# Permutation complexity of images of Sturmian words by marked morphisms 

Adam Borchert<br>Narad Rampersad*

Department of Mathematics and Statistics, University of Winnipeg, CANADA
received $2^{\text {nd }}$ Nov. 2017, revised $3^{r d}$ Apr. 2018, accepted $21^{\text {st }}$ May 2018.

We show that the permutation complexity of the image of a Sturmian word by a binary marked morphism is $n+k$ for some constant $k$ and all lengths $n$ sufficiently large.

Keywords: permutation complexity, Sturmian words, morphisms

## 1 Introduction

The permutation complexity of an infinite aperiodic word is a concept introduced by Makarov [5]. It is based on the following idea: Given an infinite word $\omega$, consider the linear order $\pi_{\omega}$ on $\mathbb{N}$ induced by the lexicographic order on the successive shifts of $\omega$. The permutation complexity of $\omega$ is the function that counts the number of distinct subpermutations of $\pi_{\omega}$ of a given length. Makarov [6] proved that any Sturmian word $s$ has $n$ subpermutations of length $n$ for all $n \geq 1$. In this paper, we determine the permutation complexity of any word $T(s)$, where $s$ is a Sturmian word, and $T$ is a marked binary morphism ("marked" means that the images of the morphism on letters begin with different letters and end with different letters).

In this paper, we only consider infinite permutations obtained from infinite words in the manner described above, but there is also a more general theory of infinite permutations [3]. Avgustinovich, Frid, and Puzynina [1] studied a subclass of these infinite permutations called equidistributed permutations and showed that within this family, the infinite permutations of minimal permutation complexity are exactly those obtained from Sturmian words.

There have been several other recent results on permutation complexity of infinite words. Here we mention only Widmer's work [10], in which he computes the permutation complexity function of the Thue-Morse word-this turns out to be a rather non-trivial task—and Valyuzhenich's work [9], which generalizes this result somewhat. We should point out that while it may seem rather unsatisfying to report a result that only applies to marked morphisms, it appears to be rather difficult to deal with arbitrary morphisms: in Valyuzhenich's work, he also restricts his attention to marked morphisms, and even in this case the proofs of his results are quite difficult.

[^0]
## 2 Preliminaries

Given an ordered alphabet $\Sigma$, the lexicographic order on $\Sigma^{*}$ is the order defined as follows: $u \leq v$ if either

- $u$ is a prefix of $v$, or
- $u=x a y, v=x b z$ for some words $x, y, z$ and letters $a<b$.

We write $u<v$ if $u \leq v$ and $u \neq v$.
Let $\omega=\omega_{0} \omega_{1} \omega_{2} \cdots$ be an infinite, aperiodic word over the alphabet $\{0,1\}$ (throughout this paper all words will be binary). We denote the $i$-th letter, $\omega_{i}$, by $\omega[i]$, and the factor $\omega_{i} \omega_{i+1} \cdots \omega_{j}$ by $\omega[i, j]$. The $i$-th shift of $\omega$ is the infinite word $\omega[i, \infty]=\omega_{i} \omega_{i+1} \omega_{i+2} \cdots$. Let the shifts of $\omega$ be ordered lexicographically (with respect to the order $0<1$ ). Let $\pi_{\omega}$ be the order on $\mathbb{N}$ defined by $\pi_{\omega}(i)<\pi_{\omega}(j)$ if $\omega[i, \infty]<\omega[j, \infty]$, and $\pi_{\omega}(j)<\pi_{\omega}(i)$ otherwise.

For $i<j$, let $\pi_{\omega}[i, j]$ denote the permutation of $\{1,2, \ldots, j-i+1\}$ for which $\pi_{\omega}[i, j](k)<\pi_{\omega}[i, j](\ell)$ exactly when $\pi_{\omega}(i+k-1)<\pi_{\omega}(i+\ell-1)$. If $j-i+1=n$ we say that the permutation $\pi_{\omega}[i, j]$ is a finite subpermutation of length $n$ of $\pi_{\omega}$. The permutation complexity of $\omega$ is the function $f_{\omega}(n)$ that associates every $n$ to the number of finite subpermutations of length $n$ of $\pi_{\omega}$.

