On neighbour sum-distinguishing {0,1}-weightings of bipartite graphs

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Let $S \subset \mathbb{Z}$ be a set of integers. A graph G is said to have the *S-property* if there exists an S-edge-weighting $w : E(G) \to S$ such that any two adjacent vertices have different sums of incident edge-weights. In this paper we characterise all bridgeless bipartite graphs and all trees without the {0,1}-property. In particular this problem belongs to P for these graphs while it is NP-complete for all graphs.

1   Introduction

The problems investigated in this paper are highly related to the well-known *1,2,3-Conjecture* formulated in [2]. One way to approach this conjecture (see for example [3]) has been to study the *{a, b}-property* of graphs for two integers a and b defined in the following way: a graph G is said to have the {a, b}-property if there exists a mapping $w : E(G) \to \{a, b\}$ such that for all pairs of adjacent vertices u and v we have $\sum_{e \in E(v)} w(e) \neq \sum_{e \in E(u)} w(e)$, where $E(v)$ and $E(u)$ denote the edges incident to v and u respectively. We call w a *neighbour sum-distinguishing edge-weighting* of G with weights a and b.

In [4] Lu investigated the problem of determining whether or not a given bipartite graph has the $\{0, 1\}$- or the $\{1, 2\}$-property. The restriction to bipartite graphs was motivated by a result by Dudek and Wajc [1] saying that the problem is NP-complete for general graphs. In particular Lu asked the natural question whether the problem is polynomial if only bipartite graphs are considered (Problem 1 in [4]). The results of the present paper answer in the affirmative for bridgeless bipartite graphs and trees. Lu also proved the following theorem:

Theorem 1.   [4] *Every 2-connected and 3-edge-connected bipartite graph has the {0, 1}- and the {1, 2}-property.*

In [6] Skowronek-Kaziów investigated the problem of determining whether a graph has a $\{1, 2\}$-edge-weighting such that the following vertex-colouring is proper: for each vertex v, assign the product of the edge-weights incident to v as v's colour. This product-property is the same as the {0, 1}-property and Skowronek-Kaziów verified this for various classes of bipartite graphs, for example bipartite graphs

of minimum degree at least 3. In [6] Skowronek-Kaziów also asked for a characterization of all bipartite graphs, in particular trees, which have such $\{1, 2\}$-edge-weightings, that is, which have the $\{0, 1\}$-property. As mentioned above the results of the present paper give such a characterization for trees and bridgeless bipartite graphs.

A bipartite graph without the $\{0,1\}$-property is said to be *bad*.

Thomassen, Wu and Zhang [7] gave a complete characterisation of all bipartite graphs without the $\{1, 2\}$-property. Any such graph is an *odd multi-cactus* defined as follows: Take a collection of cycles of length 2 modulo 4, each of which have edges coloured alternately red and green. Then form a connected simple graph by pasting the cycles together, one by one, in a tree-like fashion along green edges. Finally replace every green edge by a multiple edge of any multiplicity. The graph with one edge and two vertices is also called an odd multi-cactus. It can easily be checked that an odd multi-cactus do not have the $\{a, b\}$-property for any $a, b \in \mathbb{Z}$. As mentioned above these graphs characterise the bipartite graphs without the $\{1,2\}$-property:

Theorem 2. [7] *G is a connected bipartite graph without the $\{1, 2\}$-property if and only if G is an odd multi-cactus.*

Since an odd multi-cactus is recognisable in polynomial-time, this answers the part of Lu's problem from [4] concerning the $\{1, 2\}$-property. As pointed out in [7], Theorem 2 extends to all positive edge-weights a and b of distinct parity but not to the edge-weights 0 and 1. In [7] it is also remarked that any bipartite graph of minimum degree at least 3 has the $\{a, b\}$-property for all pairs of non-negative integers a, b of distinct parity. Thus it remains open to characterise those bipartite graphs with cut-vertices and minimum degree at most 2 which do not have the $\{0,1\}$-property.

In [4], Lu gave the following example of a bad graph with the $\{1, 2\}$-property: Two 6-cycles connected by a path of length 3 and, as noted in [7], we can construct an infinite number bad graphs with the $\{1, 2\}$-property by the following procedure: Take two graphs without the $\{0, 1\}$-property and join them by a path of length 3 modulo 4. We can even generalise this procedure further: Let $s \geq 0$ be an integer and let P be a path of length 1 modulo 4. Join each intermediate vertex in P to s bad graphs by s edges, and join the end-vertices of P to $s + 1$ bad graphs (see Figure 1). This will create a new bad graph with the $\{1, 2\}$-property.
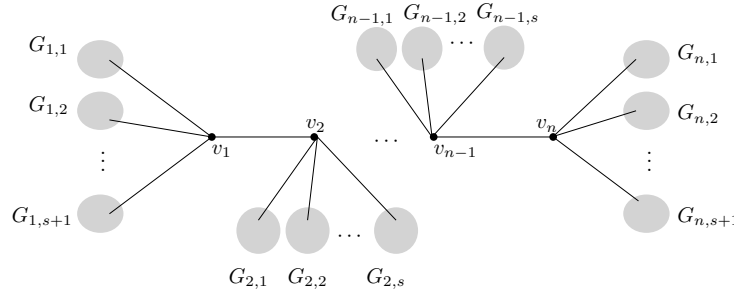

Fig. 1: A construction of bad graphs with the $\{1,2\}$-property.

Although the preceding paragraph shows a large class of bipartite graphs without the $\{0,1\}$-property, the list is still not complete. Not even for trees, as demonstrated by the tree in Figure 2.
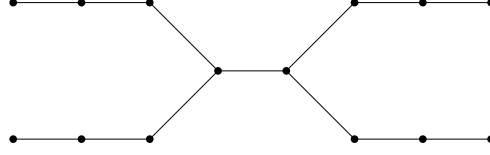
Fig. 2: A tree without the {0,1}-property.

Thus there is a large class of bad graphs which are not odd multi-cacti and it seems that the {0,1}-property is very different from the {1,2}-property. However, note that the above procedure always create bridges. This gives the hint that the {0,1}-property and the {1,2}-property might behave in a similar way if we don't allow bridges. This is indeed true as we prove in Section 2:

Theorem 3. *G is a connected bridgeless bipartite graph without the $\{0,1\}$-property if and only if G is an odd multi-cactus.*

As mentioned after Theorem 2, an odd multi-cactus is recognisable in polynomial time so this answers the part of Lu's problem from [4] concerning the {0,1}-property for bridgeless bipartite graphs. In Section 3 we provide additional operations for constructing trees without the {0,1}-property. The class of trees without the $\{0,1\}$-property we can obtain using these operations we call $\mathcal{B}$ and these are all recognisable in polynomial time. Whilst the class $\mathcal{B}$ is difficult to describe we show that this gives a full characterisation of all bad trees.

Theorem 4. *A tree T has the $\{0,1\}$-property unless T is a member of $\mathcal{B}$.*

Taken together, Theorems 3 and 4 show a marked difference between the {0,1}-problem and the {1,2}-problem. Indeed for bridgeless bipartite graphs Theorems 3 and 2 show that the class of graphs without the {0,1}-property and the class of graphs without the {1,2}-property are precisely the same. On the other hand, Theorems 2 and 4 show that this is far from the case with trees.

2 Bridgeless bipartite graphs without the {0,1}-property

Let G be a bipartite graph. A $\{0,1\}$-*weighting* of G is a map $w : E(G) \to \{0,1\}$. Given a $\{0,1\}$-weighting w of G and a vertex v of G we call the sum $\sum_{e \in E(v)} w(e)$ the *weighted degree* of v or the induced *colour* of v (induced by w). For convenience the weighted degree of a vertex v is also denoted $w(v)$. We say that a $\{0,1\}$-weighting w is *neighbour sum-distinguishing* or *proper* if for all pairs of adjacent vertices u, v it holds that $\sum_{e \in E(v)} w(e) \neq \sum_{e \in E(u)} w(e)$. That is, if the induced vertex-colouring is proper. If w is a $\{0,1\}$-weighting of G and two adjacent vertices u and v have the same weighted degree, then we say that the edge uv is a *conflict*. If two adjacent vertices u, v have the same weighted degree parity we call the edge uv a *parity conflict*. Note that a parity conflict is not necessarily a conflict. If $f : V(G) \to \mathbb{Z}_k$ is a mapping and H is a spanning subgraph of G such that for all vertices v we have $d_H(v) \equiv f(v) \mod k$ then we say that H is an *f-factor modulo k*. These factors play an important role in the investigations of $\{a, b\}$-properties for bipartite graphs, in particular because of the following result mentioned in [7]:

Lemma 5. [7] *Let G be a connected graph. If $f : V(G) \to \mathbb{Z}_2$ is a mapping satisfying $\sum_{v \in V(G)} f(v) \equiv 0 \mod 2$, then G contains an f-factor modulo 2.*

As also pointed out in [3], [6] and [7] this immediately implies that all bipartite graphs where one bi-partition set has even size have the $\{a, b\}$-property when a and b are numbers of different parity, since the weighted degree of all the vertices belonging to the even-sized bipartition set can get odd weighted degree while all other vertices get even weighted degree. So the problem is reduced to the case where both bipartition sets have odd size. Another useful tool is Lemma 6 below.

