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Ore and Erdős type conditions for long cycles in balanced bipartite graphs

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We conjecture Ore and Erdős type criteria for a balanced bipartite graph of order $2n$ to contain a long cycle C_{2n-2k}, where $0 \leq k < n/2$. For $k = 0$, these are the classical hamiltonicity criteria of Moon and Moser. The main two results of the paper assert that our conjectures hold for $k = 1$ as well.

Keywords: bipartite graph, cycle, long cycle, hamiltonicity, degree sum

1 Introduction

One of the classical problems of graph theory is the study of sufficient conditions for a graph to contain a Hamilton cycle. In this paper we are primarily interested in two types of such conditions. Namely, the ones that put constraints on degree sums of pairs of non-adjacent vertices, and those that combine bounds on the size of a graph with bounds on its minimal degree. The first approach is due to Ore (see Section 2 for notation):

Theorem 1.1 (Ore, [12])**.** *Let G be a graph of order $n \geq 3$, in which*

$$d_G(x) + d_G(y) \geq n$$

for every pair of non-adjacent vertices x and y. Then G contains a Hamilton cycle.

It follows immediately from Ore's theorem that the minimal size of a graph of order $n \geq 3$ that guarantees hamiltonicity is $\binom{n-1}{2} + 2$. Erdős generalized this condition by adding a bound on the minimal degree of a graph:

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Theorem 1.2 (Erdős, [9]). *Let G be a graph of order $n \geq 3$ and minimal degree $\delta(G) \geq r$, where $1 \leq r < n/2$. Then G contains a Hamilton cycle, provided*

$$\|G\| > \max\left\{\binom{n-r}{2} + r^2, \binom{n - \lfloor\frac{n-1}{2}\rfloor}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor^2\right\}.$$

The above conditions can, of course, be significantly strengthened in case of a balanced bipartite graph. The following two theorems are bipartite counterparts of Ore and Erdős criteria, respectively.

Theorem 1.3 (Moon and Moser, [11]). *Let G be a bipartite graph of order $2n$, with colour classes X and Y, where $|X| = |Y| = n \geq 2$. Suppose that $d_G(x) + d_G(y) \geq n + 1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Then G contains a Hamilton cycle.*

Theorem 1.4 (Moon and Moser, [11]). *Let G be a bipartite graph of order $2n$, with colour classes X and Y, $|X| = |Y| = n \geq 2$, and minimal degree $\delta(G) \geq r$, $1 \leq r \leq n/2$. Then G contains a Hamilton cycle, provided $\|G\| > n(n-r) + r^2$.*

Our goal is to generalize the above criteria to long cycles, that is, cycles of length $2n - 2k$, where $0 \leq k < n/2$. We state the following two conjectures, that include Theorems 1.3 and 1.4 as special cases ($k = 0$).

Conjecture A. *Let G be a 2-connected balanced bipartite graph of order $2n$, with colour classes X and Y, $|X| = |Y| = n \geq 5$, and let $k < n/2$ be a non-negative integer. If*

$$d_G(x) + d_G(y) \geq n - k + 1$$

for every pair of non-adjacent $x \in X$ and $y \in Y$, then G contains a cycle of length $2n - 2k$.

Conjecture B. *Let G be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 2k + 2r$ and $k \in \mathbb{Z}$. If*

$$\|G\| > n(n-k-r) + r(k+r)$$

then G contains a cycle of length $2n - 2k$.

The main two results of this paper, Theorems A and B (Section 3), assert that our conjectures hold true for $k = 1$. We believe the conjectures to be significantly harder in case $k \geq 2$.

It should be mentioned here that analogous generalizations to long cycles of Ore's and Erdős's theorems have been studied in ordinary graphs. Woodall [14, Thm. 11] gives a complete list of Erdős type conditions for a graph of order n to contain a cycle of length $n - k$ for any $0 \leq k \leq \frac{n-3}{2}$. The Ore type criterion is conjectured in [1], and follows from a result of Linial [10] in case $k \leq 1$.

Remark 1.5. Both the degree sum condition of Conjecture A and the bound on the size of Conjecture B are sharp, as can be seen in Example 1.6 below. It is also necessary to assume 2-connectedness in Conjecture A (Example 1.7). Finally, a quick look at C_6 and C_8 shows that Conjecture A would fail for $n < 5$.

Example 1.6. Let G_1 be a balanced bipartite graph, with colour classes X and Y, $|X| = |Y| = n$, where $X = A \cup B$, $Y = C \cup D$, $|A| = k + r$, $|B| = n - k - r$, $|C| = r$, and $|D| = n - r$. Moreover, assume that $N_{G_1}(x) = C$ for all $x \in A$, and $N_{G_1}(x) = Y$ for all $x \in B$. Then $d_{G_1}(x) + d_{G_1}(y) = n - k$ for every pair $x \in A$ and $y \in D$, and, in general, $d_{G_1}(x) + d_{G_1}(y) \geq n - k$ for every pair of $x \in X$ and $y \in Y$. If $n \geq 2k + 2r$, then $\delta(G_1) = r \geq 1$ and $\|G_1\| = n(n - k - r) + r(k + r)$, but G_1 does not contain a cycle of length $2n - 2k$.

