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The distribution of ascents of size \( d \) or more in compositions

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A composition of a positive integer \( n \) is a finite sequence of positive integers \( a_1, a_2, \ldots, a_k \) such that \( a_1 + a_2 + \cdots + a_k = n \). Let \( d \) be a fixed nonnegative integer. We say that we have an ascent of size \( d \) or more whenever \( a_{i+1} \geq a_i + d \). For example, there are 3 ascents of size 2 or more that occur in the compositions of 5: \( 14, 113, \) and \( 131 \).

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In Sections 2, 3 and 4, respectively, we determine the mean, variance and asymptotic distribution of the number of ascents of size $d$ or more in compositions of $n$. Ordinary ascents (the cases $d = 0$ and $d = 1$) have previously been studied by Carlitz (1) and more recently by Chinn, Heubach and Grimaldi in (2). Finally in Section 5 we investigate the maximum value of $d$ for which compositions of $n$ can expect to have an ascent of size $d$. That is, we find the average value of the largest ascent that occurs in the compositions of $n$. For example, the compositions of 4 in the table have maximum ascents of sizes 0, 2, 0, 0, 1, 1, 0, 0, respectively, giving an average largest ascent size per composition of $1/2$ when $n = 4$. We note that the asymptotic expression for the maximum ascent size in Theorem 4 involves a fluctuating function of $n$ of mean zero. A similar phenomenon has been observed when studying averages of certain other statistics for compositions, such as the largest part size (11) or the number of distinct part sizes (8), (7).

2 The average number of ascents of size $d$ or more in compositions

For fixed $d \geq 0$ we wish to find the average number of ascents of size $d$ or more per composition of $n$. We use the “adding-the-slice” technique which was originally used by Flajolet and Prodinger in (4) and more recently, for example, by Knopfmacher and Prodinger in (9). Let $j$ be the value of the last component of the composition with $k$ parts, i.e. $a_k = j$. We proceed from a composition with $k$ parts to a composition with $k + 1$ parts. We denote by $f_k(z, u, v)$ the generating function where $z$ marks the size $n$, $u$ the value of $j$ and $v$ the number of ascents of size $d$ or more in compositions with $k$ parts. In moving from a composition with $k$ parts to a composition with $k + 1$ parts, where $a_k = j$, we have an ascent whenever the new last integer has any value from $j + d$ onwards. This gives the following rule for adding a new part or “slice” to the end of the composition:

$$u^j \longrightarrow zu + (zu)^2 + (zu)^3 + \cdots + (zu)^{j+d-1} + v \left\{ (zu)^{j+d} + (zu)^{j+d+1} + \cdots \right\}$$

This implies that

$$f_{k+1}(z, u, v) = \frac{zu}{1 - zu} f_k(z, 1, v) - \frac{(zu)^d}{1 - zu} f_k(z, zu, v) + \frac{v(zu)^d}{1 - zu} f_k(z, zu, v)$$

$$= \frac{zu}{1 - zu} f_k(z, 1, v) - \frac{(1 - v)(zu)^d}{1 - zu} f_k(z, zu, v). \quad (2.1)$$

Now define $F(z, u, v) := \sum_{k \geq 1} f_k(z, u, v)$. Then summing (2.1) over $k \geq 1$ gives

$$F(z, u, v) - f_1(z, u, v) = \frac{zu}{1 - zu} F(z, 1, v) - \frac{(1 - v)(zu)^d}{1 - zu} F(z, zu, v),$$

so that

$$F(z, u, v) = \frac{zu}{1 - zu} F(z, 1, v) + \frac{zu}{1 - zu} - \frac{(1 - v)(zu)^d}{1 - zu} F(z, zu, v),$$
where we have used
\[ f_1(z, u, v) = zu + (zu)^2 + (zu)^3 + \cdots = \frac{zu}{1 - zu}. \]

At this stage we iterate the recursion for \( F(z, u, v) \).

\[
F(z, u, v) = \frac{zu}{1 - zu} F(z, 1, v) + \frac{zu}{1 - zu} - \frac{(1 - v)(zu)^d}{1 - zu} \times 
\left\{ \frac{z^2 u}{1 - z^2 u} F(z, 1, v) + \frac{z^2 u}{1 - z^2 u} - \frac{(1 - v)(z^2 u)^d}{1 - z^2 u} F(z, z^2 u, v) \right\} 
\]

\[
= \frac{zu}{1 - zu} - \frac{(1 - v)z^2 u(zu)^d}{(1 - zu)(1 - z^2 u)} [F(z, 1, v) + 1] + \frac{(1 - v)^2 (zu)^d (z^2 u)^d}{(1 - zu)(1 - z^2 u)} \times 
\left\{ \frac{z^3 u}{1 - z^3 u} F(z, 1, v) + \frac{z^3 u}{1 - z^3 u} - \frac{(1 - v)(z^3 u)^d}{1 - z^3 u} F(z, z^3 u, v) \right\} 
\]

\[
= \frac{zu}{1 - zu} - \frac{(1 - v)z^2 u(zu)^d}{(1 - zu)(1 - z^2 u)} + \frac{(1 - v)^2 z^3 u(zu)^d (z^2 u)^d}{(1 - zu)(1 - z^2 u)(1 - z^3 u)} [F(z, 1, v) + 1] 
\]

\[
- \frac{(1 - v)^3 (zu)^d (z^2 u)^d (z^3 u)^d}{(1 - zu)(1 - z^2 u)(1 - z^3 u)} F(z, z^3 u, v). 
\]