Lemma 6. [7] *Let q be a natural number such that $q \geq 4$. Let G be a connected graph and let A be an independent set of at most q vertices such that each vertex in A has degree at least $q - 1$, or, each vertex in A, except possibly one has degree at least q. Assume that no vertex in A is adjacent to a bridge in G. Then, for each vertex a of A, there is an edge e_a incident with a such that the deletion of all e_a, $a \in A$, results in a connected graph unless $|A| = q = 4$, all vertices of A have degree 3 and $G - A$ has six components each of which is joined to two distinct vertices of A.*

As can be seen in [7] and later in this paper, Lemma 5 and 6 work well together under some assumptions in the following way: Let G be a simple bipartite graph with an odd number of vertices in both bipartition sets X and Y, and let w_0 be a vertex belonging to X with at least 4 neighbours and which is not a cutvertex. Assume that no neighbour of w_0 has greater degree than w_0 (such a vertex w_0 is said to have *local maximum degree*), and such that no neighbour of w_0 having the same degree as w_0 is incident to a bridge in $G - w_0$. Furthermore, assume that we are not in the exceptional case in Lemma 6 when we remove w_0 and choose A to be the neighbours of w_0 having the same degree as w_0. Now we can find a proper $\{0, 1\}$-weighting of G as follows. We remove w_0 and an edge e_a incident to each $a \in A$ and maintain connectivity by Lemma 6. We call the resulting graph G'. First consider the case where w_0 has even degree. By Lemma 5 we find a $\{0, 1\}$-weighting of G' such that all vertices in $X \backslash \{w_0\} \cup N(w_0)$ have odd weighted degree and all vertices in $Y \backslash N(w_0)$ have even weighted degree. Now we extend this $\{0, 1\}$-weighting to the whole of G by assigning weight 1 to all edges incident to w_0 and weight 0 to all edges e_a. The parity conflicts are between w_0 and its neighbours, but because all edges e_a have weight 0, the weighted degree of w_0 is strictly greater than that of all its neighbours. In the case where w_0 has odd degree, we find a $\{0, 1\}$-weighting of G' such that all vertices in $X \backslash \{w_0\} \cup N(w_0)$ have even weighted degree and all vertices in $Y \backslash N(w_0)$ have odd weighted degree. As before we extend this $\{0, 1\}$-weighting to the whole of G by assigning weight 1 to all edges incident to w_0 and weight 0 to all edges e_a.
Note that this shows that whenever we consider a vertex w_0 which is not a cutvertex, then we can find a $\{0, 1\}$-weighting where all edges incident to w_0 have weight 1 and the only parity conflicts are between w_0 and its neighbours.

Before we prove Theorem 3, we will need three facts about simple odd multi-cacti formulated in Lemmas 7, 8 and 9 below.
If G is an odd multi-cactus then, by definition, G contains at least two cycles containing two adjacent vertices with at least three neighbours each in G while the other vertices all have two neighbours in G, unless G is a single cycle or K_2 (possibly with multiple edges). Cycles of this type are called *end-cycles* in G.

Lemma 7. *Let $G \neq K_2$ be a simple odd multi-cactus. For any vertex $v \in V(G)$ there is a $\{0, 1\}$-weighting of G such that v and all vertices in the opposite bipartition set to v get weighted degree 1 and all other vertices get weighted degree 0 or 2.*

Proof: The proof is by induction on the number of vertices n. It is easy to check that the statement is true for a single cycle of length 2 modulo 4, so assume $n > 6$. Let C be an end-cycle in G such that v is not a vertex in C with only two neighbours. We can assume C is a 6-cycles since subdividing edges with four vertices preserves the conclusion of the lemma. Thus, say that $C = v_1 v_2 \cdots v_6 v_1$, where v_1 and v_2 have at least three neighbours in G. Since v is in $G - \{v_3, v_4, v_5, v_6\}$ we can use the induction hypothesis on $G - \{v_3, v_4, v_5, v_6\}$ and extend this $\{0, 1\}$-weighting to the whole of G. $\qquad\square$

Let w be a $\{0, 1\}$-weighting of G, let v be a vertex of G and let a be a natural number. Finally, let C_w denote the vertex-colouring induced by w. Let $C_w(v, a)$ denote the colouring obtained from C_w by replacing the colour $C_w(v)$ of v with the colour $C_w(v) + a$. If $C_w(v, a)$ is a proper vertex-colouring we say that *w is a proper $\{0, 1\}$-weighting of G when the degree of v is increased by a*. This may be thought of as a neighbour sum-distinguishing edge-weighting where the vertex v has some pre-assigned weight.

Lemma 8. *Let $G \neq K_2$ be a simple odd multi-cactus. Furthermore, let u, v be any two vertices in G belonging to the same bipartition set (possibly $u = v$). There is a proper $\{0, 1\}$-weighting of G when the weighted degrees of both u and v are increased by 1 (if $u = v$ the weighted degree is increased by 2).*

Proof: First note that the case $u = v$ follows from Lemma 7, so we assume that $u \neq v$. The proof is by induction on the number of vertices n. It is easy to check that the statement holds for a single cycle of length 2 modulo 4. As in the proof of Lemma 7 we choose and end-cycle C such that one of v and u, say, u is not a vertex in C with only two neighbours in G and we may assume that $C = v_1 v_2 \cdots v_6 v_1$, where v_1 and v_2 have at least three neighbours in G. If v and u are both in $G - \{v_3, v_4, v_5, v_6\}$ then we use the induction hypothesis on $G - \{v_3, v_4, v_5, v_6\}$ and get a proper $\{0, 1\}$-weighting of $G - \{v_3, v_4, v_5, v_6\}$ if the weighted degree of both u and v are increased by 1. We can easily extend this $\{0, 1\}$-weighting to the whole of G, a contradiction. So we can assume that u is in $G - \{v_3, v_4, v_5, v_6\}$ and v is one of v_3, v_5 (the other cases are similar). If u is one of v_1, v_2, say, v_1 and v is v_5, then we use Lemma 7 on $G - \{v_3, v_4, v_5, v_6\}$ choosing v_1 as our special vertex. Then we get a $\{0, 1\}$-weighting w of $G - \{v_3, v_4, v_5, v_6\}$ where v_1 and all vertices in the opposite bipartition set to v_1 get weight 1 and all other vertices get weight 0 or 2. We extend this $\{0, 1\}$-weighting to the whole of G by defining $w(v_1 v_6) = w(v_4 v_5) = 1$ and $w(v_2 v_3) = w(v_3 v_4) = w(v_5 v_6) = 0$.

If $u = v_1$ and v is v_3, then again we use Lemma 7 on $G - \{v_3, v_4, v_5, v_6\}$ choosing v_1 as our special vertex. As before we get a $\{0, 1\}$-weighting w of $G - \{v_3, v_4, v_5, v_6\}$ we can extend to the whole of G by defining $w(v_1 v_6) = w(v_3 v_4) = 1$ and $w(v_2 v_3) = w(v_4 v_5) = w(v_5 v_6) = 0$.

This leaves us with the case where u is in $G - C$ and v is one of $\{v_3, v_4, v_5, v_6\}$. We can assume that v_1 is in the same bipartition set as u and we start by considering the case where $v = v_3$. In this case we use the induction hypothesis on $G - \{v_3, v_4, v_5, v_6\}$ choosing u and v_1 as our special vertices. We extend this $\{0, 1\}$-weighting, letting the edge $v_1 v_6$ play the role of the extra weight on v_1 by defining $w(v_1 v_6) = 1$ and $w(v_2 v_3) = 0$. Now v_1 and v_2 have different weighted degrees by the induction hypothesis so we can choose the weights on $v_3 v_4$ and $v_5 v_6$ to be different such that we avoid conflicts between v_6 and v_1, between v_3 and v_2 and between v_4 and v_5. Finally we define $w(v_4 v_5) = 0$ to avoid conflicts between v_5 and v_6, and between v_3 and v_4.

The case where $v = v_5$ remains. Here we use Lemma 7 on $G - \{v_3, v_4, v_5, v_6\}$ choosing u as our special vertex and extend this $\{0, 1\}$-weighting to G by defining $w(v_2 v_3) = w(v_1 v_6) = w(v_3 v_4) = 0$ and $w(v_5 v_6) = w(v_4 v_5) = 1$. $\qquad\square$

Using Lemma 7 and induction as in the proof of Lemma 7 we can easily derive Lemma 9 below.

Lemma 9. *Let G be an odd multi-cactus where the red-green edge-colouring is unique. If G' is obtained from G by replacing a red edge with an edge of multiplicity > 1, then G has the $\{0, 1\}$-property.*

In a graph G, a *suspended path* or *suspended cycle* is a path or cycle $v_1 v_2 ... v_q$ such that all intermediate vertices have degree 2 and the end-vertices v_1, v_q have degree at least 3. All vertices $v_1, ..., v_q$ should be distinct, except that possibly $v_1 = v_q$ (if it is a suspended cycle).
Having these small facts established we are ready for the proof of Theorem 3. The proof follows the same approach as the proof of Theorem 2 in [7], but new problems arise which have to be dealt with along the way. At the end of the proof, the reader is referred to [7].

Proof of Theorem 3: It suffices to prove that if G is a connected bridgeless bipartite graph without the $\{0, 1\}$-property, then G is an odd multi-cactus.
Suppose the theorem is false and let G be a smallest counterexample. That is, among all bridgeless bipartite graphs without the $\{0,1\}$-property which are not odd multi-cacti, G has the fewest vertices and subject to that, the fewest edges. Note that by induction and Lemma 9 we can assume that if there is an edge uv of multiplicity greater than 1, then uv must have multiplicity 2 and be a bridge in the simple graph underlying G. Let X and Y be the two bipartition sets of G. By the remark following Lemma 5 we can assume that both X and Y have odd size.
First note that if $v \in X$ is a vertex in G which is only adjacent to one other vertex v' (since G is bridgeless the multiplicity of uv is then at least 2), then for any edge $e = v'v'' \neq v'v$ in $G - v$ incident to v', the graph $G' = G - v - e$ is connected. So by Lemma 5 the graph G' contains a spanning subgraph H where all vertices in $(X \setminus \{v, v''\}) \cup \{v'\}$ have odd degree and vertices in $Y \setminus \{v'\}$ have even weighted degree. By assigning weight 1 to all edges in $E(H) \cup \{e\}$ and weight 0 to all other edges we get a proper $\{0, 1\}$-weighting of G, a contradiction. Thus, we can assume that there is no vertex v in G which is only adjacent to one other vertex v'.
Let B denote an endblock in G. Note that the above implies that there are no multiple edges in B.

Claim 1: B contains no suspended path of length 2.