Example 1.7. Let $G_2 = (X, Y; E)$ be a balanced bipartite graph obtained from the disjoint union of $H_1 = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ and $H_2 = K_{\lceil n/2 \rceil, \lceil n/2 \rceil}$ by adding a single edge joining a vertex of H_1 with a vertex of H_2. Then $d_{G_2}(x) + d_{G_2}(y) \geq n$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$, nonetheless G_2 contains no cycle of length $2n - 2$. In fact, G_2 contains no long cycle whatsoever.

The next section contains the inventory of basic definitions and results used throughout the paper. In Section 3 we state our main results, Theorems A and B, and their consequences. In particular, by combining Theorems A and B, we obtain a complete Erdős type characterisation of balanced bipartite graphs that do not contain cycles of length $2n - 2$ (Theorem 3.6). The last two sections are devoted to proofs of the two main results.

2 Notation and tools

All graphs considered are undirected, have no loops and no multiple edges. Given a graph G, we denote by $\|G\|$ the size (i.e., number of edges) of G, and by $V(G)$ the vertex set of G. A bipartite graph is often denoted by $G = (X, Y; E)$, where X and Y are the two colour classes of G, and $E = E(G)$ is the edge set of G. When $|X| = |Y|$, we say that G is *balanced*. Given a vertex $x \in V(G)$, $N_G(x)$ denotes the set of vertices adjacent to x in G, $d_G(x)$ the degree of x in G (i.e., $d_G(x) = |N_G(x)|$), and $\delta(G)$ the minimal vertex degree in G. If $L \subset V(G)$ is a vertex subset of G, then $G - L$ denotes the subgraph of G induced by $V(G) \setminus L$, and $N_G(L)$ is the set of neighbours of all the vertices in L. Given distinct vertices x and y of G, an $x - y$ *path* is a path in G with endvertices x and y. We denote by C_l a cycle of length l, and by $K_{n,n}$ a complete balanced bipartite graph of order $2n$. Finally, recall that a graph is called 2-*connected* if the removal of any single vertex does not disconnect G.

In this section we have gathered results used in the proofs of Theorems A and B. First of all, we recall two hamiltonicity criteria obtained by Moon and Moser [11].

Theorem 2.1 (Moon and Moser, [11])**.** *Let G be a balanced bipartite graph of order $2n \geq 4$, with $\delta(G) \geq \dfrac{n+1}{2}$. Then G contains a Hamilton cycle.*

Theorem 2.2 (Moon and Moser, [11])**.** *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$, and let $S_m = \{x \in X : d_G(x) \leq m\}$, $T_m = \{y \in Y : d_G(y) \leq m\}$ for $m \in \mathbb{Z}$. If, for every $1 \leq m \leq n/2$, the sets S_m and T_m are of cardinalities less than m, then G is hamiltonian.*

We shall need the following strengthening of Theorem 1.4.

Theorem 2.3 (Wojda and Woźniak, [13])**.** *Let $G(n, r)$ denote a bipartite graph with colour classes $X = P \cup Q$ and $Y = R \cup S$ such that $|P| = |R| = r$, $|Q| = |S| = n - r$, $N_{G(n,r)}(x) = R$ for all $x \in P$, and $N_{G(n,r)}(x) = Y$ for all $x \in Q$. Let G be a balanced bipartite graph of order $2n \geq 4$, minimal degree $\delta(G) \geq r \geq 1$, and size $\|G\| \geq n(n - r) + r^2$. Then G contains a Hamilton cycle, else $r \leq n/2$ and G is isomorphic to $G(n, r)$.*

A bipartite graph of order $2n$ is called *bipancyclic* if it contains cycles of lengths $2k$ for all $2 \leq k \leq n$.

Theorem 2.4 (Bagga and Varma, [5]). *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n \geq 8$. If $d_G(x)+d_G(y) \geq n+1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$, then G is bipancyclic.*

Theorem 2.5 (Entringer and Schmeichel, [8]). *Let G be a hamiltonian bipartite graph of order $2n \geq 8$. If $\|G\| > n^2/2$, then G is bipancyclic.*

We will also need to know the cycle structure of an $n/2$-regular hamiltonian bipartite graph G of order $2n$. Notice that then $\|G\| = n^2/2$, so the above theorem does not apply. We then have:

Theorem 2.6 (J. Adamus, [2]). *Let G be an $n/2$-regular hamiltonian bipartite graph of order $2n$. Then G contains a cycle C of length $2n - 2$. Moreover, if C can be chosen to omit a pair of adjacent vertices, then G is bipancyclic.*

Given a balanced bipartite graph $G = (X, Y; E)$, one defines a *k-biclosure $BCl_k(G)$* of G as the graph obtained from G by succesively joining pairs of non-adjacent vertices $x \in X$ and $y \in Y$, with degree sum of at least k, until no such pair remains. Closely related to this construction is the notion of *k-bistability*: A property $\mathcal{P}$ defined on all balanced bipartite graphs of order $2n$ is called k-bistable when, whenever $G + xy$ has the property $\mathcal{P}$ and $d_G(x) + d_G(y) \geq k$, then G itself has the property $\mathcal{P}$.