If $u$ is a factor of length $n$ of $\omega$, define

$$
\operatorname{Perm}_{\omega}(u)=\left\{\pi_{\omega}[i, i+n-1]: \omega[i, i+n-1]=u\right\}
$$

We say that $u$ has $\left|\operatorname{Perm}_{\omega}(u)\right|$ permutations. Furthermore, if $\operatorname{Perm}_{\omega}(u) \cap \operatorname{Perm}_{\omega}(v) \neq \emptyset$, we say that $u$ and $v$ are factors with the same permutation.

Our goal is to analyze the permutation complexity of the morphic image of Sturmian words. A Sturmian word is an infinite word with factor complexity $n+1$ for all $n \geq 0$ (the factor complexity of an infinite word $w$ is the function giving the number of distinct factors of $w$ of length $n$ ). Let $s$ be a Sturmian word over $\{0,1\}$ and let $T:\{0,1\} \rightarrow\{0,1\}$ be a morphism such that $T(s)$ is aperiodic. Then $T(s)$ has factor complexity $n+t$ for some constant $t$ and all $n$ sufficiently large [7]. Makarov [6] showed that $f_{s}(n)=n$ for all $n \geq 2$. We conjecture that $f_{T(s)}(n)=n+k$ for some constant $k$ and all $n$ sufficiently large; however, we are only able to prove this for "marked" morphisms (defined below).

If the first letters of $T(0)$ and $T(1)$ are both different and the last letters of $T(0)$ and $T(1)$ are both different, then we say that $T$ is a marked morphism. If $T(0)$ and $T(1)$ are powers of a common word we say that $T$ is a periodic morphism; if not we say that $T$ is an aperiodic morphism. Note that a marked morphism is necessarily aperiodic.

A factor $u$ of an infinite word $s$ is right special (resp. left special) if both $u 0$ and $u 1$ (resp. $0 u$ and $1 u$ ) are factors of $s$. If $s$ is Sturmian then for all $n \geq 0$ the word $s$ contains exactly one right special factor of length $n$ and exactly one left special factor of length $n$ (see [4, Section 2.1.3]). If $T(s)$ is aperiodic then for $n$ sufficiently large the word $T(s)$ contains exactly one right special factor of length $n$ and exactly one left special factor of length $n$.

An infinite word $s$ is uniformly recurrent if for every length $\ell$ there is another length $L$ such that every factor of $s$ of length $L$ contains every factor of $s$ of length $\ell$. If $s$ is Sturmian, then $s$ and $T(s)$ are both uniformly recurrent.

## 3 Recognizability of a morphism

We also need some results concerning the recognizability of the morphism $T$. The basic definitions are given in terms of bi-infinite words (following [2]).
Definition 1. Let $\theta: A^{*} \rightarrow B^{*}$ be a non-erasing morphism; let $x=\cdots x_{-1} x_{0} x_{1} \cdots$ be a bi-infinite word with each $x_{i} \in A$; and let $y=\theta(x)$. The set of cutting points of $(\theta, x)$ is the set

$$
C(\theta, x)=\{0\} \cup\{|\theta(x[0, i])|: i \geq 0\} \cup\{-|\theta(x[-i,-1])|: i>0\} .
$$

Definition 2. Let $\theta: A^{*} \rightarrow B^{*}$ be a non-erasing morphism; let $x \in A^{\mathbb{Z}}$; and let $y=\theta(x)$. The morphism $\theta$ is recognizable in the sense of Mossé for $x$ if there exists $\ell$ such that, for every $m \in C(\theta, x)$ and $m^{\prime} \in \mathbb{Z}$, the equality $y[m-\ell, m+\ell-1]=y\left[m^{\prime}-\ell, m^{\prime}+\ell-1\right]$ implies that $m^{\prime} \in C(\theta, x)$.

In the special case of binary morphisms we have the following.
Lemma 1. Let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be an aperiodic morphism. Then $T$ is recognizable in the sense of Mossé for any aperiodic word $x \in\{0,1\}^{\mathbb{Z}}$.