Assume that $y_1 x y_2$ is a suspended path of length 2 in B, where $x \in X$ and $y_1, y_2 \in Y$. By Lemma 5 there exists a spanning subgraph H of $G' = G - x$ such that all vertices in $X \setminus \{x\}$ have odd degree and all vertices in Y have even degree. From H we can construct a $\{0, 1\}$-weighting $w_{G'}$ of G' such that each vertex in $X \setminus \{x\}$ has odd weighted degree and each vertex in Y has even weighted degree. We do this by assigning weight 1 to the edges in H and weight 0 to the edges outside H. We extend this $\{0, 1\}$-weighting to a $\{0, 1\}$-weighting w_G of the whole graph G by defining $w_G(xy_1) = w_G(xy_2) = 0$. The only possible conflicts are xy_1 and xy_2 in the case where $w_{G'}(y_1) = 0$ or $w_{G'}(y_2) = 0$. If we can remove an edge $e_1 = y_1 z_1$ incident to y_1 and an edge $e_2 = y_2 z_2$ incident to y_2 in G' and still have a connected graph, then we can avoid this situation as follows: using Lemma 5 we redefine H to be a subgraph of $G - x - e_1 - e_2$ such that all vertices in $X - x - z_1 - z_2 \cup \{y_1, y_2\}$ have odd weighted degree and all other vertices have even weighted degree. Then define $w_{G'}$ to be the $\{0, 1\}$-weighting assigning weight 1 to all edges in $E(H) \cup \{e_1, e_2\}$ and weight 0 to all other edges. This is a proper $\{0, 1\}$-weighting of G, so we can assume that we cannot remove two edges incident to y_1 and y_2 respectively in G' and still have a connected graph.
There must be a cycle, C, going through y_1 and y_2 in G' since otherwise y_1 and y_2 lie in distinct blocks of G', and since the degree of both y_1 and y_2 is at least 3 and since G is bridgeless it is now possible to

remove an edge from both y_1 and y_2 and still have a connected graph, a contradiction. We first look at the case where $w_{G'}(y_1) = w_{G'}(y_2) = 0$. Here we swap all the weights in C (that is, we change all 1-weights to 0 weights and all 0-weights to 1-weights). This will not change the parity of the weighted degrees and now y_1 and y_2 both have weighted degree 2. We redefine $w_{G'}$ accordingly, put x back and give the edges xy_1 and xy_2 weight 0. This gives a proper {0,1}-weighting of G.

Now assume that $w_{G'}(y_1) = 0$ and $w_{G'}(y_2) \geq 2$. Actually we can assume that $w_{G'}(y_2) = 2$ since otherwise if $w_{G'}(y_2) > 2$ we just repeat the proof of the previous case (after swapping the weights in C the weighted degree of both y_1 and y_2 is at least 2). We can assume that all cycles going through y_1 in G' also go through y_2 (otherwise we simply swap the weights in a cycle containing y_1 and not y_2). The only possible case is where $G - \{x, y_1, y_2\}$ consists of precisely two connected components G_1, G_2 with bipartition sets X_i, Y_i each containing exactly one neighbour of both y_1 and y_2 (see Figure 3). Let x_1, x_2 denote the neighbours of y_1 in G_1 and G_2 respectively and let z_1, z_2 denote the neighbours of y_2 in G_1 and G_2 respectively. We allow the possibility that $x_1 = z_1$ or $x_2 = z_2$. If one of X_1, X_2 has even size, for example X_1, then the subgraph of G_1 consisting of all edges weighted 1 under $w_{G'}$ will have an odd number of odd degree vertices, which is not possible. So both X_1 and X_2 have odd size. The sets Y_1 and Y_2 must have different parity, in particular one of them, say, Y_1 has even size. Furthermore, if G_2 has a proper {0, 1}-weighting w_{G_2} then we can find a proper {0, 1}-weighting of the whole graph G with weight 0 on y_1x_2 and y_2z_2 as follows: If the weighted degrees of x_2 and z_2 have the same, say, odd parity under w_{G_2}, then because both bipartition sets in $G - G_2$ have even size, Lemma 5 implies that there is a proper {0,1}-weighing w_{G-G_2} of $G - G_2$ where y_1 and y_2 get even weighted degree. We can now define a proper {0,1}-weighting w_G of G by $w_G(e) = w_{G_2}(e)$ for $e \in E(G_2)$, $w_G(e) = w_{G-G_2}(e)$ for $e \in E(G - G_2)$ and $w_G(y_1x_2) = w_G(y_2z_2) = 0$. So the weighted degree of x_2 and z_2 do not have the same parity under w_{G_2}. Without loss of generality assume that x_2 has even weighted degree under w_{G_2} and z_2 has odd weighted degree under w_{G_2}. As before there is a proper {0,1}-weighting w_{G-G_2} of $G - G_2$ where all vertices in $X_1 \cup \{x\}$ get odd weighted degree and all other vertices get even weighted degree. When extending w_{G-G_2} and w_{G_2} to the whole of G, the only possible conflict that can arise is y_1x_2, but we can always avoid this conflict by swapping the weights of the edges in a cycle in $G - G_2$ containing y_1. The same kind of argument shows that there is no proper {0, 1}-weighting of G_2 where the weighted degree of both x_2 and z_2 are increased by 1 (increased by 2 if $x_2 = z_2$) since we can let the edges x_2y_1 and z_2y_2 play the roles of the extra weights. By the minimality of G, the subgraph G_2 must either be an odd multi-cactus or contain a bridge. Lemma 8 shows that G_2 cannot be an odd multi-cactus and hence it contains a bridge. Note that this shows that $x_2 \neq z_2$ and since B is an endblock, one of x_2, z_2, say x_2, is not a cutvertex in G_2. If x_1 is also not a cutvertex in G_1 we do the following: Weight $G_2 - x_2$ such that all vertices in $X_2 - \{x_2\}$ have odd degree and all vertices in Y_2 have even degree, and weight $G_1 - x_1$ such that all vertices in $X_1 - \{x_1\}$ have odd degree and all vertices in Y_1 have even degree. These two {0, 1}-weightings extend to the whole graph G by assigning weight 0 to all edges incident to x_1, x_2 and y_1 except that we assign weight 1 to the edges y_1x_1, and y_1x_2 and also to y_1x. So x_1 must be a cutvertex in G_1. Since B is an end-block it follows that z_2 is not a cutvertex in G_2 and if z_1 is not a cutvertex in G_1 we do the same as before (with x_1 replaced by z_1 and x_2 replaced by z_2). The only possibility is that $x_1 = z_1$ is a cutvertex. By Lemma 5 there is a {0, 1}-weighting w_{G_1} of G_1 where all vertices in $X_1 \backslash \{x_1\}$ get odd weighted degree and all other vertices get even weighted degree, and a {0, 1}-weighting w_{G_2} of G_2 where all vertices in $X_2 \backslash \{x_2\}$ get even weighted degree and all other vertices get odd weighted degree. We extend w_{G_1} and w_{G_2} to a {0, 1}-weighting of the whole of G by assigning weight 1 to the edges y_1x_2 and y_2x_1 and weight 0 to the edges y_2x, y_1x, y_2z_2 and y_1x_1. The

only possible conflicts are between y_1, y_2 and $x_1 = z_1$ if x_1 has weighted degree 1. If this is the case, then since x_1 is a cutvertex in G_1, there is a cycle in G_1 containing two edges with weight 0 incident to x_1. We can then swap the weights on such a cycle to avoid conflicts between x_1 and y_1 and y_2. This contradicts G being a bad graph.
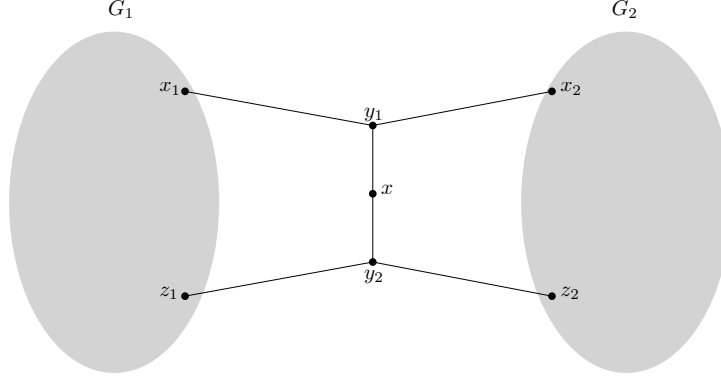

Fig. 3: An illustration of the situation in Claim 1.

Claim 2: B contains no suspended path or cycle of length 4.

Assume that $y_1x_1y_2x_2y_1$ is a suspended cycle of length 4 in B. By Lemma 5 there is a proper $\{0,1\}$-weighting of $G - \{x_1, y_2, x_2\}$ where all vertices in $X \backslash \{x_1, x_2\}$ get even weighted degree and all vertices in $Y \backslash \{y_2\}$ get odd weighted degree. This proper $\{0,1\}$-weighting can now be extended to the whole graph by assigning weight 1 to the edges y_1x_1 and x_2y_1 and weight 0 to the edges x_1y_2 and y_2x_2, a contradiction.

The case where $y_1x_1y_2x_2y_3$ is a suspended path of length 4 in G is treated in the same way as the suspended path of length 2 in the proof of the previous claim (here we just choose the graph $G - x_1 - y_2 - x_2$ as our G').

Claim 3: G contains no suspended path or cycle of length at least 5.

Suppose that $y_1x_1y_2x_2y_3x_3$ is a path in G where $x_1 \in X$ and where the degree in G of each of x_1, y_2, x_2, y_3 is 2. Now delete each of x_1, y_2, x_2, y_3 and add an edge y_1x_3 if there is not already such an edge. If the resulting graph is not an odd multi-cactus it has a proper $\{0,1\}$-weighting by the minimality of G. This $\{0,1\}$-weighting can now be used to find a proper $\{0,1\}$-weighting of G: we put back the vertices x_1, y_2, x_2, y_3. If y_3x_3 was not originally in G we give y_1x_1 and y_3x_3 the same weight as y_1x_3 and delete that edge. We give y_2x_2 the opposite colour. Then we give x_1y_2 and x_2y_3 distinct colours. Since y_1 and x_3 have different colours, there are two choices for this and one of them will give a proper $\{0,1\}$-weighting. If y_1x_3 was in G to begin with we assign weight 0 to the edges y_1x_1 and y_3x_3. We give y_2x_2 weight 1. Then we give x_1y_2 and x_2y_3 distinct colours. Again, there are two choices for this and one of them will give a proper $\{0,1\}$-weighting, a contradiction. So we can assume that G' is an odd multi-cactus. Since G is not an odd multi-cactus the only possibility is that G is obtained from an

odd multi-cactus by subdividing a green edge joining two vertices of degree at least 3 by four vertices. In this case we can find another path $y_1'x_1'y_2'x_2'y_3'x_3'$ where the degree in G of each of x_1', y_2', x_2', y_3' is 2 and define G' from that such that G' is not an odd multi-cactus, unless G consists of two vertices joined by $s \geq 3$ paths of length 5. In this case it is easy to check that G has a proper $\{0, 1\}$-weighting.

