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Ore and Erdős type conditions for long cycles in balanced bipartite graphs

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We conjecture Ore and Erdős type criteria for a balanced bipartite graph of order $2n$ to contain a long cycle $C_{2n-2k}$, where $0 \leq k < n/2$. For $k = 0$, these are the classical hamiltonicity criteria of Moon and Moser. The main two results of the paper assert that our conjectures hold for $k = 1$ as well.

Keywords: bipartite graph, cycle, long cycle, hamiltonicity, degree sum

1 Introduction

One of the classical problems of graph theory is the study of sufficient conditions for a graph to contain a Hamilton cycle. In this paper we are primarily interested in two types of such conditions. Namely, the ones that put constraints on degree sums of pairs of non-adjacent vertices, and those that combine bounds on the size of a graph with bounds on its minimal degree. The first approach is due to Ore (see Section 2 for notation):

Theorem 1.1 (Ore, [12]). Let $G$ be a graph of order $n \geq 3$, in which

$$d_G(x) + d_G(y) \geq n$$

for every pair of non-adjacent vertices $x$ and $y$. Then $G$ contains a Hamilton cycle.

It follows immediately from Ore’s theorem that the minimal size of a graph of order $n \geq 3$ that guarantees hamiltonicity is $\binom{n-1}{2} + 2$. Erdős generalized this condition by adding a bound on the minimal degree of a graph:

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Theorem 1.2 (Erdős, [9]). Let $G$ be a graph of order $n \geq 3$ and minimal degree $\delta(G) \geq r$, where $1 \leq r < n/2$. Then $G$ contains a Hamilton cycle, provided

$$\|G\| > \max \left\{ \left( \frac{n-r}{2} \right)^2 + r^2, \left( \frac{n-\lfloor \frac{n-1}{2} \rfloor}{2} \right)^2 + \left( \frac{n-1}{2} \right)^2 \right\}.$$ 

The above conditions can, of course, be significantly strengthened in case of a balanced bipartite graph.

The following two theorems are bipartite counterparts of Ore and Erdős criteria, respectively.

Theorem 1.3 (Moon and Moser, [11]). Let $G$ be a bipartite graph of order $2n$, with colour classes $X$ and $Y$, where $|X| = |Y| = n \geq 2$. Suppose that $d_G(x) + d_G(y) \geq n + 1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Then $G$ contains a Hamilton cycle.

Theorem 1.4 (Moon and Moser, [11]). Let $G$ be a bipartite graph of order $2n$, with colour classes $X$ and $Y$, $|X| = |Y| = n \geq 2$, and minimal degree $\delta(G) \geq r$, $1 \leq r \leq n/2$. Then $G$ contains a Hamilton cycle, provided $\|G\| > n(n-r) + r^2$.

Our goal is to generalize the above criteria to long cycles, that is, cycles of length $2n - 2k$, where $0 \leq k < n/2$. We state the following two conjectures, that include Theorems 1.3 and 1.4 as special cases ($k = 0$).

**Conjecture A.** Let $G$ be a 2-connected balanced bipartite graph of order $2n$, with colour classes $X$ and $Y$, $|X| = |Y| = n \geq 5$, and let $k < n/2$ be a non-negative integer. If

$$d_G(x) + d_G(y) \geq n - k + 1$$

for every pair of non-adjacent $x \in X$ and $y \in Y$, then $G$ contains a cycle of length $2n - 2k$.

**Conjecture B.** Let $G$ be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 2k + 2r$ and $k \in \mathbb{Z}$. If

$$\|G\| > n(n-k-r) + r(k+r)$$

then $G$ contains a cycle of length $2n - 2k$.

The main two results of this paper, Theorems A and B (Section 3), assert that our conjectures hold true for $k = 1$. We believe the conjectures to be significantly harder in case $k \geq 2$.

It should be mentioned here that analogous generalizations to long cycles of Ore’s and Erdős’s theorems have been studied in ordinary graphs. Woodall [14] Thm. 11] gives a complete list of Erdős type conditions for a graph of order $n$ to contain a cycle of length $n - k$ for any $0 \leq k \leq \frac{n-3}{2}$. The Ore type criterion is conjectured in [11, and follows from a result of Linial [10 in case $k \leq 1$.

**Remark 1.5.** Both the degree sum condition of Conjecture A and the bound on the size of Conjecture B are sharp, as can be seen in Example 1.6 below. It is also necessary to assume 2-connectedness in Conjecture A (Example 1.7). Finally, a quick look at $C_6$ and $C_8$ shows that Conjecture A would fail for $n < 5$. 
Example 1.6. Let $G_1$ be a balanced bipartite graph, with colour classes $X$ and $Y$, $|X| = |Y| = n$, where $X = A \cup B$, $Y = C \cup D$, $|A| = k + r$, $|B| = n - k - r$, $|C| = r$, and $|D| = n - r$. Moreover, assume that $N_{G_1}(x) = C$ for all $x \in A$, and $N_{G_1}(x) = Y$ for all $x \in B$. Then $d_{G_1}(x) + d_{G_1}(y) = n - k$ for every pair $x \in A$ and $y \in D$, and, in general, $d_{G_1}(x) + d_{G_1}(y) \geq n - k$ for every pair of $x \in X$ and $y \in Y$. If $n \geq 2k + 2r$, then $\delta(G_1) = r \geq 1$ and $||G_1|| = n(n - k - r) + r(k + r)$, but $G_1$ does not contain a cycle of length $2n - 2k$.

Example 1.7. Let $G_2 = (X, Y; E)$ be a balanced bipartite graph obtained from the disjoint union of $H_1 = K_{[n/2],[n/2]}$ and $H_2 = K_{[n/2],[n/2]}$ by adding a single edge joining a vertex of $H_1$ with a vertex of $H_2$. Then $d_{G_2}(x) + d_{G_2}(y) \geq n$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$, nonetheless $G_2$ contains no cycle of length $2n - 2$. In fact, $G_2$ contains no long cycle whatsoever.