Proof: This follows from [2, Theorems 3.1 and 2.5(1)].
Definition 3. Let $\theta: A^{*} \rightarrow B^{*}$ be a non-erasing morphism; let $x \in A^{\mathbb{Z}}$; and let $w$ be a non-empty factor of $\theta(x)$. An interpretation of $w$ in $x$ is a triple $(p, z, s)$ such that

- $z=z_{0} \cdots z_{n-1}$ is a factor of $x\left(\right.$ each $\left.z_{i} \in A\right)$,
- $p$ is a proper prefix of $\theta\left(z_{0}\right)$,
- $s$ is a proper suffix of $\theta\left(z_{n-1}\right)$, and
- $\theta(z)=p w s$.

Lemma 2. Let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be an aperiodic, marked morphism; let $x \in\{0,1\}^{\mathbb{Z}}$; let $\ell$ be the constant of Definition 2; and let w be a factor of $T(x)$ of length at least $L=2 \ell+\max \{|T(0)|,|T(1)|\}$. Then $w$ has a unique interpretation in $x$.

Proof: Consider two occurrences of $w$ in $T(x)$. In the first occurrence there is some position $m$ such that $m$ is at distance at least $\ell$ from both the beginning and the end of $w$ and is a cutting point. By Lemma 1 and Definition 2, the corresponding position in the second occurrence of $w$ is also a cutting point. Now, since $T$ is marked, the interpretations of both occurrences are uniquely determined.

## 4 Permutation complexity of $T(s)$

Let $s$ be a Sturmian word over $\{0,1\}$ and let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be a marked morphism. Let $\bar{s}$ denote the word obtained from $s$ by applying the morphism $0 \rightarrow 1,1 \rightarrow 0$, and let $\bar{T}$ denote the morphism defined by $0 \rightarrow T(1), 1 \rightarrow T(0)$. Note that $\bar{s}$ is again Sturmian and we have $T(s)=\bar{T}(\bar{s})$. Hence, without loss of generality, we suppose that $T(0)$ begins with 0 and $T(1)$ begins with 1 (replacing $T$ and $s$ with $\bar{T}$ and $\bar{s}$ if necessary).

Theorem 3. There exist constants $N$ and $k$ such that the permutation complexity of $T(s)$ is $n+k$ for $n>N$.

Plan of the proof:

- Lemma 4 handles distinct factors having the same permutation.
- Lemma 8 shows that minimal factors with multiple permutations are small.
- Lemma 9 shows that other than small exceptions, factors of $T(s)$ have at most two permutations.
- Lemmas 10 and 11 show that the number of factors with two permutations is (eventually) constant.

Once these facts are proved, we conclude that (other than small exceptions) there are exactly $l$ factors with two permutations of length $n$ for each $n$, and that every factor of $T(s)$ has exactly one or two permutations. The result follows with either $k=t+l$ or $k=t+l-1$ (depending on the result of Lemma 4) where $t$ is the integer such that $T(s)$ has $n+t$ factors of length $n$.
Example 1. Let $s$ be the Fibonacci word; i.e. $s$ is the fixed point of the morphism $0 \rightarrow 01,1 \rightarrow 0$. Let $T$ be the morphism that maps $0 \rightarrow 0110$ and $1 \rightarrow 11$. For $n \geq 14$, the word $T(s)$ has exactly 10 factors of length $n$ with two permutations.

First, we deal with distinct factors of $s$ having the same permutation. We start with some basic general results.

We need the following important fact due to Makarov [5, Lemma 1]: Let $u$ and $v$ be distinct factors of $s$ of the same length (greater than 1) that have the same permutation. Then $u$ and $v$ differ only in the last position.
Lemma 4. In $T(s)$, for $n$ sufficiently large, there is at most one pair of distinct factors of length $n$ with the same permutation. If there are such pairs for infinitely many $n$, then there are such pairs for all $n$.

Proof: Let $u$ and $v$ be factors of $T(s)$ of length $n$ and suppose that $u$ and $v$ have the same permutation. Then write $u=w 0$ and $v=w 1$; we see that $w$ is right-special. If $n$ is sufficiently large, the word $T(s)$ contains exactly one right-special factor of length $n-1$, so there can be at most one such pair $u, v$. Now if $u$ and $v$ have the same permutation, then so do any of their equal-length suffixes, so if there are such pairs for infinitely many $n$, then there are such pairs for all $n$.