By Claims $1, 2, 3$ all degree-2 vertices in the endblocks of G lie on a suspended path of length 3. In G we replace all suspended paths of length 3 with an edge to form a multi-graph G^*. Edges arising from suspended paths will be called *blue edges*. Note that G^* is bridgeless and the minimum degree in any endblock is at least 3. Now let B be an endblock of G^*. Possibly $G^* = B$. If $B \neq G^*$, then we let x_0 be the unique cutvertex of G^* contained in B. If the deletion of some pair of neighbouring vertices in B disconnects G^*, then we define a graph B' as follows: we select an edge $y_0 z_0$ in B such that $G^* - y_0 - z_0$ is disconnected and such that some component, H, not containing x_0 of $G^* - y_0 - z_0$ is smallest possible. Possibly x_0 is one of z_0 and y_0. The union of that component, H, and y_0, z_0 together with all edges connecting them is called B'. Otherwise, if the deletion of any pair of adjacent vertices in B leaves a connected graph we define $B' = H = B$. Note that in this case we must have $B' = H = B = G^*$, since the deletion of x_0 together with any of its neighbours disconnects G^*.

Claim 4: There is an end-block B of G^* such that all vertices in H have degree 3 in G^*.

The overall strategy for proving this claim is to find a vertex w_0 in H with local maximum degree, and then use the procedure explained in the remark following Lemma 6 to find a proper $\{0, 1\}$-weighting of G where all edges incident to w_0 have weight 1.

Case 1: We can choose B to be an end-block whose cutvertex is adjacent to only one other block.

Suppose the claim is false and let $w_0 \in V(H)$ (if $B' = B$ choose w_0 distinct from x_0) be a vertex having maximum degree. We want to choose w_0 such that when we remove w_0 we avoid the exceptional case in Lemma 6 (when A is set to be the neighbours of w_0 having the same degree as w_0). If $d(w_0) \geq 5$ then this exceptional case cannot occur. If $d(w_0) = 4$ and some neighbour of w_0 has degree 3 then the exceptional case is also avoided. Such a w_0 is possible to choose unless all vertices in H have degree 4. If it is impossible to avoid the exceptional case then it must be that whenever we remove a vertex w_0 in H and its four neighbours the resulting graph has six components each having exactly two neighbours in $N(w_0)$. If this is the case we choose w_0 to be such that the component arising when deleting w_0 and its neighbours (there are six components) containing y_0 or z_0 is maximal. The other components are easily seen to be isolated vertices (otherwise we redefine w_0 to be a neighbour of w_0 not joined to the component containing y_0 or z_0 and this will contradict the choice of w_0). But these isolated vertices must have degree 2, a contradiction. This shows that we can always find a w_0 of maximum degree in H and avoid the exceptional case in Lemma 6 (when A is set to be the neighbours of w_0 having the same degree as w_0). We now choose such a w_0. When we have found and defined w_0 we go back to considering the original graph G. We will look at three different subcases:

1. w_0 is not a neighbour of z_0 or y_0.

2. w_0 is a neighbour of z_0 and $z_0 \neq x_0$.

    3. w_0 is a neighbour of x_0.

Subcase 1: This subcase is dealt with as described in the remark following Lemma 6. By the minimality of H, none of the neighbours of w_0 are incident to a bridge in $G - w_0$.

Subcase 2: We can assume that z_0 has degree strictly greater than that of w_0 since otherwise we do the same as in Subcase 1. This implies that the degree of z_0 is at least 5. We can assume that all vertices in H having maximum degree are adjacent to z_0 or y_0, since otherwise we can redefine w_0 and go to Subcase 1. Note that this implies that we can never be in the exceptional case in Lemma 6 when we delete a vertex v in H of maximum degree and define A to be the neighbours of v with the same degree as v.
We can assume that z_0 has precisely one neighbour in each component other than H in $B - y_0 - z_0$, since otherwise we do the same as in Subcase 1 except we now also remove two edges from z_0 that go to the same component of $B - y_0 - z_0$ other than H. If we then end up with a colour-conflict between w_0 and z_0 we have made sure that we can swap the weights in a cycle avoiding H that contains two edges with weight 0 incident to z_0. This will then avoid the conflict between w_0 and z_0 and give a proper $\{0, 1\}$-weighting of G. So z_0 has precisely one neighbour in each component other than H in $B - y_0 - z_0$. We can also assume that there is at most one component C other than H in $B - y_0 - z_0$ which contains a neighbour of z_0 since otherwise we can remove two edges incident to z_0 going to two different components distinct from H and use the same weight-swapping argument as before to avoid a conflict between z_0 and w_0 (this time the cycle will also go through y_0).
If z_0 has no neighbour in any component of $G - z_0 - y_0$ other than H, then since $G - y_0 - z_0$ is disconnected it must be the case that $y_0 = x_0$. If this is the case we redefine w_0 to be z_0 and go to Subcase 3. So we assume that there is such a component C other than H in $G - z_0 - y_0$ containing a neighbour of z_0. We start by doing the same as before giving w_0 maximum weighted degree and assigning 0 to at least one edge e_a incident to each neighbour a of w_0 which has the same degree as w_0. We also assign weight 0 to the unique edge $z_0 z_1$ joining z_0 to the component C. Actually, since w_0 and all the neighbours of w_0 have the same weighted degree-parity, each of these neighbours of w_0 with the same degree as w_0 will be incident to at least two edges weighted 0. We can assume that w_0 and z_0 have the same weighted degree and the edge $z_0 y_0$ has weight 1 (otherwise we can swap the weights in a cycle using the edges $z_0 z_1$ and $z_0 y_0$ to avoid the conflict between w_0 and z_0). If we swap the weights in a cycle in $G - w_0$ containing two edges incident to z_0 with the same weight, then the only conflicts we can create are between w_0 and a neighbour v of w_0 with the same degree as w_0, and these conflicts can only arise when the cycle goes through the only two edges incident to v with weight 0. If v is a neighbour of w_0 with the same degree as w_0 and incident to only two edges with weight 0, then we call this pair of weight 0-edges a *forbidden pair* of edges.
We will now show that we can always find a cycle in $G - w_0$ containing two edges incident to z_0 with the same weight that does not use any forbidden pair of edges. Note that all neighbours of w_0 which are incident with a forbidden pair of edges have the same degree as w_0 and are therefore neighbours of y_0. Let $v_1, ..., v_m$ denote these neighbours of w_0. Since z_0 has weighted degree strictly greater than 3, there is a vertex $z_2 \neq w_0$ in $N(z_0) \cap V(H)$. It suffices to find a path P from y_0 to a vertex z' in $N(z_0) \cap V(H)$ in the connected graph $G - z_0 - w_0$ (connected by the minimality of H) not using any forbidden pair of edges, since then we can define our cycle to be $P \cup \{z_0 y_0, z' z_0\}$ if the weight on $z_0 z'$ is 1, or $P \cup P_c$, where P_c is a path from z_0 to y_0 in $G - H - z_0 y_0$ if the weight on $z_0 z'$ is 0. See Figure 4. Since the graph $G - z_0 - w_0$ is connected, there is a path P_1 from z_2 to y_0. We can assume that this uses forbidden

pairs of edges. Without loss of generality let av_1 and v_1b be the first forbidden pair of edges P_1 uses when starting from z_2. Since v_1 is adjacent to y_0 it follows that $b = y_0$, since otherwise we can find a path from y_0 to z_2 not using any forbidden pair of edges. This shows that there is some path from y_0 to a vertex in $N(z_0) \cap V(H)$ only using one forbidden pair of edges. Now we look at all such paths only using one pair of forbidden edges y_0v_i and v_ia (for $i \in \{1, ..., m\}$) and choose a path P among those which goes through the most neighbours of w_0. Let y_0v_i and v_ia be the pair of forbidden edges P contains. Since y_0v_i and v_ia is a forbidden pair, the vertex v_i has a neighbour $v_i' \neq w_0$ in H such that v_iv_i' has weight 1. Since $G - w_0 - v_i$ is connected it has a path P' from v_i' to a vertex in $N(z_0) \cap V(H)$. The path P' must use a forbidden pair of edges, otherwise the graph induced by $E(P) \cup E(P')$ contains a desired path from y_0 to a vertex in $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Let the first pair of forbidden edges P' uses when starting from v_i' be bv and vc. The subpath P_1' of P' from v_i' to v must be disjoint from P, since otherwise the graph induced by $E(P) \cup E(P_1')$ contains a desired path from y_0 to $N(Z_0) \cap V(H)$ avoiding forbidden pairs of edges. Furthermore, we must have that $c = y_0$ since otherwise the path P'' defined to be y_0v together with the subpath of P' from v to v_i' followed by $v_i'v_i$ and the subpath of P from v_i to $N(z_0) \cap V(H)$ is a desired path from y_0 to $N(z_0) \cap V(H)$ avoiding forbidden pairs of edges. Now the path P'' contradicts the maximality of P.

This takes care of Subcase 2 if z_0 has a neighbour in some component C other than H in $G - z_0 - y_0$. If this is not the case then, as noted above, we can go to Subcase 3 redefining w_0 to be z_0.
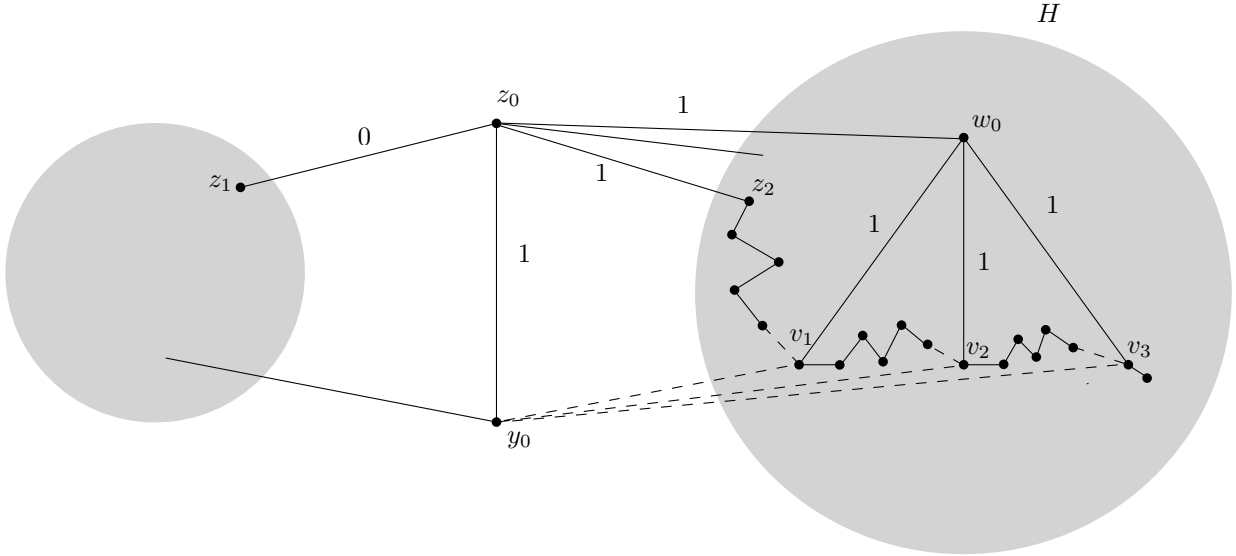

Fig. 4: An illustation of a situation in the proof of Claim 4. Dashed edges indicate pairs of forbidden edges.