Theorem 2.7 (Bondy and Chvátal, [7]). *A balanced bipartite graph G of order $2n$ is hamiltonian if and only if its $(n + 1)$-biclosure $BCl_{n+1}(G)$ is so.*

Theorem 2.8 (Amar, Favaron, Mago and Ordaz, [4]). *The property of containing a cycle of length $2n-2$ is $(n + 2)$-bistable on balanced bipartite graphs of order $2n$.*

3 Long cycles in balanced bipartite graphs

Suppose we want to know whether a balanced bipartite graph $G = (X, Y; E)$ has the property of containing a long cycle C_{2n-2k} for some $0 \leq k < n/2$. Given Theorem 1.3 of Moon and Moser, a natural question arises: Can one impose such a property by decreasing the bound on the degree sum of non-adjacent vertices by k? We believe the answer to this question be positive (Conjecture A). As shown in Example 1.6, any lower bound on the degree sum of non-adjacent vertices $x \in X$ and $y \in Y$ which ensures $C_{2n-2k} \subset G$ is at least $n - k + 1$. On the other hand, decreasing the bound below $n + 1$ imposes additional assumptions on the graph. Interestingly enough, without the 2-connectedness constraint the graph could contain no long cycles at all (see Example 1.7). The following result gives a positive answer to the above question in case $k = 1$.

Theorem A. *Let $G = (X, Y; E)$ be a 2-connected balanced bipartite graph of order $2n \geq 4$, such that $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Then G contains an even cycle of length at least $2n - 2$.*

We postpone the proof of the theorem to Section 4. Right now we will show that Theorem A implies Conjecture A for $k = 1$.

Corollary 3.1. *Conjecture A holds for $k = 1$.*

Proof: Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ that satisfies the assumptions of Conjecture A. By Theorem A above, G contains an even cycle of length at least $2n-2$, so without loss of generality one may assume that G is hamiltonian.

Let $x \in X$, say, be a vertex of minimal degree $\delta(G)$ in G. Then Y contains precisely $n-\delta(G)$ vertices non-adjacent to x, each of degree at least $n-\delta(G)$ (as $d_G(x)+d_G(y) \geq n$ for $xy \notin E$). Counting the edges incident with Y, we get

$$\|G\| \geq (n-\delta(G))\cdot(n-\delta(G)) + \delta(G)\cdot\delta(G)\,.$$

Observe that $(n-\delta(G))^2+\delta(G)^2 > n^2/2$ iff $\delta(G) \neq n/2$. Hence $\|G\| > n^2/2$, provided $\delta(G) \neq n/2$, and thus G contains C_{2n-2}, by Theorem 2.5. If, in turn, $\delta(G) = n/2$, then the result follows from Theorem 2.6. $\square$

Let us now turn to Erdős type criteria. In [3], the second author conjectured the following sufficient condition for a balanced bipartite graph to contain a long cycle C_{2n-2k} (proved in [3] under considerably stronger assumptions).

Conjecture 3.2 (L. Adamus, [3])**.** *Let G be a balanced bipartite graph of order $2n$, where $n \geq 2k+2$, $k \in \mathbb{Z}$. If $\|G\| > n(n-k-1)+k+1$, then G contains a cycle of length $2n-2k$.*

Notice that both assumptions of the conjecture are weakest possible, as shown by the following two examples.

Example 3.3. Consider a graph G_1 of Example 1.6, with $r=1$. This graph has precisely $n(n-k-1)+k+1$ edges, and it contains no cycle of length greater than $2n-2k-2$.

Example 3.4. Let $G_3 = (X, Y; E)$ be a balanced bipartite graph, with colour classes of the form $X = A \cup B$, $Y = C \cup D$, where $|A| = |D| = k+1$, $|B| = |C| = n-k-1$. Fix a vertex y_0 in C, and let $N_{G_3}(x) = C$ for all $x \in A$, and $N_{G_3}(x) = D \cup \{y_0\}$ for all $x \in B$. Then $\|G_3\| > n(n-k-1)+k+1$ for $k+3 \leq n \leq 2k+1$, yet G_3 contains no cycle of length greater than $2n-2k-2$. Hence the necessity of the assumption $n \geq 2k+2$.

Interestingly, a similar graph was recently shown in [6] to be a counterexample to Győri's conjecture on C_{2l}-free bipartite graphs.

In light of Example 3.3 above, we ask: By how much can we decrease the lower bound on the size of a given graph G ensuring the existence of a cycle of length $2m-2k$, knowing that the minimal degree of G is greater than 1? We address this question in Conjecture B. Certain special cases of Conjecture B are known true: $k=0$ is Theorem 1.4, $k=r=1$ is done in [3]. The following theorem (proved in Section 5 below) shows that the conjecture also holds for $k=1$ and arbitrary r.

Theorem B. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 4$ and $n \geq 2r+1$. Let*

$$g(n, r) = n(n-1-r) + r(1+r) + 1.$$

Then G contains a cycle of length $2n-2$, provided $\|G\| \geq g(n, r)$.

Notice that Theorems 2.1 and 1.4 can be put together as follows:

Theorem 3.5. *Let G be a balanced bipartite graph of order $2n \geq 4$, with minimal degree $\delta(G) \geq r$. Then G contains a Hamilton cycle, provided*

(1) $n \leq 2r - 1$ *or*

(2) $n \geq 2r$ *and* $\|G\| > n(n-r) + r^2$.

Along the same lines, we combine Theorem 2.4 and Theorems A and B to prove the following criterion for cycles of length $2n - 2$.

