rceil$. Indeed, Duffus, Gould and Jacobson [5] proved that every connected $\{K_{1,3}, K_3^*\}$-free graph has a Hamiltonian path. Since $\gamma(P_n) = \lceil n/3 \rceil$ for every integer n, the above inequality is a consequence of this result. Furthermore, forbidden induced subgraph conditions for domination-like invariants were widely studied (see, for example, [2, 4, 8, 10]).

In this paper, we will characterize the sets $\mathcal{H}$ of connected graphs satisfying the condition that

(A1) there exists a constant $c = c(\mathcal{H})$ such that $\gamma(G) \leq c$ for every connected $\mathcal{H}$-free graph G.

Let $n \geq 1$ be an integer. Let K_n^* denote the graph with $V(K_n^*) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and $E(K_n^*) = \{x_i x_j : 1 \leq i < j \leq n\} \cup \{x_i y_i : 1 \leq i \leq n\}$, and let S_n^* denote the graph with $V(S_n^*) = \{x\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}$ and $E(S_n^*) = \{x y_i : 1 \leq i \leq n\} \cup \{y_i z_i : 1 \leq i \leq n\}$ (see Figure 1). Our main result is the following.

Theorem 1.1 *Let $\mathcal{H}$ be a set of connected graphs. Then $\mathcal{H}$ satisfies (A1) if and only if $\mathcal{H} \leq \{K_k^*, S_\ell^*, P_m\}$ for some positive integers k, ℓ and m.*

We conclude this section by considering the case where a set $\mathcal{H}$ can contain disconnected graphs. Then the following proposition holds.

Proposition 1.2 *Let $\mathcal{H}$ be a set of graphs. Then $\mathcal{H}$ satisfies (A1) if and only if $\mathcal{H} \leq \{\overline{K_k}\}$ for some positive integer k.*

Proof: Suppose that $\mathcal{H}$ satisfies (A1). Then there exists a constant $c = c(\mathcal{H})$ such that $\gamma(G) \leq c$ for every connected $\mathcal{H}$-free graph G. Since $\gamma(\overline{K_{c+1}}) = c+1$, $\overline{K_{c+1}}$ is not $\mathcal{H}$-free, and so $\mathcal{H} \leq \{\overline{K_{c+1}}\}$.

On the other hand, if $\mathcal{H} \leq \{\overline{K_k}\}$, then every $\mathcal{H}$-free graph G satisfies $\gamma(G) \leq k-1$ because every maximal independent set of G is a dominating set. □

2 Proof of Theorem 1.1

For positive integers s and t, let $R(s,t)$ denote the *Ramsey number* with respect to s and t. For positive integers k, ℓ and i, we recursively define $g_{k,\ell}(i)$ as follows:

$$\begin{cases} g_{k,\ell}(1) = 1 \\ g_{k,\ell}(i) = R(k, (\ell-1)g_{k,\ell}(i-1)+1) - 1 \quad (i \geq 2). \end{cases}$$

Lemma 2.1 *Let k, ℓ and i be positive integers. Let G be a $\{K_k^*, S_\ell^*\}$-free graph, and let a be a vertex of G. Then for an independent set $X \subseteq N_G^i(a)$, there exists $U \subseteq N_G^{i-1}(a)$ with $|U| \leq g_{k,\ell}(i)$ that dominates X.*

Proof: We proceed by induction on i. If $i = 1$, then $U = \{a\}$ is a desired subset of $N_G^{i-1}(a) = \{a\}$. Thus we may assume that $i \geq 2$. Note that $N_G^{i-1}(a)$ dominates X. Let U be a minimal subset of $N_G^{i-1}(a)$ that dominates X. It suffices to show that $|U| \leq R(k, (\ell-1)g_{k,\ell}(i-1)+1) - 1 = g_{k,\ell}(i)$.

By way of contradiction, suppose that $|U| \geq R(k, (\ell-1)g_{k,\ell}(i-1)+1)$. For each $u \in U$, since $U - \{u\}$ does not dominate X by the minimality of U, there exists a vertex $x_u \in X$ such that $N_G(x_u) \cap U = \{u\}$. Recall that X is an independent set. If there exists a clique $U_1 \subseteq U$ with $|U_1| = k$, then the subgraph of G induced by $U_1 \cup \{x_u : u \in U_1\}$ is isomorphic to K_k^*, which contradicts the K_k^*-freeness of G. Since $|U| \geq R(k, (\ell-1)g_{k,\ell}(i-1)+1)$, this implies that there exists an independent set $U_2 \subseteq U$ with $|U_2| = (\ell-1)g_{k,\ell}(i-1)+1$. By the induction hypothesis, there exists $U' \subseteq N_G^{i-2}(a)$ with $|U'| = g_{k,\ell}(i-1)$ that dominates U_2. By the pigeon-hole principle, there exists a vertex $u' \in U'$ such that $|N_G(u') \cap U_2| \geq \ell$. Let $\tilde{U}_2 \subseteq N_G(u') \cap U_2$ be a set with $|\tilde{U}_2| = \ell$. Then the subgraph of G induced by $\{u'\} \cup \tilde{U}_2 \cup \{x_u : u \in \tilde{U}_2\}$ is isomorphic to S_ℓ^*, which is a contradiction. □