Definition 4. Let $u$ and $v$ be finite words. A morphism $T$ is order-preserving if whenever $u \leq v$ we have $T(u) \leq T(v)$. If the same holds true whenever $u$ and $v$ are infinite words we say that $T$ is order-preserving on infinite words.

These morphisms are studied further in Section 5. Since we have assumed that $T(0)$ starts with 0 and $T(1)$ starts with 1 , the morphism $T$ is order-preserving on infinite words.

We now examine when it is possible for a factor $w$ of length $n$ to have more than one distinct permutation. In this case there must exist two occurrences of $w$ in $T(s)$-say at positions $i$ and $i^{\prime}$ and integers $\ell$ and $\ell^{\prime}$ satisfying $0 \leq \ell, \ell^{\prime} \leq n-1$ such that $T(s)[i+\ell, \infty]<T(s)\left[i+\ell^{\prime}, \infty\right]$ and $T(s)\left[i^{\prime}+\ell, \infty\right]>T(s)\left[i^{\prime}+\ell^{\prime}, \infty\right]$. It follows that $T(s)[i+\ell]=T(s)\left[i+\ell^{\prime}\right]=x$, for some letter $x$. There then must exist factors $w_{0}$ and $w_{1}$ of $T(s)$, each with prefix $w$, having the following forms:

$$
\begin{align*}
& w_{0}=\rho_{1} x u 0 \gamma=\rho_{2} x u 1  \tag{1}\\
& w_{1}=p_{1} x v 1 g=p_{2} x v 0
\end{align*}
$$

for some words $\rho_{1}, \rho_{2}, \gamma, p_{1}, p_{2}, g$, where $\left|\rho_{1}\right|<\mid \rho_{2}$ and $\left|p_{1}\right|<\left|p_{2}\right|$, the common prefix $w$ extends at least to include the second $x$, and the $x$ 's have the same relative indices in $w_{0}$ and $w_{1}$. Let us assume that $|u|>|v|$. We need the following result [6, Lemma 3]:
Lemma 5 ([6]). Let $z$ be a factor of length $n+1$ of a Sturmian word $s$. Then $z$ has exactly one permutation and this permutation is uniquely determined by the prefix of $z$ of length $n$.

We also need the following well-known result about repetitions in Sturmian words.
Lemma 6. Let $s$ be a Sturmian word and let $T$ be an aperiodic binary morphism. For any integer $p \geq 1$ there is a constant $K_{0}(p)($ resp. $K(p))$ such that every factor of $s(r e s p . T(s)$ ) of period at most $p$ has length at most $K_{0}(p)(r e s p . K(p))$.

Proof: The claim is an easy consequence of the fact that $s$, and hence $T(s)$, is aperiodic but uniformly recurrent. Recall that this means that for every length $\ell$ there is another length $L$ such that every factor of $s$ of length $L$ contains every factor of $s$ of length $\ell$. If, contrary to the claim, there were unboundedly large factors of $s$ of period $p$, these factors would necessarily fail to contain some factor of $s$ of length $p$.

In the rest of the argument, we will often wish to indicate that certain types of factors have lengths that are bounded by some absolute constant depending only on $T$ and $s$. We will abbreviate this notion by saying that these factors are small.

Lemma 7. In Equation (1), the words $u$ and $v$ are small.
Proof: Let $P$ be the longest common prefix of $w_{0}$ and $w_{1}$. The assumption $|u|>|v|$ implies that $P$ has suffix $v$. Let $L$ be the constant of Lemma 2 (where the $x$ of the lemma is any aperiodic extension of $s$ to a bi-infinite word). If $|x u|<L$ then $u$ and hence $v$ are small and we are done, so suppose instead that $|x u| \geq L$.

Suppose first that $|P| \geq L$. Then by Lemma 2, the words $x u$ and $P$ each have unique interpretations in $s$. Let $(\pi, z, \sigma)$ be the interpretation of $P$ in $s$. Then there exist positions $I, J$ in $z$ such that the two $x$ 's in $w_{0}$ and $w_{1}$ occur in $T(z[I])$ and $T(z[J])$. Furthermore, by the uniqueness of the interpretations of $x u$ in $s$ we have $z[I]=z[J]$ and the $x$ 's occur at the same positions of $T(z[I])$ and $T(z[J])$. Recalling that $T$ is order-preserving on infinite words, we see that by Lemma 5, the relative orders of both pairs $\left(T(s)[i+\ell, \infty], T(s)\left[i+\ell^{\prime}, \infty\right]\right)$ and $\left(T(s)\left[i^{\prime}+\ell, \infty\right], T(s)\left[i^{\prime}+\ell^{\prime}, \infty\right]\right)$ are determined by the factor $z$ of $s$. This contradicts the assumption that these two pairs of infinite words have opposite relative orders.