Theorem 3.6. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n \geq 8$, with minimal degree $\delta(G) \geq r \geq 1$. Then G contains a cycle of length $2n - 2$, provided*

(1) $n \leq 2r - 1$ *or*

(2) $n = 2r$ *and* $\|G\| \geq 2r^2 + r + 1$ *or*

(3) $n \geq 2r + 1$ *and* $\|G\| \geq n(n - 1 - r) + r(1 + r) + 1$.

Remark 3.7. The lower bounds of conditions (2) and (3) are sharp: For an extremal graph for (2), consider the graph G_3 from Example 3.4 with $k + 1 = r$; for (3), consider G_1 from Example 1.6 with $k = 1$.

Proof of Theorem 3.6:

(1) Since $n \leq 2r - 1$ iff $r \geq (n + 1)/2$, then the degree sum is greater than or equal to $n + 1$ for every pair of vertices in G (in particular, for non-adjacent ones). By Theorem 2.4, G is then bipancyclic.

(2) The bound on the size of G together with $\delta(G) \geq r = n/2$ force 2-connectedness. Also, the degree sum is at least $2r = n$ for every pair of vertices in G. Hence, by Corollary 3.1, G contains C_{2n-2}.

(3) This is Theorem B. □

4 Proof of Theorem A

As 2-connectedness of a graph G implies $\delta(G) \geq 2$, the assertion of the theorem holds true for $n \leq 3$, by Theorem 2.1. Suppose then there exists $n \geq 4$ for which the assertion fails. Let $G = (X, Y; E)$ be a maximal 2-connected balanced bipartite graph of order $2n$, in which $d_G(x) + d_G(y) \geq n$ for all non-adjacent $x \in X$, $y \in Y$, without a cycle of length at least $2n - 2$. By maximality of G, $G + xy$ contains a cycle of length at least $2n-2$, and hence G contains an $x - y$ path of length $2n-3$ or $2n-1$ for every pair of non-adjacent $x \in X$, $y \in Y$.

We shall show first that G contains a Hamilton path. Suppose not. Let $x \in X$, $y \in Y$ be non-adjacent vertices and let P be an $x - y$ path in G of length $2n - 3$; say, $P = u_1v_1u_2v_2 \dots u_{n-1}v_{n-1}$, where $X = \{u_1, \dots, u_n\}$, $Y = \{v_1, \dots, v_n\}$, $u_1 = x$ and $v_{n-1} = y$. Put $I_P = \{1 \leq i \leq n - 1 \mid u_1v_i \in E\}$ and $J_P = \{1 \leq i \leq n - 1 \mid u_iv_{n-1} \in E\}$. Then $I_P \cap J_P = \emptyset$, for if $i_0 \in I_P \cap J_P$, then G contains a cycle $u_1v_{i_0}u_{i_0+1} \dots v_{n-1}u_{i_0}v_{i_0-1} \dots v_1u_1$ of length $2n - 2$; a contradiction.

As $|I_P| = d_{G[V(P)]}(x)$ and $|J_P| = d_{G[V(P)]}(y)$, we obtain

$$d_{G[V(P)]}(x) + d_{G[V(P)]}(y) = |I_P| + |J_P| = |I_P \cup J_P| \leq n-1\,,$$

where $G[V(P)]$ denotes the subgraph of G induced by the vertex set of P. This shows that at least one of the vertices u_1 and v_{n-1} has a neighbour among the remaining vertices u_n, v_n of $G-P$; say, $v_{n-1}u_n \in E$. Notice that then $u_nv_n \notin E$, for otherwise $u_1 \dots v_{n-1}u_nv_nu_1$ would be a Hamilton path. Similarly, $u_1v_n \notin E$. Hence, in particular, I_P contains indices of all the neighbours of u_1 in G, so $|I_P| = d_G(u_1)$. Let now $K_P = \{1 \leq i \leq n-1 \mid u_iv_n \in E\}$. Then $|K_P| = d_G(v_n)$, and as $d_G(u_1) + d_G(v_n) \geq n$, it follows that there exists $i_0 \in I_P \cap K_P$. Then $v_nu_{i_0}v_{i_0-1}\dots u_1v_{i_0}u_{i_0+1}\dots v_{n-1}u_n$ is a Hamilton path in G; a contradiction.

Let now $x \in X$ and $y \in Y$ be a pair of non-adjacent vertices such that G contains a Hamilton $x-y$ path P; say, $P = u_1v_1\dots u_nv_n$, where $X = \{u_1,\dots,u_n\}$, $Y = \{v_1,\dots,v_n\}$, $x = u_1$ and $y = v_n$. Put $I_G = \{1 \leq i \leq n \mid u_1v_i \in E\}$ and $J_G = \{1 \leq i \leq n \mid u_iv_n \in E\}$. Then $|I_G| = d_G(x)$, $|J_G| = d_G(y)$ and $I_G \cap J_G = \emptyset$, for if $i_0 \in I_G \cap J_G$, then $u_1v_{i_0}u_{i_0+1}\dots v_nu_{i_0}v_{i_0-1}\dots v_1u_1$ is a Hamilton cycle in G. Hence

$$n \geq |I_G \cup J_G| = |I_G| + |J_G| = d_G(x) + d_G(y) \geq n,$$

so that, for every $1 \leq i \leq n$,

$$\text{either } \; u_i \in N_G(y) \; \text{ or else } \; v_i \in N_G(x). \tag{$\star$}$$

Let $d = d_G(y)$. Denote by $x_1,\dots,x_d$ those of the vertices $u_1,\dots,u_n$ that are adjacent to y, ordered according to the orientation of P (from x to y). Let $y_1,\dots,y_d$ be the vertices of Y that lie on P next to the respective $x_1,\dots,x_d$; then $y_d = y$.