For positive integers k, ℓ and i with $i \geq 2$, let $f_{k,\ell}(i) = R(k,\ell)g_{k,\ell}(i)$.

Lemma 2.2 *Let k, ℓ and i be positive integers with $i \geq 2$. Let G be a $\{K_k^*, S_\ell^*\}$-free graph, and let a be a vertex of G. Then there exists $\hat{U} \subseteq V(G)$ with $|\hat{U}| \leq f_{k,\ell}(i)$ that dominates $N_G^i(a)$.*

Proof: Let X be a maximal independent subset of $N_G^i(a)$. By Lemma 2.1, there exists $U \subseteq N_G^{i-1}(a)$ with $|U| \leq g_{k,\ell}(i)$ that dominates X. By the maximality of X, X dominates $N_G^i(a)$, and so X dominates $N_G^i(a) - N_G[U]$. Let X_0 be a minimal subset of X that dominates $N_G^i(a) - N_G[U]$.

Claim 2.1 *We have $|X_0| \leq (R(k,\ell)-1)g_{k,\ell}(i)$.*

Proof: Suppose that $|X_0| \geq (R(k,\ell)-1)g_{k,\ell}(i)+1$. Since U dominates X_0 and $|U| \leq g_{k,\ell}(i)$, there exists a vertex $u' \in U$ such that $|N_G(u') \cap X_0| \geq R(k,\ell)$. For each $x \in X_0$, since $X_0 - \{x\}$ does not dominate $N_G^i(a) - N_G[U]$ by the minimality of X_0, there exists a vertex $y_x \in N_G^i(a) - N_G[U]$ such that $N_G(y_x) \cap X_0 = \{x\}$. Set $Y = \{y_x : x \in N_G(u') \cap X_0\}$, and for each $y \in Y$, write $N_G(y) \cap X_0 = \{x_y\}$. Note that $\{x_y : y \in Y\} \subseteq N_G(u') \cap X_0$ and $y_{x_y} = y$ for each $y \in Y$. Since $|Y| = |N_G(u') \cap X_0| \geq R(k,\ell)$, there exists a clique $Y_1 \subseteq Y$ with $|Y_1| = k$ or an independent set $Y_2 \subseteq Y$ with $|Y_2| = \ell$. Recall that $Y \subseteq N_G^i(a) - N_G[U]$, and so $N_G(u') \cap Y = \emptyset$. If there exists a clique $Y_1 \subseteq Y$ with $|Y_1| = k$, then the subgraph of G induced by $Y_1 \cup \{x_y : y \in Y_1\}$ is isomorphic to K_k^*; if there exists an independent set $Y_2 \subseteq Y$ with $|Y_2| = \ell$, then the subgraph of G induced by $\{u'\} \cup \{x_y : y \in Y_2\} \cup Y_2$ is isomorphic to S^*_ℓ. In either case, we obtain a contradiction. □

Recall that X_0 dominates $N^i_G(a) - N_G[U]$. Hence $U \cup X_0$ dominates $N^i_G(a)$. Furthermore, by the definition of U and Claim 2.1,

$$|U \cup X_0| = |U| + |X_0| \leq g_{k,\ell}(i) + (R(k,\ell) - 1)g_{k,\ell}(i) = f_{k,\ell}(i).$$

Thus $\hat{U} = U \cup X_0$ is a desired set. □

Proof of Theorem 1.1: We first prove the "only if"part. Let $\mathcal{H}$ be a set of connected graphs satisfying (A1). Then there exists a constant $c = c(\mathcal{H})$ such that $\gamma(G) \leq c$ for every connected $\mathcal{H}$-free graph G. Since we can easily verify that $\gamma(K^*_{c+1}) = \gamma(S^*_{c+1}) = \gamma(P_{3c+1}) = c+1$, none of K^*_{c+1}, S^*_{c+1} and P_{3c+1} is $\mathcal{H}$-free. This implies that $\mathcal{H} \leq \{K^*_{c+1}, S^*_{c+1}, P_{3c+1}\}$, as desired.