Now suppose that $|P|<L$. Since $|x u| \geq L>|P|$ and $P$ contains both occurrences of $x$, the two occurrences of $x u$ in $w_{0}$ must overlap. Let $Q$ be the factor of $w_{0}$ consisting of exactly these two overlapping occurrences of $x u$. The word $Q$ has a period which is at most the distance between the two $x$ 's, and since $P$ contains both $x$ 's, this period is therefore at most $|P|$. Then by Lemma 6 we have $|Q| \leq K(|P|)$, and so a fortiori, we have $|u| \leq K(|P|)$.

In both cases, we get an upper bound on the lengths of both $u$ and $v$ that depends only on $T$ and $s$.
In the next lemma, by minimal we mean that no proper factor has two permutations.
Lemma 8. Minimal factors of $T(s)$ with two permutations are small.
Proof: Let $w$ be a minimal factor of $T(s)$ with two permutations. Let $w_{0}$ and $w_{1}$ be as in Equation (1) with $|u|>|v|$. By the minimality of $w$ we may suppose that $w_{0}$ and $w_{1}$ begin with the first occurrence of xu0.

By Lemma 7, the word $u$ is small. If $w$ is sufficiently large, then (by the uniform recurrence of $T(s)$ ) it contains a second occurrence of $x u 0$ (which begins with $x v 1$ ), which contradicts the minimality of $w$.

Next, we handle factors with more than two permutations.
Lemma 9. Factors of $T(s)$ with more than two permutations are small.
Proof: Suppose $w$ has three permutations in $T(s)$. Let $w_{0}, w_{1}, w_{2}$ be minimal length factors of $T(s)$ with prefix $w$ extended far enough to the right for the permutations of $w$ to be determined. Assume also that $w$ has a different permutation in each. Suppose further that the longest common prefix of $w_{0}$ and $w_{1}$ is shorter than the longest common prefix of $w_{0}$ and $w_{2}$. As in Equation 1, we may write

$$
\begin{aligned}
& w_{0}=\rho_{1} x u 0 \gamma=\rho_{2} x u 1 \\
& w_{1}=p_{1} x v 1 g=p_{2} x v 0
\end{aligned}
$$

where $|u|>|v|$. Thus $v 1$ is a prefix of $u$, and thus the common prefix of $w_{0}, w_{1}$ ending in the second $x v$ (call this $P_{0}$ ) is right special. By Lemma 7, $\left|P_{0}\right|-|w|$ is small. Similarly, since $w$ has different permutations in $w_{0}$ and $w_{2}$, these two words have a common right special prefix $P_{1}$, where again $\left|P_{1}\right|-|w|$ is small. Now if $w$ is large, then so are $P_{0}$ and $P_{1}$, and hence $T(s)$ contains exactly one right special factor of length $\left|P_{0}\right|$. Consequently, the suffix of $P_{1}$ of length $\left|P_{0}\right|$ is in fact equal to $P_{0}$. Since $\left|P_{0}\right|-|w|$ and $\left|P_{1}\right|-|w|$ are both small, the quantity $\left|P_{1}\right|-\left|P_{0}\right|$ is also small. It follows that $P_{1}$ has period $\left|P_{1}\right|-\left|P_{0}\right|$, and therefore, by Lemma 6, we have $\left|P_{1}\right| \leq K\left(\left|P_{1}\right|-\left|P_{0}\right|\right)$, as required.

Lemma 10. If $T(s)$ has $k$ factors of length $n$ with two permutations for $n$ sufficiently large, then $T(s)$ has at least $k$ factors of length $n+1$ with two permutations.