Observe that if $x_1 = u_i$ with $i < n-d+1$, then there exists $1 \leq j \leq d-1$ such that $y_j = v_l$, where $u_{l+1} \notin N_G(y)$. Then $v_{l+1} \in N_G(x)$ and we obtain a cycle $u_1v_{l+1}u_{l+2}\dots v_nu_lv_{l-1}\dots v_1u_1$ of length $2n-2$ in G; a contradiction.

Therefore $x_1 = u_{n-d+1}$, and hence $N_G(y)$ coincides with the set $\{u_{n-d+1},\dots,u_n\}$, call it U. Then $\{y_1,\dots,y_d\}$ coincides with $V := \{v_{n-d+1},\dots,v_n\}$, and by ($\star$), $N_G(x) = Y \setminus V$.

Suppose now that, for every $v \in V$, $N_G(v) \subset U$. Then, for all $u \in X \setminus U$ and $v \in V$, u and v are non-adjacent, hence $N_G(u) \subset Y \setminus V$. Consequently, $d_G(u_i) \leq n-d$ ($i \leq n-d$), and $d_G(v_j) \leq d$ ($j \geq n-d+1$). But u_i and v_j being non-adjacent, we also have $d_G(u_i) + d_G(v_j) \geq n$, which implies that $d_G(u_i) = n-d$ and $d_G(v_j) = d$, and hence

$$N_G(u_i) = Y \setminus V \;\text{ and }\; N_G(v_j) = U \quad \text{for all } \; i \leq n-d,\; j \geq n-d+1.$$

Thus G contains a complete bipartite graph $K_{d,d}$ spanned on the vertices of U and V, and a complete bipartite $K_{n-d,n-d}$ spanned on $X \setminus U$ and $Y \setminus V$.

Now, G being 2-connected, it must contain two independent edges $u_{i_1}v_{j_1}$ and $u_{i_2}v_{j_2}$ for some $i_1, i_2 \geq n-d+1$ and $j_1, j_2 \leq n-d$. One immediately verifies that such a graph contains a cycle of length $2n-2$, again contradicting the choice of G.

We can therefore conclude that there exists a vertex v_j, with $n-d+1 \leq j \leq n-1$, adjacent to a u_i, where $i \leq n-d$. Then $u_1v_i\dots u_jv_nu_n\dots v_ju_iv_{i-1}\dots v_1u_1$ is a Hamilton cycle in G. This contradiction completes the proof of the theorem. $\square$

5 Proof of Theorem B

Throughout this section we will frequently refer to the exceptional graph $G(n,r)$ of Theorem 2.3. Recall that by $G(n,r)$ we denote a balanced bipartite graph of order $2n$, with colour classes $X = P \cup Q$ and $Y = R \cup S$, where $|P| = |R| = r$, $|Q| = |S| = n-r$, $N_{G(n,r)}(x) = R$ for all $x \in P$, and $N_{G(n,r)}(x) = Y$ for all $x \in Q$.

Let, as before, $g(n,r) = n(n-1-r) + r(1+r) + 1$. We shall first show the following lemma.

Lemma 5.1. *Let $G = (X,Y;E)$ be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 4$ and $n \geq 2r+1$. Let $\|G\| \geq g(n,r)$, and assume there exists a pair of vertices $x \in X$ and $y \in Y$ such that $d_G(x) + d_G(y) \leq n$ and $\delta(G - \{x,y\}) \geq r$. Then G contains a cycle of length $2n-2$.*

Proof: Suppose G contains no cycle of length $2n-2$. Then $G - \{x,y\}$ contains no such cycle either, and as $\delta(G - \{x,y\}) \geq r$, Theorem 2.3 implies that

$$\|G - \{x,y\}\| \leq (n-1)(n-1-r) + r^2 = n^2 - 2n - nr + r^2 + r + 1.$$

On the other hand,

$$\|G - \{x,y\}\| \geq g(n,r) - (d_G(x) + d_G(y)) \geq n^2 - 2n - nr + r^2 + r + 1.$$

Hence $d_G(x) + d_G(y) = n$, the vertices x and y are non-adjacent, $G - \{x,y\}$ equals $G(n-1,r)$, and $r \leq (n-1)/2$. Without loss of generality, we may assume that x belongs to the colour class of G containing $P \cup Q$ of $G(n-1,r)$.

Now, either $d_G(x) \geq r+1$ or $d_G(x) = r$. In the first case, x must have at least two neighbours in S or else at least one neighbour in both S and R. One easily verifies that then G contains a cycle of length $2n-2$, omitting y and a single vertex of P; a contradiction.

If, in turn, $d_G(x) = r$, then $d_G(y) = n - r$ and y must have neighbours in both P and Q, since $r \leq (n-1)/2 < n/2$. Consequently, G contains a cycle of length $2n-2$, omitting x and a vertex of S, which again contradicts the choice of G. □

We are now in position to prove Theorem B.