The next section contains the inventory of basic definitions and results used throughout the paper. In Section 3 we state our main results, Theorems A and B, and their consequences. In particular, by combining Theorems A and B, we obtain a complete Erdős type characterisation of balanced bipartite graphs that do not contain cycles of length $2n - 2$ (Theorem 3.6). The last two sections are devoted to proofs of the two main results.

2 Notation and tools

All graphs considered are undirected, have no loops and no multiple edges. Given a graph $G$, we denote by $|G|$ the size (i.e., the number of edges) of $G$, and by $V(G)$ the vertex set of $G$. A bipartite graph is often denoted by $G = (X, Y; E)$, where $X$ and $Y$ are the two colour classes of $G$, and $E = E(G)$ is the edge set of $G$. When $|X| = |Y|$, we say that $G$ is balanced. Given a vertex $x \in V(G)$, $N_G(x)$ denotes the set of vertices adjacent to $x$ in $G$, $d_G(x)$ the degree of $x$ in $G$ (i.e., $d_G(x) = |N_G(x)|$), and $\delta(G)$ the minimal vertex degree in $G$. If $L \subset V(G)$ is a vertex subset of $G$, then $G - L$ denotes the subgraph of $G$ induced by $V(G) \setminus L$, and $N_G(L)$ is the set of neighbours of all the vertices in $L$. Given distinct vertices $x$ and $y$ of $G$, an $x - y$ path is a path in $G$ with endvertices $x$ and $y$. We denote by $C_l$ a cycle of length $l$, and by $K_{n,n}$ a complete balanced bipartite graph of order $2n$. Finally, recall that a graph is called 2-connected if the removal of any single vertex does not disconnect $G$.

In this section we have gathered results used in the proofs of Theorems A and B. First of all, we recall two hamiltonicity criteria obtained by Moon and Moser [11].

Theorem 2.1 (Moon and Moser, [11]). Let $G$ be a balanced bipartite graph of order $2n \geq 4$, with $\delta(G) \geq \frac{n + 1}{2}$. Then $G$ contains a Hamilton cycle.

Theorem 2.2 (Moon and Moser, [11]). Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$, and let $S_m = \{x \in X : d_G(x) \leq m\}$, $T_m = \{y \in Y : d_G(y) \leq m\}$ for $m \in \mathbb{Z}$. If, for every $1 \leq m \leq n/2$, the sets $S_m$ and $T_m$ are of cardinalities less than $m$, then $G$ is hamiltonian.

We shall need the following strengthening of Theorem 1.4.

Theorem 2.3 (Wojda and Woźniak, [13]). Let $G(n, r)$ denote a bipartite graph with colour classes $X = P \cup Q$ and $Y = R \cup S$ such that $|P| = |R| = r$, $|Q| = |S| = n - r$, $N_{G(n, r)}(x) = R$ for all $x \in P$, and $N_{G(n, r)}(x) = Y$ for all $x \in Q$. Let $G$ be a balanced bipartite graph of order $2n \geq 4$, minimal degree $\delta(G) \geq r \geq 1$, and size $||G|| \geq n(n - r) + r^2$. Then $G$ contains a Hamilton cycle, else $r \leq n/2$ and $G$ is isomorphic to $G(n, r)$.
A bipartite graph of order $2n$ is called bipancyclic if it contains cycles of lengths $2k$ for all $2 \leq k \leq n$.

Theorem 2.4 (Bagga and Varma, [5]). Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n \geq 8$. If $d_G(x) + d_G(y) \geq n + 1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$, then $G$ is bipancyclic.

Theorem 2.5 (Entringer and Schmeichel, [8]). Let $G$ be a hamiltonian bipartite graph of order $2n \geq 8$. If $\|G\| > n^2/2$, then $G$ is bipancyclic.

We will also need to know the cycle structure of an $n/2$-regular hamiltonian bipartite graph $G$ of order $2n$. Notice that then $\|G\| = n^2/2$, so the above theorem does not apply. We then have:

Theorem 2.6 (J. Adamus, [2]). Let $G$ be an $n/2$-regular hamiltonian bipartite graph of order $2n$. Then $G$ contains a cycle $C$ of length $2n - 2$. Moreover, if $C$ can be chosen to omit a pair of adjacent vertices, then $G$ is bipancyclic.

Given a balanced bipartite graph $G = (X, Y; E)$, one defines a $k$-biclosure $\text{BCl}_k(G)$ of $G$ as the graph obtained from $G$ by successively joining pairs of non-adjacent vertices $x \in X$ and $y \in Y$, with degree sum of at least $k$, until no such pair remains. Closely related to this construction is the notion of $k$-bistability:

A property $P$ defined on all balanced bipartite graphs of order $2n$ is called $k$-bistable when, whenever $G + xy$ has the property $P$ and $d_G(x) + d_G(y) \geq k$, then $G$ itself has the property $P$.

Theorem 2.7 (Bondy and Chvátal, [7]). A balanced bipartite graph $G$ of order $2n$ is hamiltonian if and only if its $(n + 1)$-biclosure $\text{BCl}_{n+1}(G)$ is so.

Theorem 2.8 (Amar, Favaron, Mago and Ordaz, [4]). The property of containing a cycle of length $2n - 2$ is $(n + 2)$-bistable on balanced bipartite graphs of order $2n$.