Subcase 3: The situation is more or less the same as in Subcase 2 except now $z_0 = x_0$. If some vertex $w \in N(w_0) \backslash \{x_0\}$ has the same degree as w_0, then we can assume that we are in the exceptional case in Lemma 6 when we remove w and define A to be the set of neighbours of w with the same degree as

w (otherwise we redefine w_0 to be w and go to Subcase 1 or 2). So in this case the degree of both w_0 and w is 4 and so is the degree of all neighbours of w. Choose w in H to be a non-neighbour of x_0 with the same degree as w_0 such that the component arising when deleting w and its neighbours (there are six components) containing y_0 or z_0 is maximal. The other components are easily seen to be isolated vertices, and this contradicts that the minimum degree in H is 3.

So we can assume that w_0 is a strict local degree maximum in $V(B)\backslash\{x_0\}$ and x_0 has strictly greater degree than w_0 in G. This implies that x_0 has degree at least 5. Let Y denote the bipartition set containing w_0 and let X denote the opposite bipartition set. As before we find a $\{0,1\}$-weighting of G where all edges incident to w_0 have weight 1, all vertices in X have the same weighted degree parity as w_0, and all vertices in $Y\backslash\{w_0\}$ have weighted degree parity different from w_0. Now we can only have a conflict between x_0 and w_0. Recall that x_0 is only incident with two blocks B and B_1. There must be precisely two neighbours w_1 and w_2 of x_0 in B_1, since otherwise we can avoid the conflict between x_0 and w_0 by swapping weights in a cycle in B_1 using two edges incident to x_0 with the same weight. By the same argument we can assume that the weights on the two edges x_0w_1 and x_0w_2 are different. Since $d(x_0) \geq 5$, this implies that x_0 must have at least two neighbours w_3 and w_4 in $B - w_0$ joined to x_0 by an edge weighted 1. The graph $B - x_0 - w_0$ is connected by the minimality of H so we can find a cycle in B through the two edges x_0w_3 and x_0w_4 avoiding w_0. We swap the weights on this cycle and thereby avoid the conflict between x_0 and w_0.

This completes Case 1.

Since we can now assume that we are not in Case 1 we can go to Case 2 below by considering a longest path in the block-tree of G^*.

Case 2: We can choose B to be an end-block incident to endblocks $B_1, ..., B_n$ where $n \geq 1$, and the union of all other blocks B_{-1} satisfies that $B_{-1} - x_0$ is connected.

In this case the proofs in Subcases 1 and 2 are exactly the same as before (the situation is now only different when w_0 is incident to x_0). For $i = -1, ..., n$ define $G_i = B_i - x_0$. As before let Y denote the bipartition set containing w_0 and let X denote the opposite bipartition set. For $i = 0, ..., n$ let w_i denote the vertex defined in the same way as w_0 just in B_i instead of in B. As before we can assume that all neighbours of w_i different than x_0 have strictly lower degree than that of w_i and, furthermore, that x_0 has precisely two neighbours $v_{i,1}$ and $v_{i,2}$ in each $G_{-1}, ..., G_n$. We can assume that $w_i = v_{i,1}$ for $i = 0, 1, ..., n$ and that the degree of $v_{i,2}$ is at most that of w_i, since otherwise we redefine w_i to be $v_{i,2}$. For each $i = -1, ..., n$, let X_i and Y_i denote the part of G_i belonging to X and Y respectively. We can assume that we will get a conflict between x_0 and w_i whenever we weight as before giving w_i maximal weighted degree. As noted above, x_0 will get precisely weight 1 from each G_j for $j \neq i$. So, for each $i = 0, ..., n$, the degree of w_i is either $n + 2$ or $n + 3$. We look at five different subcases:

(a) $d(w_0) = n + 2$ and $d(w_1) = n + 3$ and n is even.

(b) $d(w_0) = n + 2$ and $d(w_1) = n + 3$ and n is odd.

(c) $d(w_i) = n + 2$ for $i = 0, 1, ..., n$.

(d) $d(w_i) = n + 3$ for $i = 0, 1, ..., n$ and n is even.

(e) $d(w_i) = n + 3$ for $i = 0, 1, ..., n$ and n is odd.

(a): In this subcase n is at least 2. Recall that when weighting G as before giving w_0 weighted degree $n + 2$ the vertex x_0 will have precisely one edge weighted 1 going to each $G_{-1}, G_0, ..., G_n$, and when weighting G as before giving w_1 weighted degree $n + 3$, the vertex x_0 will get precisely weight 1 from all $G_{-1}, G_0, G_2, G_3, ..., G_n$ and weight 2 from G_1. The $\{0, 1\}$-weighting giving w_0 maximum weighted degree implies that all the sets $Y_{-1}, Y_1, Y_2, ..., Y_n$ have odd size and Y_0 has even size, since otherwise if Y_i has even size for $i = -1, 1, 2, ..., n$ or if Y_0 has odd size, then the subgraph of G_i consisting of the edges with weight 1 has an odd number of vertices of odd degree. Similarly the $\{0, 1\}$-weighting giving w_1 maximum weighted degree implies that all the sets $X_{-1}, X_0, X_1, X_2, X_3, ..., X_n$ have odd size. We find a proper $\{0, 1\}$-weighting of G as follows. For $i = -1, 1, 2, 3, ..., n$ we weight each B_i by Lemma 5 such that all vertices in $X_i \cup x_0$ get odd weighted degree and all vertices in Y_i get even weighted degree. We also find a $\{0, 1\}$-weighting of B_0 such that all vertices in Y_0 get odd weighted degree and all vertices in $X_0 \cup \{x_0\}$ get even weighted degree. We can assume that x_0 gets weighted degree 2 (if the weighted degree of x_0 is 0 we swap the weights on a cycle containing the two edges incident to x_0). The union of these $\{0, 1\}$-weightings gives a $\{0, 1\}$-weighting of G such that the only parity conflicts are between x_0 and its neighbours in B_0. However, the weighted degree of x_0 is $n + 3$ while the neighbours of x_0 in B_0 have degree at most $n + 2$.

(b): In this subcase n is at least 3. By the same argument as in Subcase (a), all the sets $X_{-1}, X_1, X_2, ..., X_n$ have odd size, X_0 has even size and all the sets $Y_{-1}, Y_0, Y_1, Y_2, ..., Y_n$ have odd size. We find a proper $\{0, 1\}$-weighting of G as follows. For $i = -1, 1, 2, 3, ..., n$ we weight each B_i by Lemma 5 such that all vertices in $X_i \cup x_0$ get odd weighted degree and all vertices in Y_i get even weighted degree. We also find a $\{0, 1\}$-weighting of B_0 such that all vertices in $Y_0 \cup \{x_0\}$ get even weighted degree and all vertices in X_0 get odd weighted degree. As in Subcase (a) we can assume that the weighted degree of x_0 is 2. The union of these $\{0, 1\}$-weightings gives a proper $\{0, 1\}$-weighting of G (analogously to Subcase (a)).

(c): First assume that n is even. Then n is at least 2. As before we deduce from the $\{0, 1\}$-weighting of G where w_0 gets weighted degree $n + 2$ that all the sets $Y_{-1}, Y_1, Y_2, ..., Y_n$ have odd size and Y_0 has even size. The same argument for the $\{0, 1\}$-weighting of w_1 shows that all the sets $Y_{-1}, Y_0, Y_2, ..., Y_n$ have odd size and Y_1 has even size, a contradiction. An analogous argument holds when n is odd.

(d): In this subcase n is at least 2 and all the sets $X_{-1}, X_0, X_1, X_2, X_3, ..., X_n$ have odd size. We weight each B_i for $i = 0, 1, 2$ such that w_i gets maximum weighted degree, x_0 gets weighted degree 2 and there are only parity conflicts around w_0. In all other blocks B_j, $j \neq i$, we weight such that all vertices in $X_j \cup \{x_0\}$ get odd weighted degree and all other vertices get even weighted degree. The union of these $\{0, 1\}$-weightings gives a proper $\{0, 1\}$-weighting of G.

(e): In this subcase n is at least 1 and all the sets $Y_{-1}, Y_0, Y_1, Y_2, ..., Y_n$ have odd size. One of the sets $X_{-1}, X_0, ..., X_n$ must have even size. If X_{-1} has even size we weight as follows: In B_{-1} we weight such that all vertices in X_{-1} get odd weighted degree and all vertices in $Y_{-1} \cup \{x_0\}$ get even weighted degree and, furthermore, such that x_0 has weighted degree 2. In B_0 we weight such that w_0 gets maximum weighted degree and all vertices in $X_0 \cup \{w_0, x_0\}$ get even weighted degree and all vertices in $Y_0 - \{w_0\}$ get odd weighted degree and, furthermore, such that the degree of x_0 is 2. In all other blocks

B_j, $j \neq -1, 0$ we weight such that all vertices in $Y_j \cup \{x_0\}$ get odd weighted degree and all other vertices get even weighted degree. The union of these $\{0, 1\}$-weightings gives a proper $\{0, 1\}$-weighting of G. Hence we can assume that X_{-1} has odd size. One of $X_0, X_1, ..., X_n$, say, X_0 has even size and we now weight as follows: In B_0 we weight such that all vertices in X_0 get odd weighted degree and all vertices in $Y_0 \cup \{x_0\}$ get even weighted degree and, furthermore, such that x_0 has weighted degree 2. In B_1 we weight such that:

- w_1 gets maximum weighted degree.