Proof: Suppose that $n$ is sufficiently large that $T(s)$ has exactly one right special factor of each length for lengths $n$ and larger. Let $w$ be a factor of length $n$ with two permutations in $T(s)$. Note that we can uniquely extend $w$ to the right until the result becomes a right special factor of $T(s)$. As in Equation 1, write

$$
\begin{aligned}
& w_{0}=\rho_{1} x v 1 q 0 \gamma=\rho_{2} x v 1 q 1 \\
& w_{1}=p_{1} x v 1 g=p_{2} x v 0
\end{aligned}
$$

where $v 1 q=u$. Let $P_{0}$ be the common prefix of $w_{0}, w_{1}$ ending with the second $v$. Note that $P_{0}$ is right special. If both $a P_{0} 1 q 1$ and $a P_{0} 0$ occur in $T(s)$ for $a=0$ or $a=1$, then $a w$ has two permutations. Assume this is not the case. Let $a$ be such that $a P_{0}$ is right special (such an $a$ exists, since otherwise there would be two right special factors of length $\left|P_{0}\right|$ ). Then any occurrence of $a P_{0} 1$ is not followed by $q 1$. Hence, $P_{0} 1 y$ is right special for $y$ some prefix of $q 1$. By Lemma $7, v$ and $q$ (and thus $y$ ) are small. Set $P_{1}=P_{0} 1 y$ and apply the same argument as in the end of the proof of Lemma 9. We find that $\left|P_{1}\right| \leq K\left(\left|P_{1}\right|-\left|P_{0}\right|\right)$, contradicting the assumption that $w$ is large.

Lemma 11. If $T(s)$ has $k$ factors of length $n$ with two permutations for $n$ sufficiently large, then $T(s)$ has at most $k$ factors of length $n+1$ with two permutations.

Proof: Suppose that $n$ is sufficiently large that $T(s)$ has exactly one right special factor of each length for lengths $n$ and larger. If $T(s)$ has no factors with two permutations the result is trivial, so assume otherwise. Let $a w$ be a factor of $T(s)$ of length $n+1$ with two permutations where $|a|=1$. If $w$ does not have two permutations, then the same argument as in the proof of Lemma 8 applied to $a w$. where $a$ necessarily plays the role of $x$ (since otherwise, if the $x$ 's were contained in $w$, then $w$ would have two permutations), shows that in this case $a w$ is small, which is a contradiction. So in fact $w$ does have two permutations.

This shows that there are at most $k$ factors of length $n+1$ with two permutations except in one particular circumstance: $w$ is left special and both $a w$ and $b w$ have two permutations, where $a$ and $b$ are different letters. Write

$$
\begin{aligned}
& w_{0}=a \rho_{1} x v c q d \gamma=\rho_{2} x v c q c \\
& w_{1}=a p_{1} x v c g=p_{2} x v d
\end{aligned}
$$

where $x \in\{0,1\}, c$ and $d$ are different letters, and the relative positions of the $x$ 's in $w_{0}$ and $w_{1}$ are the same. Similarly, write

$$
\begin{aligned}
& w_{0}^{\prime}=b \rho_{1}^{\prime} x^{\prime} v^{\prime} c^{\prime} q^{\prime} d^{\prime} \gamma^{\prime}=\rho_{2}^{\prime} x^{\prime} v^{\prime} c^{\prime} q^{\prime} c^{\prime} \\
& w_{1}^{\prime}=b p_{1}^{\prime} x^{\prime} v^{\prime} c^{\prime} g^{\prime}=p_{2}^{\prime} x^{\prime} v^{\prime} d^{\prime}
\end{aligned}
$$

where $x^{\prime} \in\{0,1\}, c^{\prime}$ and $d^{\prime}$ are different letters, and the relative positions of the $x^{\prime}$ 's in $w_{0}^{\prime}$ and $w_{1}^{\prime}$ are the same.