## 3 Long cycles in balanced bipartite graphs

Suppose we want to know whether a balanced bipartite graph $G = (X, Y; E)$ has the property of containing a long cycle $C_{2n-2k}$ for some $0 \leq k < n/2$. Given Theorem 1.3 of Moon and Moser, a natural question arises: Can one impose such a property by decreasing the bound on the degree sum of non-adjacent vertices by $k$? We believe the answer to this question be positive (Conjecture A). As shown in Example 1.6, any lower bound on the degree sum of non-adjacent vertices $x \in X$ and $y \in Y$ which ensures $C_{2n-2k} \subset G$ is at least $n - k + 1$. On the other hand, decreasing the bound below $n + 1$ imposes additional assumptions on the graph. Interestingly enough, without the 2-connectedness constraint the graph could contain no long cycles at all (see Example 1.7). The following result gives a positive answer to the above question in case $k = 1$.

Theorem A. Let $G = (X, Y; E)$ be a 2-connected balanced bipartite graph of order $2n \geq 4$, such that $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Then $G$ contains an even cycle of length at least $2n - 2$.

We postpone the proof of the theorem to Section 4. Right now we will show that Theorem A implies Conjecture A for $k = 1$.


Theorem 2.6. Let \( G = (X, Y; E) \) be a balanced bipartite graph of order \( 2n \) that satisfies the assumptions of Conjecture A. By Theorem A above, \( G \) contains an even cycle of length at least \( 2n - 2 \), so without loss of generality one may assume that \( G \) is hamiltonian.

Let \( x \in X \), say, be a vertex of minimal degree \( \delta(G) \) in \( G \). Then \( Y \) contains precisely \( n - \delta(G) \) vertices non-adjacent to \( x \), each of degree at least \( n - \delta(G) \) (as \( d_G(x) + d_G(y) \geq n \) for \( xy \notin E \)). Counting the edges incident with \( Y \), we get

\[
\|G\| \geq (n - \delta(G)) \cdot (n - \delta(G)) + \delta(G) \cdot \delta(G).
\]

Observe that \((n - \delta(G))^2 + \delta(G)^2 > n^2/2\) iff \( \delta(G) \neq n/2 \). Hence \( \|G\| > n^2/2 \), provided \( \delta(G) \neq n/2 \), and thus \( G \) contains \( C_{2n-2} \), by Theorem 2.5. If, in turn, \( \delta(G) = n/2 \), then the result follows from Theorem 2.6.

Let us now turn to Erdős type criteria. In [3], the second author conjectured the following sufficient condition for a balanced bipartite graph to contain a long cycle \( C_{2n-2k} \) (proved in [3] under considerably stronger assumptions).

Conjecture 3.2 (L. Adamus, [3]). Let \( G \) be a balanced bipartite graph of order \( 2n \), where \( n \geq 2k + 2 \), \( k \in \mathbb{Z} \). If \( \|G\| > n(n - k - 1) + k + 1 \), then \( G \) contains a cycle of length \( 2n - 2k \).

Notice that both assumptions of the conjecture are weakest possible, as shown by the following two examples.

Example 3.3. Consider a graph \( G_1 \) of Example 3.1 with \( r = 1 \). This graph has precisely \( n(n - k - 1) + k + 1 \) edges, and it contains no cycle of length greater than \( 2n - 2k - 2 \).

Example 3.4. Let \( G_3 = (X, Y; E) \) be a balanced bipartite graph, with colour classes of the form \( X = A \cup B \), \( Y = C \cup D \), where \( |A| = |D| = k + 1 \), \( |B| = |C| = n - k - 1 \). Fix a vertex \( y_0 \) in \( C \), and let \( N_{G_3}(x) = C \) for all \( x \in A \), and \( N_{G_3}(x) = D \cup \{y_0\} \) for all \( x \in B \). Then \( \|G_3\| > n(n - k - 1) + k + 1 \) for \( k + 3 \leq n \leq 2k + 1 \), yet \( G_3 \) contains no cycle of length greater than \( 2n - 2k - 2 \). Hence the necessity of the assumption \( n \geq 2k + 2 \).

Interestingly, a similar graph was recently shown in [6] to be a counterexample to Győri’s conjecture on \( C_{2r} \)-free bipartite graphs.

In light of Example 3.3 above, we ask: By how much can we decrease the lower bound on the size of a given graph \( G \) ensuring the existence of a cycle of length \( 2n - 2k \), knowing that the minimal degree of \( G \) is greater than \( 1 \)? We address this question in Conjecture B. Certain special cases of Conjecture B are known true: \( k = 0 \) is Theorem 1.4, \( k = r = 1 \) is done in [3]. The following theorem (proved in Section 5 below) shows that the conjecture also holds for \( k = 1 \) and arbitrary \( r \).

Theorem B. Let \( G = (X, Y; E) \) be a balanced bipartite graph of order \( 2n \) and minimal degree \( \delta(G) \geq r \geq 1 \), where \( n \geq 4 \) and \( n \geq 2r + 1 \). Let

\[
g(n, r) = n(n - 1 - r) + r(1 + r) + 1.
\]

Then \( G \) contains a cycle of length \( 2n - 2 \), provided \( \|G\| \geq g(n, r) \).

Notice that Theorems 2.1 and 1.4 can be put together as follows:
Theorem 3.5. Let \( G \) be a balanced bipartite graph of order \( 2n \geq 4 \), with minimal degree \( \delta(G) \geq r \). Then \( G \) contains a Hamilton cycle, provided

1. \( n \leq 2r - 1 \) or
2. \( n \geq 2r \) and \( \|G\| > n(n-r) + r^2 \).

Along the same lines, we combine Theorem 2.4 and Theorems A and B to prove the following criterion for cycles of length \( 2n - 2 \).

Theorem 3.6. Let \( G = (X,Y;E) \) be a balanced bipartite graph of order \( 2n \geq 8 \), with minimal degree \( \delta(G) \geq r \geq 1 \). Then \( G \) contains a cycle of length \( 2n - 2 \), provided

1. \( n \leq 2r - 1 \) or
2. \( n = 2r \) and \( \|G\| \geq 2r^2 + r + 1 \) or
3. \( n \geq 2r + 1 \) and \( \|G\| \geq n(n-1-r) + r(1+r) + 1 \).