We keep iterating, noting that \( F(z, z^m u, v) \to 0 \) as \( m \to \infty \) for \( |z| < \frac{1}{2} \) and \( u, v \) in a suitable small neighbourhood of 1, and put \( u = 1 \) to obtain

\[
F(z, 1, v) = \sum_{i \geq 1} \frac{(-1)^i z^i (1 - v)^{i-1} z^d(i)}{(1 - z)(1 - z^2) \cdots (1 - z^i)} [F(z, 1, v) + 1]. \tag{2.2} 
\]

By adding the term 1 for the empty composition we obtain the bivariate generating function for compositions according to the number of ascents of size \( d \) or more as

\[
F(z, v) := 1 + F(z, 1, v) = \frac{1}{1 - \tau(z, v)}, \tag{2.3} 
\]

where

\[
\tau(z, v) := \sum_{i \geq 1} \frac{(-1)^i z^i (1 - v)^{i-1} z^d(i)}{(1 - z)(1 - z^2) \cdots (1 - z^i)}. \tag{2.4} 
\]

The expected value of the number of ascents of size \( d \) or more is \( \frac{|z^n F(z, v)|}{2^n v^n} \bigg|_{v=1} \). For this we shall need

\[
\tau(z, 1) = \sum_{i \geq 1} \frac{(-1)^i z^i (1 - v)^{i-1} z^d(i)}{(1 - z)(1 - z^2) \cdots (1 - z^i)} \bigg|_{v=1} = \frac{z}{1 - z}, \tag{2.5} 
\]

and

\[
\frac{\partial \tau(z, v)}{\partial v} \bigg|_{v=1} = \sum_{i \geq 2} \frac{(-1)^i z^i (i - 1)(1 - v)^{i-2} z^d(i)}{(1 - z)(1 - z^2) \cdots (1 - z^i)} \bigg|_{v=1} = \frac{z^{2+d}}{(1 - z)(1 - z^2)}. \tag{2.6} 
\]
In particular, the generating function for all compositions is
\[ F(z, 1) = \frac{1}{1 - \tau(z, 1)} = \frac{1 - z}{1 - 2z}. \]
Now
\[
\frac{\partial F(z, v)}{\partial v} \bigg|_{v=1} = \frac{\partial \tau(z, v)}{\partial v} \bigg|_{v=1} = \frac{z^{2+d}}{(1 + z)(1 - 2z)^2}.
\]
So that
\[
\left[ z^n \right] \frac{\partial F(z, v)}{\partial v} \bigg|_{v=1} = \frac{(-1)^{n-d}}{9} + \frac{(n - d + 1)2^{n-d}}{6} - \frac{5 2^{n-d}}{18}.
\]
After dividing by $2^{n-1}$, the total number of compositions of $n$, we have

**Theorem 1** The expected number of ascents of size $d$ or more in the compositions of $n$ is
\[
E(n) := \frac{2^{-d}}{9} (3n - 3d - 2) + \frac{2 (-1)^{n-d}}{2^n}, \quad \text{for } n \geq d.
\]
Hence for fixed $d$, as $n \to \infty$,
\[
E(n) = \frac{2^{-d}}{3} n + O(1).
\]
Previously Chinn, Heubach and Grimaldi found the number of ascents for $d = 0$ and $d = 1$ in (2). The case $d = 0$ corresponds to the number of rises plus the number of levels, whereas $d = 1$ corresponds to the number of rises.

### 3 Variance of the number of ascents of size $d$ or more in compositions

To find the variance we first need to compute $\left. \frac{\partial^2 F(z, v)}{\partial v^2} \right|_{v=1}$. In addition to formulas (2.3) to (2.6) from Section 2 we require
\[
\frac{\partial^2 \tau(z, v)}{\partial v^2} \bigg|_{v=1} = \sum_{i \geq 3} \frac{(-1)^{i-1} z^i (i - 1)(i - 2)(1 - v)^i - 3 z^i (i - 1)(i - 2)}{(1 - z)(1 - z^2) \cdots (1 - z^i)} \bigg|_{v=1} = \frac{2 z^{3(1+d)}}{(1 - z)(1 - z^2)(1 - z^3)}.
\]
Then
\[
\frac{\partial^2 F(z, v)}{\partial v^2} \bigg|_{v=1} = \frac{(1 - \tau) \frac{\partial^2 \tau(z, v)}{\partial v^2} + 2 \tau^2}{(1 - \tau)^3} \bigg|_{v=1}.
\]
Computing the $n$th coefficient of the second derivative amounts to expanding and combining binomial series.
Finally, after adding the expectation and subtracting the square of the expectation we find...
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Theorem 2 The variance of the expected number of ascents of size $d$ or more in the compositions of $n$ is