Let $a P_{0}$ (resp. $b P_{1}$ ) be the longest common prefix of $w_{0}$ and $w_{1}$ (resp. $w_{0}^{\prime}$ and $w_{1}^{\prime}$ ). Then $a P_{0}$ ends with the second $x v$ and $b P_{1}$ ends with the second $x^{\prime} v^{\prime}$. Furthermore, both $P_{0}$ and $P_{1}$ are right special and both have $w$ as a prefix. If $\left|P_{0}\right|=\left|P_{1}\right|$, then $a P_{0}$ and $b P_{1}$ are distinct right special factors of the same length, which is a contradiction. So suppose that $\left|P_{1}\right|>\left|P_{0}\right|$. As in the end of the proof Lemma 9, we argue that since $T(s)$ only contains one right special factor of length $\left|P_{0}\right|$, the suffix of $P_{1}$ is equal to $P_{0}$. However, unlike in Lemma 9, we cannot say that $P_{0}$ is also a prefix of $P_{1}$. So let $P_{1}^{\prime}$ be the prefix of $P_{1}$ of length $\left|P_{1}\right|-\left|P_{0}\right|+|w|$. Then $P_{1}^{\prime}$ begins and ends with $w$. Consequently, $P_{1}^{\prime}$ has period $\left|P_{1}^{\prime}\right|-|w|$. By Lemma 7, the quantity $\left|P_{1}\right|-|w|$ and hence $\left|P_{1}^{\prime}\right|-|w|$ is small. Applying Lemma 6, we find that $\left|P_{1}^{\prime}\right| \leq K\left(\left|P_{1}^{\prime}\right|-|w|\right)$, which is again a contradiction.

This completes the proof of Theorem 3.

## 5 Effect of an arbitrary aperiodic morphism on the order

Clearly, one would like to show that Theorem 3 holds without the assumption that $T$ is marked. There are two difficulties: first, we need to establish that $T$ always preserves (or reverses) the order on infinite words; and second, we need recognizability properties similar to the marked case. The latter issue seems difficult to resolve, but we can establish the necessary properties regarding the order.

Richomme [8, Lemma 3.13] characterized the order-preserving binary morphisms.
Lemma 12 ([8]). Let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be a non-empty morphism. Then $T$ is order-preserving if and only if $T(01)<T(1)$.
Let $u$ and $v$ be infinite words. A morphism $T$ is order-reversing on infinite words if whenever $u<v$ we have $T(u)>T(v)$.

Lemma 13. Let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be a morphism such that $T(1)$ is not a prefix of $T(01)$. Suppose that $T(01)>T(1)$. Then $T$ is order-reversing on infinite words.

Proof: By hypothesis $T(1)$ is not a prefix of $T(01)$, so $T(0) \neq \epsilon$, and therefore we can write $T(01)=$ $X 1 Y$ and $T(1)=X 0 z$ for some words $X, Y$, and $z$. Let $k$ be maximal such that $T\left(0^{k}\right)$ is a prefix of $T(1)$. Then $T\left(0^{k}\right)$ is a prefix of $T(01)$ and hence a prefix of $X$. Write $X=T\left(0^{k}\right) x$. Then $T(1)=T\left(0^{k}\right) x 0 z$ and $T(01)=T\left(0^{k+1}\right) x 0 z=T\left(0^{k}\right) x 1 Y$. Hence $T(0) x 0 z=x 1 Y$. By the maximality of $k$, the word $T(0)$ is not a prefix of $x$, so $x 1$ is a prefix of $T(0)$. We therefore have $T(0)=x 1 y$ for some word $y$ and $T(1)=T\left(0^{k}\right) x 0 z$.

Now let $u$ and $v$ be infinite words such that $u<v$. Without loss of generality we may assume that $u$ begins with 0 and $v$ begins with 1 . Then $T(u)$ begins with $T\left(0^{k+1}\right)=T\left(0^{k}\right) x 1 y$ and $T(v)$ begins with $T(1)=T\left(0^{k}\right) x 0 z$, and so $T(u)>T(v)$. Hence $T$ is order-reversing on infinite words, as required.

Lemma 14. Let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be an aperiodic morphism such that $T(1)$ is a prefix of $T(01)$. Then $T$ is either order-preserving on infinite words or $T$ is order-reversing on infinite words.