Remark 3.7. The lower bounds of conditions (2) and (3) are sharp: For an extremal graph for (2), consider the graph \( G_3 \) from Example 3.4 with \( k+1 = r \); for (3), consider \( G_1 \) from Example 1.6 with \( k = 1 \).

Proof of Theorem 3.6

1. Since \( n \leq 2r - 1 \) iff \( r \geq (n+1)/2 \), then the degree sum is greater than or equal to \( n+1 \) for every pair of vertices in \( G \) (in particular, for non-adjacent ones). By Theorem 2.4, \( G \) is then bipancyclic.

2. The bound on the size of \( G \) together with \( \delta(G) \geq r = n/2 \) force 2-connectedness. Also, the degree sum is at least \( 2r = n \) for every pair of vertices in \( G \). Hence, by Corollary 3.1, \( G \) contains \( C_{2n-2} \).

3. This is Theorem B. \( \square \)

4 Proof of Theorem A

As 2-connectedness of a graph \( G \) implies \( \delta(G) \geq 2 \), the assertion of the theorem holds true for \( n \leq 3 \), by Theorem 2.4. Suppose then there exists \( n \geq 4 \) for which the assertion fails. Let \( G = (X,Y;E) \) be a maximal 2-connected balanced bipartite graph of order \( 2n \), in which \( d_G(x) + d_G(y) \geq n \) for all non-adjacent \( x \in X, y \in Y \), without a cycle of length at least \( 2n - 2 \). By maximality of \( G \), \( G + xy \) contains a cycle of length at least \( 2n - 2 \), and hence \( G \) contains an \( x - y \) path of length \( 2n - 3 \) or \( 2n - 1 \) for every pair of non-adjacent \( x \in X, y \in Y \).

We shall show first that \( G \) contains a Hamilton path. Suppose not. Let \( x \in X, y \in Y \) be non-adjacent vertices and let \( P \) be an \( x - y \) path in \( G \) of length \( 2n - 3 \); say, \( P = u_1v_1u_2v_2 \ldots u_{n-1}v_{n-1}, \) where \( X = \{u_1, \ldots, u_n\}, Y = \{v_1, \ldots, v_n\}, u_1 = x \) and \( v_{n-1} = y \). Put \( I_P = \{1 \leq i \leq n-1 \mid u_iv_i \in E\} \) and \( J_P = \{1 \leq i \leq n-1 \mid u_iv_{n-1} \in E\} \). Then \( I_P \cap J_P = \emptyset \), for if \( i_0 \in I_P \cap J_P \), then \( G \) contains a cycle \( u_1v_{i_0}u_{i_0+1}v_{i_0+1} \ldots v_{n-1}u_{i_0}v_{i_0-1} \ldots v_1u_1 \) of length \( 2n - 2 \); a contradiction.
Ore and Erdős type conditions for long cycles in balanced bipartite graphs

As \(|I_P|=d_{G[V(P)]}(x)|\) and \(|J_P|=d_{G[V(P)]}(y)|\), we obtain

\[
d_{G[V(P)]}(x) + d_{G[V(P)]}(y) = |I_P| + |J_P| = |I_P \cup J_P| \leq n - 1,
\]

where \(G[V(P)]\) denotes the subgraph of \(G\) induced by the vertex set of \(P\). This shows that at least one of the vertices \(u_1\) and \(v_{n-1}\) has a neighbour among the remaining vertices \(u_n, v_n\) of \(G-P\); say, \(v_{n-1}u_n \in E\).

Notice that then \(u_n v_n \notin E\), for otherwise \(u_1 \ldots v_n \in G\) would be a Hamilton path. Similarly, \(u_1 v_n \notin E\). Hence, in particular, \(I_P\) contains indices of all the neighbours of \(u_1\) in \(G\), so \(|I_P|=d_G(u_1)|\).

Let now \(K_P = \{1 \leq i \leq n - 1 \mid u_i v_n \in E\}\). Then \(|K_P| = d_G(v_n)|\), and as \(d_G(u_1) + d_G(v_n)|\) \(\geq n\), it follows that there exists \(i_0 \in I_P \cap K_P\). Then \(v_n u_{i_0} \ldots v_{i_0+1} u_{i_0} \ldots v_{n-1} u_1\) is a Hamilton path in \(G\); a contradiction.

Let now \(x \in X\) and \(y \in Y\) be a pair of non-adjacent vertices such that \(G\) contains a Hamilton \(x-y\) path \(P\); say, \(P = u_1 v_1 \ldots u_n v_n\), where \(X = \{u_1, \ldots, u_n\}\), \(Y = \{v_1, \ldots, v_n\}\), \(x = u_1\) and \(y = v_n\). Put \(I_G = \{1 \leq i \leq n \mid u_i v_1 \in E\}\) and \(J_G = \{1 \leq i \leq n \mid u_i v_n \in E\}\). Then \(|I_G| = d_G(x)|\), \(|J_G| = d_G(y)|\) and \(I_G \cap J_G = \emptyset\) for if \(i_0 \in I_G \cap J_G\), then \(u_1 u_{i_0} \ldots u_{i_0} u_{i_0+1} \ldots v_n v_1\) a Hamilton cycle in \(G\).

Hence

\[
n \geq |I_G \cup J_G| = |I_G| + |J_G| = d_G(x) + d_G(y) \geq n,\]

so that, for every \(1 \leq i \leq n\),

\[
either u_i \in N_G(y) \text{ or else } v_l \in N_G(x). \quad (\star)
\]

Let \(d = d_G(y)\). Denote by \(x_1, \ldots, x_d\) those of the vertices \(u_1, \ldots, u_n\) that are adjacent to \(y\), ordered according to the orientation of \(P\) (from \(x\) to \(y\)). Let \(y_1, \ldots, y_d\) be the vertices of \(Y\) that lie on \(P\) next to the respective \(x_1, \ldots, x_d\); then \(y_d = y\).