- All vertices in $X_1 \cup \{w_1, x_0\}$ get even weighted degree.

- All vertices in $Y_1 - \{w_0\}$ get odd weighted degree.

- The weighted degree of x_0 is 2.

In B_{-1} we weight such that all vertices in $X_{-1} \cup \{x_0\}$ get odd weighted degree and all vertices in Y_{-1} get even weighted degree. In all other blocks B_j, $j \notin \{-1, 0, 1\}$ we weight such that all vertices in $Y_j \cup \{x_0\}$ get odd weighted degree and all other vertices get even weighted degree. The union of these $\{0, 1\}$-weightings gives a proper $\{0, 1\}$-weighting of G.

This completes the proof of Claim 4.

If the removal of any pair of adjacent vertices leaves a connected graph we must have that G^* is 3-regular and we will simply work in G^* from now on. Otherwise we choose to work in an endblock B of G^* and the subgraph H of B defined before Claim 4. By Claim 4, all vertices of H have degree 3. Suppose first that all vertices in H are adjacent to z_0 or y_0. A small argument shows that unless B is isomorphic to $K_{3,3}$, there is a vertex $w_0 \in V(H)$, such that removing w_0 and all the neighbours w_0 would leave a connected graph. In this case we can find a proper $\{0, 1\}$-weighting of G by Lemma 5. If B is isomorphic to $K_{3,3}$ we remove all vertices in B except x_0. The resulting subgraph of G has an odd number of vertices so by Lemma 5 it has a proper $\{0, 1\}$-weighting without parity-conflicts. Some edges in B may be blue, but it can be checked that no matter how these blue edges are arranged in B this $\{0, 1\}$-weighting can be extended to the whole of G. So we can assume that there is some vertex in H not adjacent to x_0 or z_0.

The rest of the proof is as that of Theorem 2 in [7] (choose w_0 to be a non-neighbour of z_0 and y_0, in H). This completes the proof of the theorem. $\qquad\square$

3   Trees without the {0,1}-property

In this section we will give a complete characterisation of all bad trees. The characterisation consists of a recursive construction using three other classes of trees with certain properties, and immediately gives a polynomial-time algorithm for recognising bad trees. We begin by defining these properties for general bipartite graphs. The first of these three classes is described as follows. Let v be a vertex in a connected bipartite graph G with an even number of vertices in each bipartition set. We say that G is a $G_v(-)$-graph if there is no proper $\{0,1\}$-weighting of G when the weighted degree of v is increased by 1. This definition is motivated by the following easy proposition.

Proposition 10. *Let G be a graph and let v be a vertex in G. Let G' be the graph obtained from G by adding two vertices v_1 and v_2 and the the edges vv_1 and v_1v_2. The graph G is a $G_v(-)$-graph if and only if G' is bad.*

The following two lemmas show a recursive way to construct new bad bipartite graphs from other bad bipartite graphs with vertices of degree 1. These two results hold for all bipartite graphs and not just for trees.

Lemma 11. *Let G be a simple connected bipartite graph without the $\{0,1\}$-property. If v is a vertex of degree 1, and v' is the unique neighbour of v, then all edges incident to v' are bridges in G.*

Proof: By Lemma 5 there is a $\{0,1\}$-weighting of $G - v$ with no parity conflicts. The only problem we can have in extending this $\{0,1\}$-weighting to G is that the weighted degree of v' might be 0. If v' is contained in a cycle we would always be able to avoid this. $\qquad\square$

Lemma 12. *Let G be a simple connected bipartite graph and assume that v is a vertex of degree 1. Let v' denote the neighbour of v and let $e_0, e_1, ..., e_n$ be the edges incident to v' where $e_0 = vv'$. Assume that all edges incident to v' are bridges and for each $i > 0$, let G_i be the unique component of $G - e_i$ not containing v. For each $i > 0$, let G'_i denote the connected graph obtained from G_i by adding the vertices v, v' and the edges e_0, e_i. The graph G is bad if and only if all the graphs $G'_1, ..., G'_n$ are bad.*

Proof: Figure 5 shows an illustration of the situation. For $i \in \{1, ..., n\}$, let v_i be the vertex of G_i which is adjacent to v' in G'_i. If all $G'_1, ..., G'_n$ are bad then by Proposition 10 each G_i is a $G_{v_i}(-)$-graph. It follows that in any proper $\{0,1\}$-weighting of G, each edge e_i must receive weight 0. But now v and v' have the same weighted degree. Thus no such $\{0,1\}$-weighting of G exists, that is, G is bad.
Now assume that G is bad. Let X, Y denote the bipartition sets of G such that $v \in X, v' \in Y$ and for each $i = 1, ..., n$, let X_i, Y_i denote the bipartition sets of G_i. By Lemma 5 we can assume that both X and Y have odd size. By Lemma 5 there is a $\{0,1\}$-weighting of $G - v$ with no parity conflicts, where all vertices in $X - v$ get odd degree and all vertices in Y get even degree. The only problem we can have in extending this $\{0,1\}$-weighting to G is that the weight of v' can be 0. If this is the case then all X_i have even size. There must be an even number m of the sets Y_i which have an odd number of vertices. If $m \geq 2$, say $Y_1, ..., Y_m$ have even size, then by Lemma 5 there is a proper $\{0,1\}$-weighting of G where v' gets weighted degree $m + 1$ (apply Lemma 5 to $G - v$ to find a $\{0,1\}$-weighting of $G - v$ where all vertices in $Y \setminus \{v'\}$ get odd weighted degree and all vertices in $(X - \{v\}) \cup \{v'\}$ get even weighted degree. In such a weighting the weights on all the edges $e_1, ..., e_m$ are 1 and the weights on the other e_i's are zero. Now extend this weighting to the whole of G by assigning weight 1 to $e_0 = vv'$.) This contradicts that G is bad. So all Y_i's have even size. By Proposition 10 each G'_i is bad if and only if G_i is bad when the weight on the vertex incident to e_i is increased by 1. So for a contradiction assume that there is a proper $\{0,1\}$-weighting of some G_i when the weight on the vertex incident to e_i is increased by 1. By use of Lemma 5 this proper $\{0,1\}$-weighting can now easily be extended to G, a contradiction. $\qquad\square$

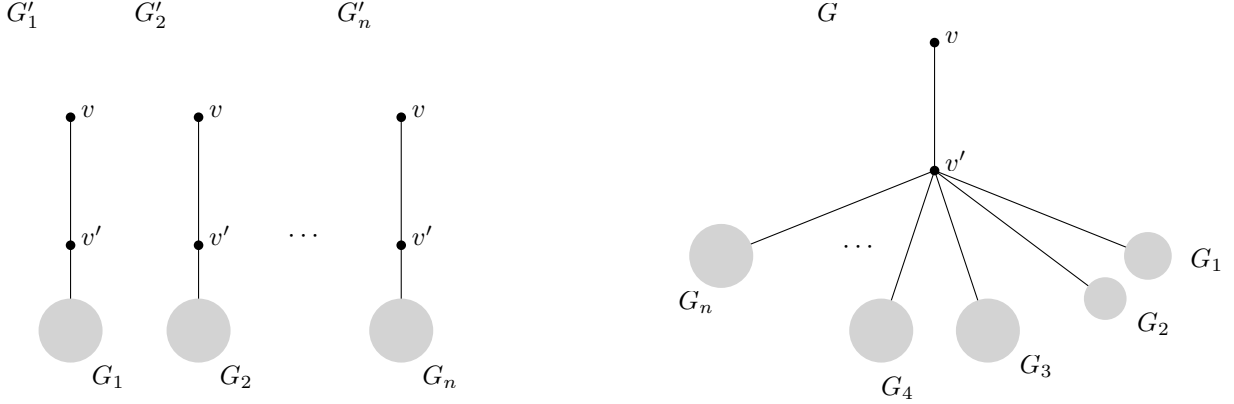G'_1 G'_2 G'_n G

Fig. 5: An illustration of the situation explained in Lemma 12.

We now describe the second and third class of trees we will use to characterise all bad trees. They are special cases of the graphs defined as follows.

Let v be a vertex in a connected bipartite graph G with an odd number of vertices and let a, b be two non-negative integers. We say that G is a $G_v(a, b)$-*graph* if v must get weighted degree a in all proper $\{0, 1\}$-weightings of G and v must get weighted degree b in all proper $\{0, 1\}$-weightings of G where the weight of v is increased by 1.

The classes of $G_v(s, s + 1)$- and $G_v(s, s + 3)$-trees where s is a non-negative integer are two interesting special cases when we want to characterise all bad trees. We will need the following two lemmas describing the local structure around v in a $G_v(s, s + 1)$- and a $G_v(s, s + 3)$-tree.

Lemma 13. *Let s be a non-negative integer and let G be a $G_v(s, s + 3)$-tree. Then G is obtained from the disjoint union of a $G_{v_1}(s + 1, s + 2)$- and a $G_{v_2}(s + 1, s + 2)$-tree together with a number of trees of type $G_{v_3}(-), G_{v_4}(-), ..., G_{v_m}(-)$ and bad trees $G_{m+1}, ..., G_{m+s}$ by adding a vertex v and all edges $vv_1, vv_2, ..., vv_m$ and also an edge from v to all the bad trees $G_{m+1}, ..., G_{m+s}$.*

Proof: Figure 6 illustrates the situation. Assume that s is even (the case where s is odd is similar). Let X, Y be the bipartition sets of G where $v \in Y$. Let $e_1, ..., e_n$ denote the edges incident to v and let $G_1, ..., G_n$ denote the corresponding components of $G - v$. Let X_i, Y_i denote the bipartition sets of G_i. Let s' denote the number of trees with an odd number of vertices in both bipartition sets and let m' denote the number trees with an even number of vertices in both bipartition sets among $G_1, ..., G_n$. Let n_1 denote the number of trees among $G_1, ..., G_n$ that have an even number of vertices in their X-bipartition and an odd number of vertices in their Y-bipartition, and assume that the ordering of $G_1, ..., G_n$ is such that $G_1, ..., G_{n_1}$ denote these trees. Let n_2 be the number of trees among $G_1, ..., G_n$ that have an odd number of vertices in their X-bipartition and an even number of vertices in their Y-bipartition and assume that the ordering of $G_1, ..., G_n$ is such that $G_{n_1+1}, ..., G_{n_1+n_2}$ denote these trees. Furthermore assume that the trees $G_{n_1+n_2+1}, ..., G_{n_1+n_2+s'}$ have an odd number of vertices in both bipartition sets and that the trees $G_{n_1+n_2+s'+1}, ..., G_{n_1+n_2+s'+m'} = G_n$ have an even number of vertices in both bipartition sets. For $i = 1, ..., n$ let v_i be the neighbour of v in G_i.