Proof: Case 1: $T(1)$ is not a prefix of $T(0)$. Note that $T(1)$ has period $|T(0)|$, and so we can write $T(0)=x^{k}$ and $T(1)=x^{\ell} y$, where $x$ is primitive, the exponent $\ell$ is maximal, and $y$ is a non-empty proper prefix of $x$. Let $u$ and $v$ be infinite words with $u<v$. Without loss of generality we may assume that $u$ begins with 0 and $v$ begins with 1 . Note that $T(u)$ begins with $x^{\ell+1} y$ (since $y$ is a prefix of $x$ ) and $T(v)$ begins with $x^{\ell} y x$. If $x^{\ell+1} y \neq x^{\ell} y x$, then either $T(u)<T(v)$ for all $u<v$ or $T(u)>T(v)$ for all $u<v$. Thus $T$ is either order-preserving on infinite words or order-reversing on infinite words. If $x^{\ell+1} y=x^{\ell} y x$, then we have $x y=y x$, which implies that $x$ and $y$ are powers of a common word, which contradicts the primitivity of $x$.

Case 2: $T(1)$ is a prefix of $T(0)$. Write $T(1)=x^{k}$ and $T(0)=x^{\ell} y$, where $x$ is primitive, the exponent $\ell$ is maximal, and $y$ is non-empty. Let $u$ and $v$ be infinite words with $u<v$. Without loss of generality we may assume that $u$ begins with 0 and $v$ begins with 1 . Note that $T(u)$ begins with $x^{\ell} y$ and $T(v)$ begins with $x^{\ell+1}$. If $y$ is not a prefix of $x$ then either $T(u)<T(v)$ for all $u<v$ or $T(u)>T(v)$ for all $u<v$. Thus $T$ is either order-preserving on infinite words or order-reversing on infinite words. If $y$ is a prefix of $x$, we apply the argument from Case 1 to obtain the desired conclusion.

Theorem 15. Let $T:\{0,1\}^{*} \rightarrow\{0,1\}^{*}$ be an aperiodic morphism. Then $T$ is either order-preserving on infinite words or $T$ is order-reversing on infinite words.

Proof: For an arbitrary morphism $T$, one of the following properties must hold:

1. $T(01)<T(1)$ (in which case $T$ is order-preserving by Lemma 12);
2. $T(01)>T(1)$, but $T(1)$ is not a prefix of $T(01)$ (in which case $T$ is order-reversing on infinite words by Lemma 13);
3. $T(1)$ is a prefix of $T(01)$ and $T$ is aperiodic (in which case $T$ is either order-preserving on infinite words or order-reversing on infinite words by Lemma 14);
4. $T(1)$ is a prefix of $T(01)$ and $T$ is periodic.

The only case where we don't have the desired conclusion is when $T$ is periodic.

## Acknowledgments

We would like to thank Lucas Mol for his very helpful feedback on earlier drafts of this paper. We also thank the anonymous referees for their excellent and useful comments.

## References

[1] S. V. Avgustinovich, A. E. Frid, S. Puzynina, "Minimal complexity of equidistributed permutations", European J. Combin. 65 (2017), 24-36.
[2] V. Berthé, W. Steiner, J. Thuswaldner, R. Yassawi, "Recognizability for sequences of morphisms". Preprint available at https://arxiv.org/abs/1705.00167.
[3] D. G. Fon-der-Flaass, A. E. Frid, "On periodicity and low complexity of infinite permutations", European J. Combin. 28 (2007), 2106-2114.
[4] M. Lothaire, Algebraic Combinatorics on Words, Cambridge, 2002.
[5] M. A. Makarov, "On permutations generated by infinite binary words", Sib. Elektron. Mat. Izv. 3 (2006), 304-311.
[6] M. A. Makarov, "On permutations generated by Sturmian words", Sibiirsk. Mat. Zh. 50 (2009), 850857; translation in Sib. Math. J. 50 (2009), 674-680.
[7] M. Paul, "Minimal symbolic flows having minimal block growth", Math. Systems Theory 8 (1974/75), 309-315.
[8] G. Richomme, "Lyndon morphisms", Bull. Belg. Math. Soc. 10 (2003), 761-785.
[9] A. Valyuzhenich, "On permutation complexity of fixed points of some uniform binary morphisms", Discrete Math. Theor. Comput. Sci. 16 (2014), 95-128.
[10] S. Widmer, "Permutation complexity of the Thue-Morse word", Adv. Appl. Math. 47 (2011), 309329.


[^0]:    *The second author is supported by an NSERC Discovery Grant.
    ISSN 1365-8050 © 2018 by the author(s) Distributed under a Creative Commons Attribution 4.0 International License