For a proof by contradiction, consider a graph G satisfying the assumptions of Theorem B, that does not contain a cycle of length $2n-2$. Observe first that $\|G\| > n^2/2$. Indeed, the difference $g(n,r) - n^2/2$ is always positive. Hence, by Theorem 2.5, G is not hamiltonian. Consequently, Theorem 2.2 implies that there exists a positive integer $m \leq n/2$ such that at least one of the sets $S_m = \{x \in X : d_G(x) \leq m\}$, $T_m = \{y \in Y : d_G(y) \leq m\}$ has cardinality greater than or equal to m.

Let l be the least such m. Without loss of generality, we may assume that l is realized in X; i.e., $|\{x \in X : d_G(x) \leq l\}| \geq l$. Order the vertices of $X = \{x_1, \dots, x_n\}$ so that $r \leq d_G(x_1) \leq \dots \leq d_G(x_n)$. Then, by minimality of l, we have $l = \min\{i : d_G(x_i) \leq i\}$. Of course, $r \leq l \leq n/2$. Put $L = \{x_1, \dots, x_l\}$.

The rest of the proof proceeds in two cases, depending on l being equal to or greater than r.

Case 1:

$l = r$. We will first show that all the vertices of Y have degrees greater than r. Suppose to the contrary that there exists $y_1 \in Y$ with $d_G(y_1) = r$. Then

$$\|G - \{x_1, y_1\}\| \geq g(n,r) - 2r = n^2 - n - nr + r^2 - r + 1,$$

and $\delta(G - \{x_1, y_1\}) \geq r - 1$. On the other hand, by Theorem 2.3,

$$\|G - \{x_1, y_1\}\| \leq (n-1)(n-r) + (r-1)^2 = n^2 - n - nr + r^2 - r + 1.$$

Hence $d_G(x_1) + d_G(y_1) = 2r$ so that $x_1y_1 \notin E$ and $G - \{x_1, y_1\}$ equals $G(n-1, r-1)$. By comparison of degrees, one readily verifies that x_1 belongs to that colour class of G that contains $P \cup Q$ of $G(n-1, r-1)$; in fact, $L = \{x_1\} \cup P$. Consider the sets R and S of the other colour class of $G(n-1, r-1)$. As $|N_G(x_1)| = r > |R|$ and $x_1y_1 \notin E$, it follows that either x_1 has neighbours in both R and S or else it has at least two neighbours in S. In any case, as in the proof of Lemma 5.1, one easily finds a cycle of length $2n-2$ in G, omitting y_1 and a vertex of P; a contradiction. Thus $d_G(y) \geq r+1$ for every $y \in Y$.

Next observe that every vertex of Y has a neighbour in L. Suppose otherwise, and let $y_1 \in Y$ be such that $N_G(y_1) \subset X \setminus L$. Notice that all vertices of $X \setminus L$ have degrees greater than r, for otherwise $g(n,r) \leq \|G\| \leq (r+1)r + (n-r-1)n = g(n,r) - 1$. Consequently, by removing y_1 and a vertex of L, say x_1, we do not decrease the minimal degree in the remainder of G. But, as $N_G(y_1) \subset X \setminus L$, we have $d_G(y_1) \leq n - r$, hence $d_G(x_1) + d_G(y_1) \leq r + (n-r) = n$, and by Lemma 5.1, G contains a cycle of lenth $2n-2$; a contradiction.

Consider the graph $G - L$. Notice that

$$\|G - L\| \geq g(n,r) - r^2 = n^2 - n - nr + r + 1.$$

Moreover, we claim that $d_{G-L}(x) + d_{G-L}(y) \geq n$ for every pair of non-adjacent $x \in X \setminus L$ and $y \in Y$. For if $d_{G-L}(x) + d_{G-L}(y) \leq n-1$ for a pair of non-adjacent $x \in X \setminus L$ and $y \in Y$, then, by the above inequality,

$$\|(G - L) - \{x, y\}\| \geq n^2 - 2n - nr + r + 2 > (n-r-1)(n-1),$$

which contradicts $(G - L) - \{x, y\}$ being a bipartite graph with colour classes of cardinality $n - r - 1$ and $n - 1$.

Taking into account that every vertex in Y has a neighbour in L, we now obtain that

$$d_G(x) + d_G(y) \geq n + 1 \quad \text{for all non-adjacent } y \in Y \text{ and } x \in X \setminus L.$$

Let $\widetilde{G}$ be the bipartite graph obtained from G by joining all the non-adjacent vertices of Y and $X \setminus L$. As $|X \setminus L| = n - r$ and every $y \in Y$ has a neighbour in L, we get that $d_{\widetilde{G}}(y) \geq n - r + 1$ for all $y \in Y$. Hence $d_{\widetilde{G}}(x) + d_{\widetilde{G}}(y) \geq n + 1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Therefore, joining all the non-adjacent vertices of X and Y in $\widetilde{G}$ with degree sum of at least $n+1$ yields a complete bipartite graph $K_{n,n}$. As $\widetilde{G}$ was obtained from G also by joining certain non-adjacent vertices of X and Y with degree sum of at least $n+1$, this shows that the $(n+1)$-biclosure of G equals $K_{n,n}$. Thus, by Theorem 2.7, G contains a Hamilton cycle, which, as we observed at the begining of this proof, is impossible.

Case 2:

$l \geq r+1$. In this case $n \geq 2r+2$ (as $l \leq n/2$) and $r \geq 2$ (for otherwise $l = r = 1$, by minimality); hence $|L| \geq 3$. Moreover, $d_G(x_{l-1}) = d_G(x_l) = l$, by minimality of l.