Observe that if \(x_1 = u_i\) with \(i < n - d + 1\), then there exists \(1 \leq j \leq d - 1\) such that \(y_j = v_l\), where \(u_{i+1} \notin N_G(y)\). Then \(v_{i+1} \in N_G(x)\), and we obtain a cycle \(u_1 v_{i+1} u_{i+2} \ldots v_n v_l v_{l-1} \ldots v_1 u_1\) of length \(2n - 2\) in \(G\); a contradiction.

Therefore \(x_1 = u_{n-d+1}\), and hence \(N_G(y)\) coincides with the set \(\{u_{n-d+1}, \ldots, u_n\}\), call it \(U\). Then \(\{y_1, \ldots, y_d\}\) coincides with \(V := \{v_{n-d+1}, \ldots, v_n\}\), and by \((\star)\), \(N_G(x) = Y \setminus V\).

Suppose now that, for every \(v \in V\), \(N_G(v) \subset U\). Then, for all \(u \in X \setminus U\) and \(v \in V\), \(u\) and \(v\) are non-adjacent, hence \(N_G(u) \subset Y \setminus V\). Consequently, \(d_G(u) \leq n - d (i \leq n - d)\), and \(d_G(v_j) \leq d (j \geq n - d + 1)\). But \(u_i\) and \(v_j\) being non-adjacent, we also have \(d_G(u_i) + d_G(v_j) \geq n\), which implies that \(d_G(u_i) = n - d\) and \(d_G(v_j) = d\), and hence

\[
N_G(u_i) = Y \setminus V \text{ and } N_G(v_j) = U \text{ for all } i \leq n - d, j \geq n - d + 1.
\]

Thus \(G\) contains a complete bipartite graph \(K_{d,d}\) spanned on the vertices of \(U\) and \(V\), and a complete bipartite \(K_{n-d,n-d}\) spanned on \(X \setminus U\) and \(Y \setminus V\).

Now, \(G\) being 2-connected, it must contain two independent edges \(u_{i_1} v_{j_1}\) and \(u_{i_2} v_{j_2}\), for some \(i_1, i_2 \geq n - d + 1\) and \(j_1, j_2 \leq n - d\). One immediately verifies that such a graph contains a cycle of length \(2n - 2\), again contradicting the choice of \(G\).

We can therefore conclude that there exists a vertex \(v_j\), with \(n - d + 1 \leq j \leq n - 1\), adjacent to a \(u_i\), where \(i \leq n - d\). Then \(u_1 v_1 \ldots u_j v_{n-n} \ldots v_j u_i v_{i-1} \ldots v_1 u_1\) is a Hamilton cycle in \(G\). This contradiction completes the proof of the theorem. \(\square\)
5 Proof of Theorem B

Throughout this section we will frequently refer to the exceptional graph $G(n, r)$ of Theorem 2.3. Recall that by $G(n, r)$ we denote a balanced bipartite graph of order $2n$, with colour classes $X = P \cup Q$ and $Y = R \cup S$, where $|P| = |R| = r$, $|Q| = |S| = n - r$, $N_{G(n, r)}(x) = R$ for all $x \in P$, and $N_{G(n, r)}(x) = Y$ for all $x \in Q$.

Let, as before, $g(n, r) = n(n - 1 - r) + r(1 + r) + 1$. We shall first show the following lemma.

**Lemma 5.1.** Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 4$ and $n \geq 2r + 1$. Let $\|G\| \geq g(n, r)$, and assume there exists a pair of vertices $x \in X$ and $y \in Y$ such that $d_G(x) + d_G(y) \leq n$ and $\delta(G - \{x, y\}) \geq r$. Then $G$ contains a cycle of length $2n - 2$.

**Proof:** Suppose $G$ contains no cycle of length $2n - 2$. Then $G - \{x, y\}$ contains no such cycle either, and as $\delta(G - \{x, y\}) \geq r$, Theorem 2.3 implies that

$$\|G - \{x, y\}\| \leq (n - 1)(n - 1 - r) + r^2 = n^2 - 2n - nr + r^2 + r + 1.$$ 

On the other hand,

$$\|G - \{x, y\}\| \geq g(n, r) - (d_G(x) + d_G(y)) \geq n^2 - 2n - nr + r^2 + r + 1.$$ 

Hence $d_G(x) + d_G(y) = n$, the vertices $x$ and $y$ are non-adjacent, $G - \{x, y\}$ equals $G(n - 1, r)$, and $r \leq (n - 1)/2$. Without loss of generality, we may assume that $x$ belongs to the colour class of $G$ containing $P \cup Q$ of $G(n - 1, r)$.

Now, either $d_G(x) \geq r + 1$ or $d_G(x) = r$. In the first case, $x$ must have at least two neighbours in $S$ or else at least one neighbour in both $S$ and $R$. One easily verifies that then $G$ contains a cycle of length $2n - 2$, omitting $y$ and a single vertex of $P$; a contradiction.

If, in turn, $d_G(x) = r$, then $d_G(y) = n - r$ and $y$ must have neighbours in both $P$ and $Q$, since $r \leq (n - 1)/2 < n/2$. Consequently, $G$ contains a cycle of length $2n - 2$, omitting $x$ and a vertex of $S$, which again contradicts the choice of $G$.

We are now in position to prove Theorem B.

For a proof by contradiction, consider a graph $G$ satisfying the assumptions of Theorem B, that does not contain a cycle of length $2n - 2$. Observe first that $\|G\| > n^2/2$. Indeed, the difference $g(n, r) - n^2/2$ is always positive. Hence, by Theorem 2.5 $G$ is not hamiltonian. Consequently, Theorem 2.2 implies that there exists a positive integer $m \leq n/2$ such that at least one of the sets $S_m = \{x \in X : d_G(x) \leq m\}$, $T_m = \{y \in Y : d_G(y) \leq m\}$ has cardinality greater than or equal to $m$.