Next we prove the "if" part. Let $\mathcal{H}$ be a set of connected graphs such that $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for some positive integers k, ℓ and m. Choose k, ℓ and m so that $k + \ell + m$ is as small as possible. Then k, ℓ and m are uniquely determined. In particular, the value $1 + \sum_{2 \leq i \leq m-2} f_{k,\ell}(i)$ only depends on $\mathcal{H}$. Furthermore, every $\mathcal{H}$-free graph is also $\{K^*_k, S^*_\ell, P_m\}$-free. Thus it suffices to show that every connected $\{K^*_k, S^*_\ell, P_m\}$-free graph G satisfies $\gamma(G) \leq 1 + \sum_{2 \leq i \leq m-2} f_{k,\ell}(i)$. Let $a \in V(G)$. Since G is P_m-free, $N^i_G(a) = \emptyset$ for all $i \geq m-1$. Since G is connected, this implies that $V(G) = \bigcup_{0 \leq i \leq m-2} N^i_G(a)$. Since G is $\{K^*_k, S^*_\ell\}$-free, it follows from Lemma 2.2 that for each i with $2 \leq i \leq m-2$, there exists a set $\hat{U}_i \subseteq V(G)$ with $|\hat{U}_i| \leq f_{k,\ell}(i)$ that dominates $N^i_G(a)$. Since $\{a\}$ dominates $N^0_G(a) \cup N^1_G(a)$, $\{a\} \cup (\bigcup_{2 \leq i \leq m-2} \hat{U}_i)$ is a dominating set of G, and so

$$\gamma(G) \leq |\{a\}| + \sum_{2 \leq i \leq m-2} |\hat{U}_i| \leq 1 + \sum_{2 \leq i \leq m-2} f_{k,\ell}(i),$$

as desired.

This completes the proof of Theorem 1.1. □

3 Concluding remark

In this paper, we characterized the sets $\mathcal{H}$ of connected graphs satisfying (A1). For similar problems concerning many domination-like invariants, we can use the sets appearing in Theorem 1.1.

Let μ be an invariant of graphs, and assume that

(D1) there exist two constants $c_1, c_2 \in \mathbb{R}^+$ such that $c_1\gamma(G) \leq \mu(G) \leq c_2\gamma(G)$ for all connected graphs G.

Note that many important domination-like invariants (for example, total domination number γ_t, paired domination number $\gamma_{\rm pr}$, Roman domination number γ_R, rainbow domination number $\gamma_{{\rm r}k}$, etc.) satisfy (D1). Furthermore, we focus on the condition that

(A'1) there exists a constant $c' = c'(\mu, \mathcal{H})$ such that $\mu(G) \leq c$ for every connected $\mathcal{H}$-free graph G.

We first suppose that a set $\mathcal{H}$ of connected graphs satisfies (A'1). Note that

- $\mu(K^*_{\lceil (c'+1)/c_1 \rceil}) \geq c_1\gamma(K^*_{\lceil (c'+1)/c_1 \rceil}) = c_1 \cdot \lceil (c'+1)/c_1 \rceil \geq c'+1$,
- $\mu(S^*_{\lceil (c'+1)/c_1 \rceil}) \geq c_1\gamma(S^*_{\lceil (c'+1)/c_1 \rceil}) = c_1 \cdot \lceil (c'+1)/c_1 \rceil \geq c'+1$, and
- $\mu(P_{3\lceil (c'+1)/c_1 \rceil+1}) \geq c_1\gamma(P_{3\lceil (c'+1)/c_1 \rceil+1}) = c_1 \cdot \lceil (c'+1)/c_1 \rceil \geq c'+1$.

Thus, by similar argument to the proof of "only if" part of Theorem 1.1, we have $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for some positive integers k, ℓ and m.

On the contrary, suppose that a set $\mathcal{H}$ of connected graphs satisfies $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for some positive integers k, ℓ and m. Then by Theorem 1.1, (A1) holds, and hence for a connected $\mathcal{H}$-free graph G, we have

$$\mu(G) \leq c_2\gamma(G) \leq c_2 \cdot c(\mathcal{H}).$$

Consequently (A'1) holds (for $c' = c_2 \cdot c(\mathcal{H})$). Therefore, we obtain the following theorem.

Theorem 3.1 *Let μ be an invariant for graphs satisfying (D1), and let $\mathcal{H}$ be a set of connected graphs. Then $\mathcal{H}$ satisfies (A'1) if and only if $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for positive integers k, ℓ and m.*

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