Suppose first that $d_G(x) + d_G(y) \geq n+2$ for every pair of non-adjacent $x \in X \setminus L$ and $y \in Y$. Let G' be the bipartite graph obtained from G by joining all the non-adjacent vertices of $X \setminus L$ and Y. We claim that every $y \in Y$ has a neighbour in L (in G'). Suppose otherwise, and let $y_1 \in Y$ be such that $N_{G'}(y_1) \subset X \setminus L$. Then $d_{G'}(y_1) \leq n-l$, hence $d_{G'}(x_1)+d_{G'}(y_1) \leq n$. Moreover, $\delta(G'-\{x_1,y_1\}) \geq r$, as all the vertices in $X \setminus L$ have degrees of at least $l \geq r+1$, and $d_{G'}(y) \geq n-l \geq l \geq r+1$ for all $y \in Y$. Then Lemma 5.1 implies that G' contains a cycle of length $2n-2$, and hence, by Theorem 2.8, so does G; a contradiction.

Notice that G' was obtained from G by joining only pairs of vertices with degree sum of at least $n+2$. Also, as every vertex $y \in Y$ has a neighbour in L (in G'), we have $d_{G'}(y) \geq n-l+1$. Recall that $d_{G'}(x_l) = d_G(x_l) = l$ and $d_{G'}(x_{l-1}) = d_G(x_{l-1}) = l$. Hence

$$d_{G'}(x_l) + d_{G'}(y) \geq n+1 \quad \text{and} \quad d_{G'}(x_{l-1}) + d_{G'}(y) \geq n+1 \quad \text{for all } y \in Y.$$

Let $G^{(2)}$ be the graph obtained from G' by joining x_l and x_{l-1} with all the vertices of Y. Then $d_{G^{(2)}}(y) \geq n-l+2$ for all $y \in Y$, and as $d_{G^{(2)}}(x_{l-2}) = d_G(x_{l-2}) \geq l-1$ (by minimality of l), we get that

$$d_{G^{(2)}}(x_{l-2}) + d_{G^{(2)}}(y) \geq n+1 \quad \text{for all } y \in Y.$$

Let now $G^{(3)}$ be the graph obtained from $G^{(2)}$ by joining x_{l-2} with all the non-adjacent vertices of Y. In general, let $G^{(m)}$ ($m \geq 3$) be obtained from $G^{(m-1)}$ by joining x_{l-m+1} with all the non-adjacent vertices of Y. Then $G^{(l)} = K_{n,n}$, and $G^{(m)}$ is obtained from $G^{(m-1)}$ by joining only pairs of vertices with degree sum of at least $n+1$. Thus $G^{(l)} = BCl_{n+1}(G)$, so that the $(n+1)$-biclosure of G is a complete bipartite graph. Now Theorem 2.7 implies that G contains a Hamilton cycle, which again leads to contradiction.

To complete the proof, it remains to consider the case when there is a pair of non-adjacent $x^0 \in X \setminus L$ and $y^0 \in Y$ with $d_G(x^0) + d_G(y^0) \leq n+1$. This however can only happen when $n = 2r+2$ or $n = 2r+3$. For let us suppose that $n \geq 2r+4$, and put $f(l) = l^2 + (n-l-1)(n-1) + n + 2$. We show $\|G\| < f(l)$ and $f(l) \leq g(n,r)$, and thus obtain a contradiction with the assumption $\|G\| \geq g(n,r)$. If G contains a pair of non-adjacent vertices $x \in X \setminus L$ and $y \in Y$ with $d_G(x) + d_G(y) \leq n+1$, then

$$\|G\| \leq |L| \cdot l + |X \setminus (L \cup \{x\})| \cdot |Y \setminus \{y\}| + d_G(x) + d_G(y) \leq f(l) - 1.$$

As the derivative of f equals $f'(l) = -n + 2l + 1$, it follows that $f(l)$ is decreasing for $l \leq (n-1)/2$, and hence maximal at $l = r+1$. One immediately verifies that $f(r+1) \leq g(n,r)$ for $n \geq 2r+4$. If, on the other hand, $l > (n-1)/2$, then $l = n/2$ (since $l \leq n/2$), and it is again immediate to check that $f(n/2) \leq g(n,r)$ for $n \geq 2r+4$.

Subcase 2.1:

$n = 2r+2$. Then $r+1 \leq l \leq n/2$ yields $l = r+1$, and we obtain

$$\left\|G - \{x^0, y^0\}\right\| \geq g(2r+2, r) - (2r+3) = 3r^2 + 3r\,. \tag{1}$$

On the other hand,

$$\left\|G - \{x^0, y^0\}\right\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 3r + 1\,. \tag{2}$$

Hence

$$3r^2 + 3r \leq \left\|G - \{x^0, y^0\}\right\| \leq 3r^2 + 3r + 1 \text{ and } 2r + 2 \leq d_G(x^0) + d_G(y^0) \leq 2r + 3\,.$$

Suppose first that $\left\|G - \{x^0, y^0\}\right\| = 3r^2 + 3r + 1$. Then, by (2), $d_G(x) = l$ for all $x \in L$, and $N_G(y^0) \cap L = \emptyset$; in particular, $d_G(x_1) + d_G(y^0) \leq l + (n-l) = n$. Moreover, $N_G(y) \supset X \setminus (L \cup \{x^0\})$ for all $y \in Y \setminus \{y^0\}$, and $d_G(x) \geq r+1$ for all $x \in X$, so that $\delta(G - \{x_1, y^0\}) \geq r$, and by Lemma 5.1, G contains a cycle of length $2n-2$; a contradiction.