Let $l$ be the least such $m$. Without loss of generality, we may assume that $l$ is realized in $X$; i.e., $|\{x \in X : d_G(x) \leq l\}| \geq l$. Order the vertices of $X = \{x_1, \ldots, x_n\}$ so that $r \leq d_G(x_1) \leq \cdots \leq d_G(x_n)$. Then, by minimality of $l$, we have $l = \min\{i : d_G(x_i) \leq i\}$. Of course, $r \leq l \leq n/2$. Put $L = \{x_1, \ldots, x_l\}$.

The rest of the proof proceeds in two cases, depending on $l$ being equal to or greater than $r$. 


Case 1:

$l = r$. We will first show that all the vertices of $Y$ have degrees greater than $r$. Suppose to the contrary that there exists $y_1 \in Y$ with $d_G(y_1) = r$. Then

$$\| G - \{x_1, y_1\} \| \geq g(n, r) - 2r = n^2 - n - nr + r^2 - r + 1,$$

and $\delta(G - \{x_1, y_1\}) \geq r - 1$. On the other hand, by Theorem 2.3

$$\| G - \{x_1, y_1\} \| \leq (n - 1)(n - r) + (r - 1)^2 = n^2 - n - nr + r^2 - r + 1.$$ 

Hence $d_G(x_1) + d_G(y_1) = 2r$ so that $x_1y_1 \notin E$ and $G - \{x_1, y_1\}$ equals $G(n-1, r-1)$. By comparison of degrees, one readily verifies that $x_1$ belongs to that colour class of $G$ that contains $P \cup Q$ of $G(n-1, r-1)$; in fact, $L = \{x_1\} \cup P$. Consider the sets $R$ and $S$ of the other colour class of $G(n-1, r-1)$. As $|N_G(x_1)| = r > |R|$ and $x_1y_1 \notin E$, it follows that either $x_1$ has neighbours in both $R$ and $S$ or else it has at least two neighbours in $S$. In any case, as in the proof of Lemma 5.1, one easily finds a cycle of length $2n - 2$ in $G$, omitting $y_1$ and a vertex of $P$; a contradiction. Thus $d_G(y) \geq r + 1$ for every $y \in Y$.

Next observe that every vertex of $Y$ has a neighbour in $L$. Suppose otherwise, and let $y_1 \in Y$ be such that $N_G(y_1) \subset X \setminus L$. Notice that all vertices of $X \setminus L$ have degrees greater than $r$, for otherwise $g(n, r) \leq \| G \| \leq (r + 1)r + (n - r - 1)n = g(n, r) - 1$. Consequently, by removing $y_1$ and a vertex of $L$, say $x_1$, we do not decrease the minimal degree in the remainder of $G$. But, as $N_G(y_1) \subset X \setminus L$, we have $d_G(y_1) \leq n - r$, hence $d_G(x_1) + d_G(y_1) \leq r + (n - r) = n$, and by Lemma 5.1 $G$ contains a cycle of length $2n - 2$; a contradiction.

Consider the graph $G - L$. Notice that

$$\| G - L \| \geq g(n, r) - r^2 = n^2 - n - nr + r + 1.$$ 

Moreover, we claim that $d_{G - L}(x) + d_{G - L}(y) \geq n$ for every pair of non-adjacent $x \in X \setminus L$ and $y \in Y$. For if $d_{G - L}(x) + d_{G - L}(y) \leq n - 1$ for a pair of non-adjacent $x \in X \setminus L$ and $y \in Y$, then, by the above inequality,

$$\|(G - L) - \{x, y\}\| \geq n^2 - 2n - nr + r + 2 > (n - r - 1)(n - 1),$$ 

which contradicts $(G - L) - \{x, y\}$ being a bipartite graph with colour classes of cardinality $n - r - 1$ and $n - 1$.

Taking into account that every vertex in $Y$ has a neighbour in $L$, we now obtain that

$$d_G(x) + d_G(y) \geq n + 1 \quad \text{for all non-adjacent } y \in Y \text{ and } x \in X \setminus L.$$ 

Let $\tilde{G}$ be the bipartite graph obtained from $G$ by joining all the non-adjacent vertices of $Y$ and $X \setminus L$. As $|X \setminus L| = n - r$ and every $y \in Y$ has a neighbour in $L$, we get that $d_{\tilde{G}}(y) \geq n - r + 1$ for all $y \in Y$. Hence $d_{\tilde{G}}(x) + d_{\tilde{G}}(y) \geq n + 1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Therefore, joining all the non-adjacent vertices of $X$ and $Y$ in $\tilde{G}$ with degree sum of at least $n + 1$ yields a complete bipartite graph $K_{n,n}$. As $\tilde{G}$ was obtained from $G$ also by joining certain non-adjacent vertices of $X$ and $Y$ with degree sum of at least $n + 1$, this shows that the $(n + 1)$-biclosure of $G$ equals $K_{n,n}$. Thus, by Theorem 2.7 $G$ contains a Hamilton cycle, which, as we observed at the beginning of this proof, is impossible.
**Case 2:**

Let \( l \geq r + 1 \). In this case \( n \geq 2r + 2 \) (as \( l \leq n/2 \) and \( r \geq 2 \) (for otherwise \( l = r = 1 \), by minimality); hence \( |L| \geq 3 \). Moreover, \( d_G(x_{l-1}) = d_G(x_l) = l \), by minimality of \( l \).