Since $|V(G)|$ is odd, one of $|X|$, $|Y|$ is even. However, if $|Y|$ is even, then by Lemma 5, G has a proper $\{0,1\}$-weighting such that v gets odd weighted degree. This contradicts G being a $G_v(s,s+3)$-tree. Thus $|X|$ is even and $|Y|$ is odd, and G has a $\{0,1\}$-weighting such that all vertices in Y get even weighted degree and all vertices in X get odd weighted degree. In such a $\{0,1\}$-weighting all the edges $vv_1,...,vv_{n_1}$ must get weight 0, since otherwise if say vv_1 is weighted 1, then the subgraph consisting of edges weighted 1 in G_1 has an odd number of odd degree vertices. By a similar argument, all the edges $vv_{n_1+1},...,vv_{n_1+n_2}$ get weight 1, all the edges $vv_{n_1+n_2+1},...,vv_{n_1+n_2+s'}$ also get weight 1 and all the edges $vv_{n_1+n_2+s'},...,vv_n$ get weight 0. It follows that $n_2+s'=s$. By Lemma 5, there is also a $\{0,1\}$-weighting of G where all vertices in $Y\setminus\{v\}$ get odd weighted degree and all vertices in X get even weighted degree. This means that there is a $\{0,1\}$-weighting of G where all vertices in Y get odd weighted degree and all vertices in X get even weighted degree when the weight on v is increased by 1. We argue as before and see that in such a $\{0,1\}$-weighting all the edges $vv_1,...,vv_{n_1}$ must get weight 1, all the edges $vv_{n_1+1},...,vv_{n_1+n_2}$ get weight 0, all the edges $vv_{n_1+n_2+1},...,vv_{n_1+n_2+s'}$ get weight 1 and all the edges $vv_{n_1+n_2+s'},...,vv_n$ get weight 0. Since G is a $G_v(s,s+3)$-tree it follows that $n_1+s'+1=s+3$ and hence $n_1=n_2+2$.

We start by showing that all the trees $G_{n_1+n_2+s'+1},...,G_n$ must be trees of type $G_{v_j}(-)$. Assume that this is not the case and let G_j be a tree among $G_{n_1+n_2+s'+1},...,G_n$ such that there is a $\{0,1\}$-weighting of $G[V(G_j)\cup\{v\}]$ where the weight on vv_j is 1 and the only possible conflict is between v and v_j. Now we weight $G-G_j$ as before such that all vertices in $Y-Y_j$ get odd weighted degree and all vertices in $X-X_j$ get even weighted degree when the weight on v is increased by 1. We now put back G_j and let vv_j play the role of the extra weight on v which then has weight $s+3$. The only possible conflict is between v and v_j, and since G is a $G_v(s,s+3)$-tree we must have a conflict, so v_j will also get weighted degree $s+3$. Now we weight $G-G_j$ such that all vertices in $Y-Y_j$ get even weighted degree and all vertices in $X-X_j$ get odd weighted degree. Now we put back G_j and let vv_j play the role of an extra weight on v and we also increase the weight 1 on v. The weight on v is then $s+2$ and we have no conflicts anywhere, a contradiction. So all the trees $G_{n_1+n_2+s'+1},...,G_n$ must be trees of type $G_{v_j}(-)$. Similar arguments show all the trees $G_{n_1+n_2+1},...,G_{n_1+n_2+s'}$ must be bad trees.

It remains to show that $n_1=2$ and $n_2=0$, and that the two graphs G_1 and G_2 are trees of type $G_{v_1}(s+1,s+2)$ and $G_{v_2}(s+1,s+2)$. We start by showing that $n_1=2$ and $n_2=0$. Clearly $n_1\geq 2$. First assume that $n_1>2$ is even. For any $j=1,...,n_1$ there is a $\{0,1\}$-weighting of G where the weight on vv_j is 1, the weight on all vv_i for $i\neq j$ and $i\in\{1,...,n_1\}$ is 0 and the weight on all edges $vv_{n_1+1},...,vv_{2n_1-1}$ is 1 and the only possible conflict when the weight on v is increased by 1 is vv_j. So for each G_i, $i=1,...,n_1$ the weight of v_i must be $s'+n_2+2=s'+n_1$ when the weight on v_i is increased by 1, otherwise there is a proper $\{0,1\}$-weighting of G where the weight on v is increased by 1 up to $s'+n_1$. But now there is a proper $\{0,1\}$-weighting of G with weight 1 on all edges $vv_1,...,vv_{n_1+n_2}$ such that v gets weighted degree $s'+n_1+n_2$, and this contradicts G being a $G_v(s,s+3)$-tree. The case where $n_1>2$ and n_1 is odd is similar.

We conclude that $n_1=2$ and $n_2=0$ and it remains to show that G_1 and G_2 are trees of type $G_{v_1}(s+1,s+2)$ and $G_{v_2}(s+1,s+2)$. By Lemma 5 there is a $\{0,1\}$-weighting of G such that the weight on the edges $vv_2,...,vv_{2+s'}$ is 1 and the weight on the other edges incident to v is 0, and where the only possible conflict is between v and v_1. This must be a conflict since G is a $G_v(s,s+3)$-tree. So the weighted degree of v_1 in any proper $\{0,1\}$-weighting of G_1 must be $s+1$. If we use the same $\{0,1\}$-weighting, except we now swap the weighted degree parities in the trees $G_3,...,G_n$ and increase the weighted degree of v by 1 we can similarly conclude that the weighted degree of v_2 in any proper $\{0,1\}$-weighting of G_2 where

the degree of v_2 is increased by 1 must be $s + 2$. Interchanging G_1 and G_2 in the argument above implies that the weighted degree of v_2 in any proper $\{0, 1\}$-weighting of G_2 must be $s + 1$ and that the weighted degree of v_1 in any proper $\{0, 1\}$-weighting of G_1 where the degree of v_1 is increased by 1 must be $s + 2$. Hence G_1 and G_2 are trees of type $G_{v_1}(s + 1, s + 2)$ and $G_{v_2}(s + 1, s + 2)$. $\qquad\square$
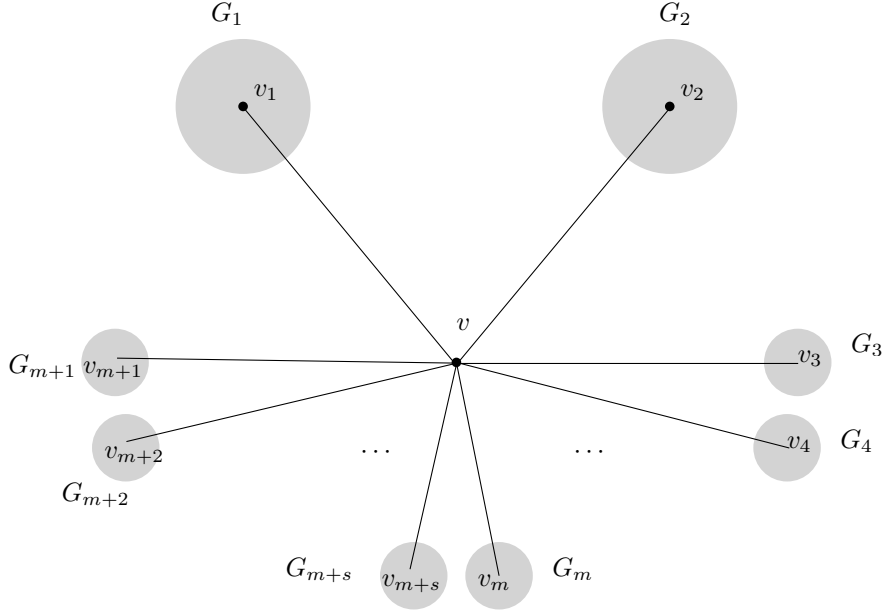

Fig. 6: An illustration of the situation explained in Lemma 13.

Similarly to what we did in the proof of Lemma 13 we can describe the local structure around v in a $G_v(s, s + 1)$-tree.

Lemma 14. *If G is a $G_v(s, s + 1)$-tree, then either*

(a) *G is obtained from the disjoint union of a $G_{v_1}(s - 1, s + 2)$- and a $G_{v_2}(s, s + 1)$-tree together with a number of trees of type $G_{v_3}(-), G_{v_4}(-), ..., G_{v_m}(-)$ and bad trees $G_{m+1}, ..., G_{m+s-1}$ by adding a vertex v and all edges $vv_1, vv_2, ..., vv_m$ and also an edge from v to all the bad trees $G_{m+1}, ..., G_{m+s-1}$, or*

(b) *G is obtained from the disjoint union of s bad graphs $B_1, ..., B_s$ and a number of graphs of type $G_{v_1}(-), G_{v_2}(-), ..., G_{v_n}(-)$ by adding a vertex v and all edges $vv_1, vv_2, ..., vv_n$ and also bridges joining v to each of the bad graphs.*
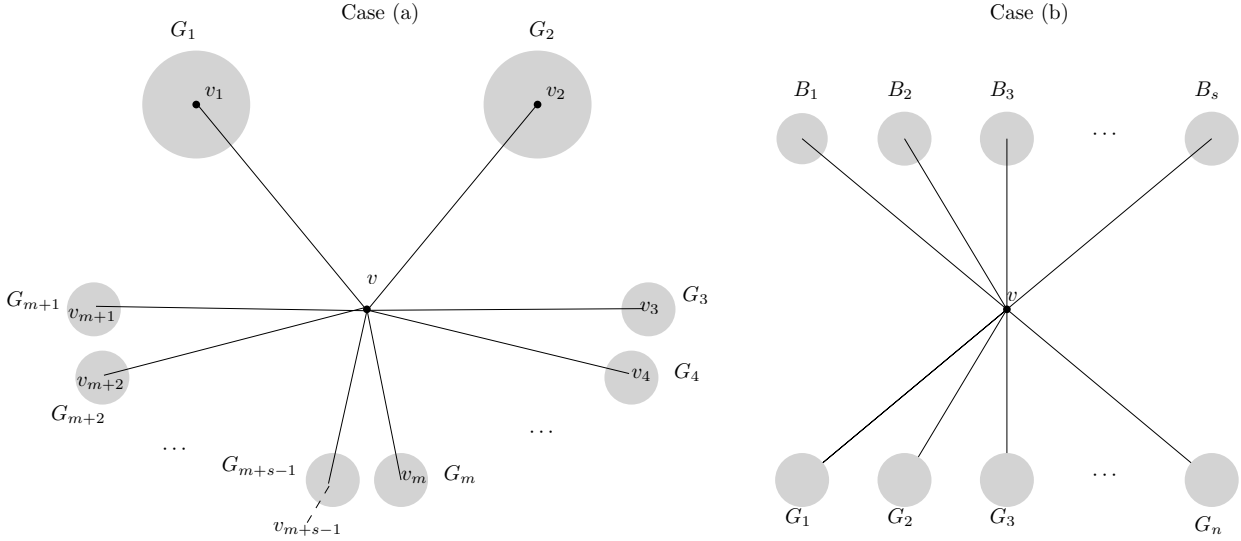
Fig. 7: An illustration of the two possible situations explained in Lemma 14.