Therefore we may assume that $\left\|G - \{x^0, y^0\}\right\| = 3r^2 + 3r$. By (1), $d_G(x^0) + d_G(y^0) = 2r + 3$, and what's more, $d_G(x) + d_G(y) \geq 2r + 3$ for all non-adjacent $x \in X \setminus L$ and $y \in Y$. Indeed, if $d_G(x^1) + d_G(y^1) \leq 2r + 2$ for some non-adjacent $x^1 \in X \setminus L$ and $y^1 \in Y$, then by (1) and (2), $\left\|G - \{x^1, y^1\}\right\| = 3r^2 + 3r + 1$, which leads to contradiction, as above.

We will now show that $|N_G(L)| > r + 1$. Suppose otherwise, that is, suppose $|N_G(L)| = l = r + 1$. Then $N_G(y^0) \cap L = \emptyset$, for else $N_G(L) \ni y^0$ implies

$$\left\|G - \{x^0, y^0\}\right\| \leq |L| \cdot (l-1) + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 2r\,,$$

which is impossible. Therefore $d_G(y^0) = n - l - 1 = r$; in particular, $d_G(x_1) + d_G(y^0) \leq l + r < n$. Notice that, as $G - \{x^0, y^0\}$ only has one edge less than the right-hand side of (2), every neighbour of y^0 in G has degree at least $n - 2 = 2r$, and every neighbour of x_1 has at least $l - 1 = r$ other neighbours in L (x_1 being the only vertex whose degree could be less than l). Thus $\delta(G - \{x_1, y^0\}) \geq r$, and we get a contradiction, by Lemma 5.1. Thus $|N_G(L)| > r + 1$.

It is now not difficult to see that $BCl_{n+1}(G) = K_{n,n}$: Recall that we have verified that $d_G(x) + d_G(y) \geq 2r + 3 = n + 1$ for all non-adjacent $x \in X \setminus L$ and $y \in Y$. Let G' be the graph obtained from G by joining all the non-adjacent vertices of $X \setminus L$ and Y. Next observe that, by minimality of $l = r + 1$, $d_{G'}(x_{r+1}) = d_G(x_{r+1}) = r + 1$, and as $|N_G(L)| > r + 1$, at least one non-neighbour of x_{r+1}, say y', has a neighbour among the other vertices of L. Hence $|N_{G'}(y')| \geq |X \setminus L| + 1$, so that $d_{G'}(x_{r+1}) + d_{G'}(y') \geq (r+1) + (r+2) = n + 1$. Let $G^{(2)}$ be obtained from G' by joining x_{r+1} with y', and hence increasing the degree of x_{r+1} to $r + 2$. Then $d_{G^{(2)}}(x_{r+1}) + d_{G^{(2)}}(y) \geq n + 1$ for all $y \in Y$. Let $G^{(3)}$ be obtained from $G^{(2)}$ by joining x_{r+1} with all the non-adjacent vertices of Y. Now $d_{G^{(3)}}(y) \geq r + 2$ for all $y \in Y$. By minimality of l again, $d_{G^{(3)}}(x_r) = d_G(x_r) = r + 1$, and hence $d_{G^{(3)}}(x_r) + d_{G^{(3)}}(y) \geq 2r + 3$ for all $y \in Y$. Let $G^{(4)}$ be obtained from $G^{(3)}$ by joining x_r with all the non-adjacent vertices of Y. Then $d_{G^{(4)}}(y) \geq r + 3$ for all $y \in Y$, and hence, as $\delta(G^{(4)}) \geq \delta(G) \geq r$, $d_{G^{(4)}}(x) + d_{G^{(4)}}(y) \geq 2r + 3$ for all non-adjacent $x \in X$ and $y \in Y$. Joining all the non-adjacent pairs $x \in X$, $y \in Y$ of $G^{(4)}$ with degree sum of at least $n + 1$ we thus obtain $K_{n,n}$. Since at each stage we only joined pairs of vertices with degree sum of at least $n + 1$, this shows that $K_{n,n} = BCl_{n+1}(G)$. By Theorem 2.7, G contains a Hamilton cycle; a contradiction.

Subcase 2.2:
$n = 2r + 3$. Again, $r + 1 \leq l \leq n/2$ yields $l = r + 1$, and we have

$$\left\|G - \{x^0, y^0\}\right\| \geq g(2r + 3, r) - (2r + 4) = 3r^2 + 6r + 3\,,$$

and, on the other hand,

$$\left\|G - \{x^0, y^0\}\right\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 6r + 3\,.$$

Therefore both inequalities must, in fact, be equalities; in particular, $d_G(x_1) = l$ and $d_G(x) \geq r + 1$ for all $x \in X$, $N_G(y^0) \cap L = \emptyset$, so that $d_G(y^0) \leq n - l$, and finally $|N_G(y)| \geq |X \setminus (L \cup \{x^0\})| = r + 1$ for all $y \in Y \setminus \{y^0\}$. Thus, again, G with the vertices x_1, y^0 satisfies the assumptions of Lemma 5.1, hence G contains a cycle of length $2n - 2$; a contradiction. This completes the proof of Theorem B. $\square$

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