Suppose first that \( d_G(x) + d_G(y) \geq n + 2 \) for every pair of non-adjacent \( x \in X \setminus L \) and \( y \in Y \). Let \( G' \) be the bipartite graph obtained from \( G \) by joining all the non-adjacent vertices of \( X \setminus L \) and \( Y \). We claim that every \( y \in Y \) has a neighbour in \( L \) (in \( G' \)). Suppose otherwise, and let \( y_1 \in Y \) be such that \( N_{G'}(y_1) \subset X \setminus L \). Then \( d_{G'}(y_1) \leq n - l \), hence \( d_{G'}(x_1) + d_{G'}(y_1) \leq n \). Moreover, \( \delta(G'-\{x_1,y_1\}) \geq r \), as all the vertices in \( X \setminus L \) have degrees of at least \( l \geq r + 1 \), and \( d_{G'}(y) \geq n - l \geq l \geq r + 1 \) for all \( y \in Y \). Then Lemma[5] implies that \( G' \) contains a cycle of length \( 2n - 2 \), and hence, by Theorem[2.8] so does \( G \); a contradiction.

Notice that \( G' \) was obtained from \( G \) by joining only pairs of vertices with degree sum of at least \( n + 2 \). Also, as every vertex \( y \in Y \) has a neighbour in \( L \) (in \( G' \)), we have \( d_{G'}(y) \geq n - l + 1 \). Recall that \( d_G(x_l) = d_G(x_1) = l \) and \( d_G(x_{l-1}) = d_G(x_1) = l \). Hence

\[
d_{G'}(x_l) + d_{G'}(y) \geq n + 1 \quad \text{and} \quad d_{G'}(x_{l-1}) + d_{G'}(y) \geq n + 1 \quad \text{for all} \quad y \in Y.
\]

Let \( G^{(2)} \) be the graph obtained from \( G' \) by joining \( x_l \) and \( x_{l-1} \) with all the vertices of \( Y \). Then \( d_{G^{(2)}}(y) \geq n - l + 2 \) for all \( y \in Y \), and as \( d_{G^{(2)}}(x_{l-2}) = d_G(x_{l-2}) \geq l - 1 \) (by minimality of \( l \)), we get that

\[
d_{G^{(2)}}(x_{l-2}) + d_{G^{(2)}}(y) \geq n + 1 \quad \text{for all} \quad y \in Y.
\]

Let now \( G^{(3)} \) be the graph obtained from \( G^{(2)} \) by joining \( x_{l-2} \) with all the non-adjacent vertices of \( Y \). In general, let \( G^{(m)} \) (\( m \geq 3 \)) be obtained from \( G^{(m-1)} \) by joining \( x_{l-m+1} \) with all the non-adjacent vertices of \( Y \). Then \( G^{(l)} = K_{n,n} \), and \( G^{(m)} \) is obtained from \( G^{(m-1)} \) by joining only pairs of vertices with degree sum of at least \( n + 1 \). Thus \( G^{(l)} = BCl_{n+1}(G) \), so that the \( (n+1) \)-biclosure of \( G \) is a complete bipartite graph. Now Theorem[2.7] implies that \( G \) contains a Hamilton cycle, which again leads to contradiction.

To complete the proof, it remains to consider the case when there is a pair of non-adjacent \( x^0 \in X \setminus L \) and \( y^0 \in Y \) with \( d_G(x^0) + d_G(y^0) \leq n + 1 \). This however can only happen when \( n = 2r + 2 \) or \( n = 2r + 3 \). For let us suppose that \( n \geq 2r + 4 \), and put \( f(l) = l^2 + (n - l - 1)(n - 1) + n + 2 \). We show \( \|G\| < f(l) \) and \( f(l) \leq g(n,r) \), and thus obtain a contradiction with the assumption \( \|G\| \geq g(n,r) \). If \( G \) contains a pair of non-adjacent vertices \( x \in X \setminus L \) and \( y \in Y \) with \( d_G(x) + d_G(y) \leq n + 1 \), then

\[
\|G\| \leq |L| \cdot l + |X \setminus (L \cup \{x\})| \cdot |Y \setminus \{y\}| + d_G(x) + d_G(y) \leq f(l) - 1.
\]

As the derivative of \( f \) equals \( f'(l) = -n + 2l + 1 \), it follows that \( f(l) \) is decreasing for \( l \leq (n - 1)/2 \), and hence maximal at \( l = r + 1 \). One immediately verifies that \( f(r + 1) \leq g(n,r) \) for \( n \geq 2r + 4 \). If, on the other hand, \( l > (n - 1)/2 \), then \( l = n/2 \) (since \( l \leq n/2 \)), and it is again immediate to check that \( f(n/2) \leq g(n,r) \) for \( n \geq 2r + 4 \).

**Subcase 2.1:**

\( n = 2r + 2 \). Then \( r + 1 \leq l \leq n/2 \) yields \( l = r + 1 \), and we obtain

\[
\|G - \{x^0, y^0\}\| \geq g(2r + 2, r) - (2r + 3) = 3r^2 + 3r.
\]
Ore and Erdős type conditions for long cycles in balanced bipartite graphs

On the other hand,

\[ \|G - \{x^0, y^0\}\| \leq |L| \cdot 1 + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 3r + 1. \]  

(2)

Hence

\[ 3r^2 + 3r \leq \|G - \{x^0, y^0\}\| \leq 3r^2 + 3r + 1 \text{ and } 2r + 2 \leq d_G(x^0) + d_G(y^0) \leq 2r + 3. \]

Suppose first that \( \|G - \{x^0, y^0\}\| = 3r^2 + 3r + 1 \). Then, by (2), \( d_G(x) = l \) for all \( x \in L \), and \( N_G(y^0) \cap L = \emptyset \); in particular, \( d_G(x_1) + d_G(y^0) \leq l + (n - l) = n \). Moreover, \( N_G(y) \supset X \setminus (L \cup \{x^0\}) \) for all \( y \in Y \setminus \{y^0\} \), and \( d_G(x) \geq r + 1 \) for all \( x \in X \), so that \( \delta(G - \{x_1, y^0\}) \geq r \), and by Lemma 5.1 \( G \) contains a cycle of length \( 2n - 2 \); a contradiction.