Lemma 15. *Any bad tree distinct from K_2 is obtained from either:*

(a) *a $G_v(s, s+1)$-tree where $s > 1$ by adding a vertex v' joined to v by an edge and to s K_2-graphs by bridges, or*

(b) *from two bad trees G_1 and G_2 by gluing together two edges $v_1'v_2'$ and $v_1''v_2''$ in G_1 and G_2 respectively where both v_1' and v_1'' have degree 1 in G_1 and G_2 respectively and both v_2' and v_2'' have degree 2 in G_1 and G_2 respectively.*

Proof: Suppose the lemma is false and look at a smallest counterexample G. It is easy to check that the statement holds for all bad trees of diameter at most 3. So we can assume that the diameter of G is at least 4 and by Lemma 12 we can also assume that all vertices of degree 1 are adjacent to vertices of degree 2. We let v be the fourth last vertex in a longest path in G and let v' be the third last vertex. Then the two subtrees obtained by removing the edge vv' form the desired construction of G. $\qquad\square$

We list a recursive way to construct bad trees below in Figures 8, 9, 10 and 11. The class of bad trees which can be obtained in this way starting with K_2 as the smallest bad graph is denoted $\mathcal{B}$.

Construction of $G_v(s, s+3)$-trees

Two $G_v(s+1, s+2)$-trees:

Some $G_{v'}(-)$-trees:
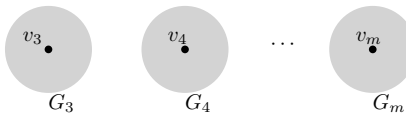
s bad trees:
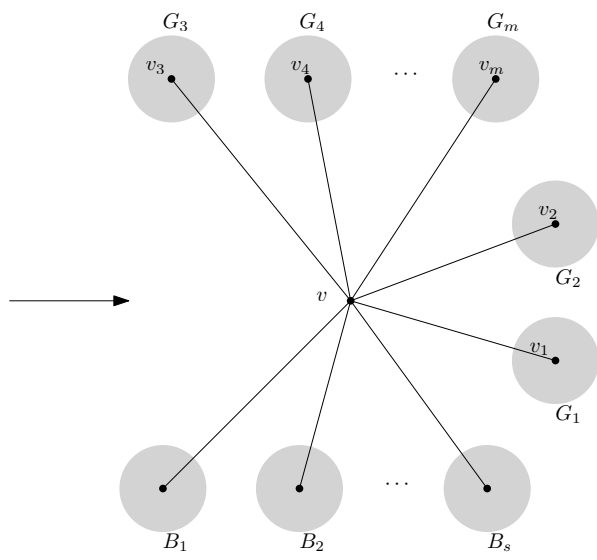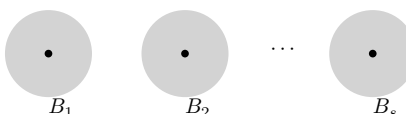
Fig. 8: This illustrates a recursive way how to construct all $G_v(s, s+3)$-trees.

Construction of $G_v(-)$-trees

One bad tree with a vertex of degree1 joined to a vertex of degree 2:

Fig. 9: This illustrates how to obtain all $G_v(-)$-trees from bad trees with 2 more vertices.

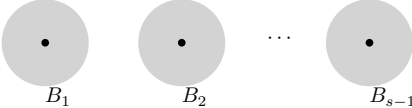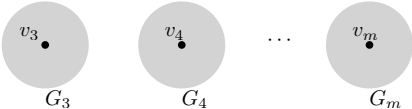**Construction of $G_v(s, s+1)$-trees**

One $G_v(s, s+1)$-tree:

One $G_v(s-1, s+2)$-tree:
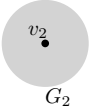
Some $G_{v'}(-)$-trees:
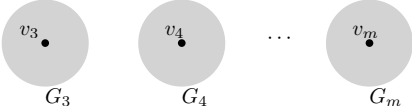
$s - 1$ bad trees:

Or

s bad trees:
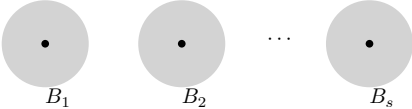
Some $G_{v'}(-)$-trees:
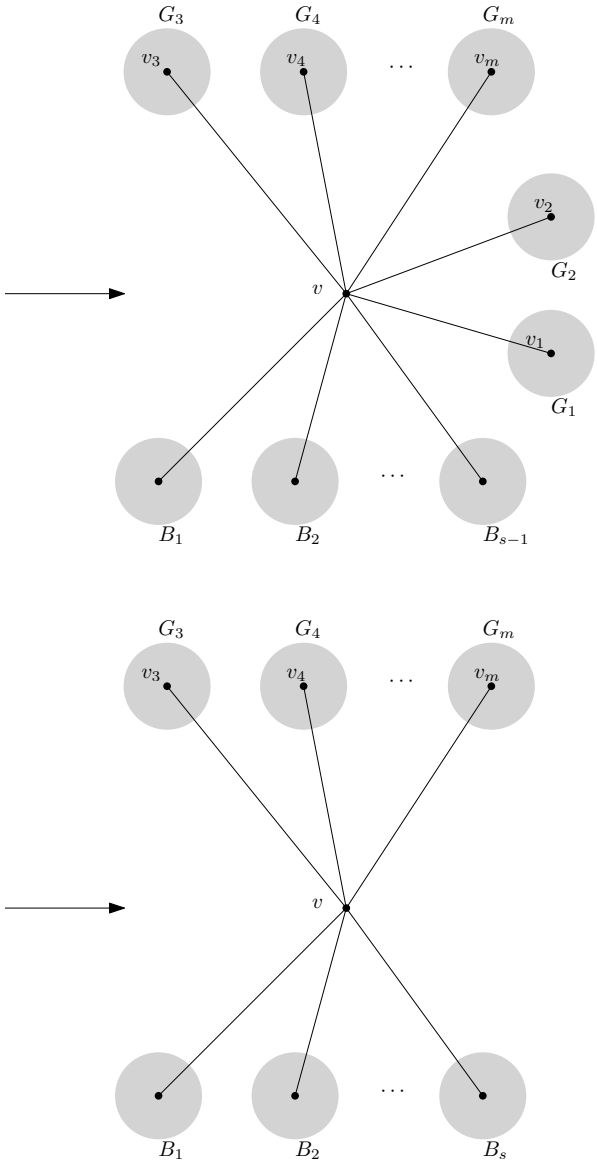
Fig. 10: This illustrates a recursive way how to construct all $G_v(s, s+1)$-trees.
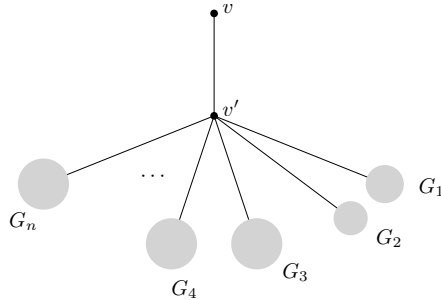
Construction of bad trees
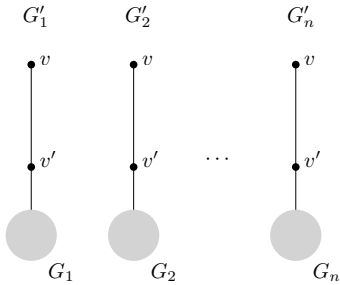
One $G_v(s, s+1)$-tree:

Fig. 11: This illustrates a recursive way how to construct all bad trees.

The above constructions do indeed describe all bad trees:

Proof of Theorem 4: Suppose the theorem is false and let G be a smallest bad tree which cannot be constructed by the above recursion. It is easy to check that the diameter of G must be at least 4. Let n be the number of vertices in G. By Proposition 10 and Lemmas 13 and 14 we can assume that all trees of type $G_v(-)$ with at at most $n-4$ vertices and all trees of type $G_v(s, s+1)$ and $G_v(s, s+3)$ with at most $n-3$ vertices can be constructed using the above recursion. Furthermore, since G is a smallest counterexample all bad trees with fewer vertices can also be constructed by the recursion. Lemma 12 implies that G cannot have a vertex of degree 1 which is adjacent to a vertex of degree at least 3. So by Lemma 15 our counterexample G is obtained from a $G_v(s, s+1)$-tree G' where $s > 0$ and a vertex joined to s K_2-graphs by bridges. But G' has at most $n-3$ vertices so G' can be constructed by the recursion, and then so can G, a contradiction. $\square$

4 Concluding remarks

We have provided a characterisation of all bridgeless bipartite graphs without the {0,1}-property and all trees without the {0,1}-property. Actually, since the {0,1}-property is equivalent to the $\{0,a\}$-property for any non-zero integer a these characterisations extend to the $\{0,a\}$-property. The characterisations also provide polynomial time algorithms checking the $\{0,a\}$-property. This, together with Theorem 2 from [7], answers Problem 1 in [4] except for bipartite graphs with bridges. So it remains to characterise all the bipartite graphs with bridges and without the {0,1}-property. It would be interesting to investigate whether the methods used in Section 3 can be extended to characterise all bipartite graphs without the {0,1}-property.

Acknowledgements

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