Therefore we may assume that \( \|G - \{x^0, y^0\}\| = 3r^2 + 3r \). By (1), \( d_G(x^0) + d_G(y^0) = 2r + 3 \), and what’s more, \( d_G(x) + d_G(y) \geq 2r + 3 \) for all non-adjacent \( x \in X \setminus L \) and \( y \in Y \). Indeed, if \( d_G(x^1) + d_G(y^1) \leq 2r + 2 \) for some non-adjacent \( x^1 \in X \setminus L \) and \( y^1 \in Y \), then by (1) and (2), \( \|G - \{x^1, y^1\}\| = 3r^2 + 3r + 1 \), which leads to contradiction, as above.

We will now show that \( |N_G(L)| > r + 1 \). Suppose otherwise, that is, suppose \( |N_G(L)| = l = r + 1 \). Then \( N_G(y^0) \cap L = \emptyset \), for else \( N_G(L) \ni y^0 \) implies

\[ \|G - \{x^0, y^0\}\| \leq |L| \cdot (l - 1) + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 2r, \]

which is impossible. Therefore \( d_G(y^0) = n - l - 1 = r \); in particular, \( d_G(x_1) + d_G(y^0) \leq l + r < n \). Notice that, as \( G - \{x^0, y^0\} \) only has one edge less than the right-hand side of (2), every neighbour of \( y^0 \) in \( G \) has degree at least \( n - 2 = 2r \), and every neighbour of \( x_1 \) has at least \( l - 1 = r \) other neighbours in \( L \) being the only vertex whose degree could be less than \( l \). Thus \( \delta(G - \{x_1, y^0\}) \geq r \), and we get a contradiction, by Lemma 5.1. Thus \( |N_G(L)| > r + 1 \).

It is now not difficult to see that \( BCL_{n+1}(G) = K_{n,n} \): Recall that we have verified that \( d_G(x) + d_G(y) \geq 2r + 3 = n + 1 \) for all non-adjacent \( x \in X \setminus L \) and \( y \in Y \). Let \( G' \) be the graph obtained from \( G \) by joining all the non-adjacent vertices of \( X \setminus L \) and \( Y \). Next observe that, by minimality of \( l = r + 1 \), \( d_{G'}(x_{r+1}) = d_G(x_{r+1}) = r + 1 \), and as \( |N_G(L)| > r + 1 \), at least one non-neighbour of \( x_{r+1} \), say \( y' \), has a neighbour among the other vertices of \( L \). Hence \( |N_{G'}(y')| \geq |X \setminus L| + 1 \), so that \( d_{G'}(x_{r+1}) + d_{G'}(y') \geq (r + 1) + (r + 2) = n + 1 \). Let \( G^{(2)} \) be obtained from \( G' \) by joining \( x_{r+1} \) with \( y' \), and hence increasing the degree of \( x_{r+1} \) to \( r + 2 \). Then \( d_{G^{(2)}}(x_{r+1}) + d_{G^{(2)}}(y) \geq n + 1 \) for all \( y \in Y \). Let \( G^{(3)} \) be obtained from \( G^{(2)} \) by joining \( x_{r+1} \) with all the non-adjacent vertices of \( Y \). Now \( d_{G^{(3)}}(y) \geq r + 2 \) for all \( y \in Y \). By minimality of \( l \) again, \( d_{G^{(3)}}(x_r) = d_G(x_r) = r + 1 \), and hence \( d_{G^{(4)}}(x_r) + d_{G^{(4)}}(y) \geq 2r + 3 \) for all \( y \in Y \). Let \( G^{(4)} \) be obtained from \( G^{(3)} \) by joining \( x_r \) with all the non-adjacent vertices of \( Y \). Then \( d_{G^{(4)}}(y) \geq r + 3 \) for all \( y \in Y \), and hence, as \( \delta(G^{(4)}) \geq \delta(G) \geq r \), \( d_{G^{(5)}}(x) + d_{G^{(5)}}(y) \geq 2r + 3 \) for all non-adjacent \( x \in X \) and \( y \in Y \). Joining all the non-adjacent pairs \( x \in X, y \in Y \) of \( G^{(4)} \) with degree sum of at least \( n + 1 \) we thus obtain \( K_{n,n} \). Since at each stage we only joined pairs of vertices with degree sum of at least \( n + 1 \), this shows that \( K_{n,n} = BCL_{n+1}(G) \). By Theorem 2.7 \( G \) contains a Hamilton cycle; a contradiction.
Subcase 2.2:
n = 2r + 3. Again, \( r + 1 \leq l \leq n/2 \) yields \( l = r + 1 \), and we have

\[
\|G - \{x^0, y^0\}\| \geq g(2r + 3, r) - (2r + 4) = 3r^2 + 6r + 3,
\]

and, on the other hand,

\[
\|G - \{x^0, y^0\}\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 6r + 3.
\]

Therefore both inequalities must, in fact, be equalities; in particular, \( d_G(x_1) = l \) and \( d_G(x) \geq r + 1 \) for all \( x \in X \), \( N_G(y^0) \cap L = \emptyset \), so that \( d_G(y^0) \leq n - l \), and finally \( |N_G(y^0)| \geq |X \setminus (L \cup \{x^0\})| = r + 1 \) for all \( y \in Y \setminus \{y^0\} \). Thus, again, \( G \) with the vertices \( x_1, y^0 \) satisfies the assumptions of Lemma 5.1, hence \( G \) contains a cycle of length \( 2n - 2 \); a contradiction. This completes the proof of Theorem B.

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References