\[
\mathbb{V}(n) := 2^{-d} \left\{ -\frac{2}{9} + \frac{n}{3} - \frac{d}{3} \right\} + 2^{-2d} \left\{ \frac{80}{81} + \frac{10d}{9} + \frac{d^2}{3} - \frac{13n}{27} - \frac{2nd}{9} \right\} \\
+ 2^{-3d} \left\{ -\frac{352}{441} + \frac{8n}{21} - \frac{8d}{7} \right\} + 2^{-n} \left\{ (-1)^n \left( \frac{5}{27} + \frac{2n}{27} - \frac{4d}{27} \right) - \frac{1}{3} + 2\alpha(n) \right\} \\
+ 2^{-n-d}(-1)^n-d \left\{ \frac{4d}{27} - \frac{4n}{81} + \frac{8}{81} \right\} - 2^{-2n} \frac{4}{81},
\]

for $n \geq 3d$, where

\[
\alpha(n) = \begin{cases} 
\frac{26}{147} & \text{if } n = 3m, \\
\frac{-4}{147} & \text{if } n = 3m - 2, \\
\frac{-22}{147} & \text{if } n = 3m - 1, \text{ for } m \in \mathbb{N}.
\end{cases}
\]

For fixed $d$ we have

\[
\mathbb{V}(n) \sim n \left\{ \frac{2^{-d}}{3} - 2^{-2d} \left( \frac{13}{27} + \frac{2d}{9} \right) + 2^{-3d} \frac{8}{21} \right\} \text{ as } n \to \infty.
\]

4 Limiting distribution

We are interested in finding the limiting distribution of our random variable. We make use of Theorem IX.9 from Flajolet and Sedgewick (5). A short version is as follows:

Let $F(z, u)$ be a bivariate function that is bivariate analytic at $(z, u) = (0, 0)$ and has nonnegative coefficients there. Assume that $F(z, 1)$ is meromorphic in $z \leq r$ with only a simple pole at $z = \rho$ for some positive $\rho < r$. Then, under further conditions stated in (5), the random variable with probability generating function

\[
p_n(u) = \left[ z^n \right] F(z, u) / \left[ z^n \right] F(z, 1)
\]

converges in distribution to a Gaussian variable with a speed of convergence that is $O(n^{-1/2})$.

Let us introduce the notation

\[
c_{i,j} := \left. \frac{\partial^{i+j}}{\partial z^i \partial u^j} C(z, u) \right|_{(\rho, 1)}.
\]

From Theorem IX.9 we need to show that

\[c_{0,1} c_{1,0} \neq 0.\]  \hspace{1cm} (4.2)

In addition, we must show that

\[\rho c_{1,0}^2 - \rho c_{1,1} c_{0,1} + \rho c_{2,0} c_{0,1} + c_{0,1}^2 \neq 0.\]  \hspace{1cm} (4.3)

For our specific problem

\[F(z, v) = \frac{1}{1 - \tau(z, v)} = \frac{B(z, v)}{C(z, v)},\]
so that
\[ C(z, v) = 1 - \sum_{i \geq 1} \frac{(-1)^{i-1} z^i (1 - v)^{i-1} z^d(i)}{(1 - z)(1 - z^2) \cdots (1 - z^i)}. \]

We have \( \rho(1) = \frac{1}{2} \) and using (4.1),
\[ c_{0,1} = -\frac{2^{1-d}}{3}, \quad c_{1,0} = -4, \quad c_{1,1} = -\frac{3d + 11}{9} 2^{2-d}, \quad c_{0,2} = -\frac{2^{4-3d}}{21}, \quad c_{2,0} = -16. \]

We are now in a position to check the conditions listed in the theorem. Equation (4.2) is satisfied since
\[ c_{0,1} c_{1,0} = 8 \frac{2^{-d}}{3} \neq 0. \]

Equation (4.3) is equivalent to
\[ 2^{4-3d} \left( \frac{2^3}{21} + 2^d \frac{3d + 11}{27} - \frac{2^{1+d}}{9} - \frac{2^{2+2d}}{9} \right) \neq 0 \]
for non-negative integer values of \( d \). Thus we deduce

**Theorem 3** The distribution of the number of ascents of size \( d \) or more in compositions of \( n \) converges to a Gaussian distribution with a speed of convergence of \( O(n^{-1/2}) \) with the mean \( \mu_n \) and the variance \( \sigma_n^2 \) are as given in Theorems 1 and 2.

**Remark** In Flajolet and Sedgewick (5) it is also shown that under the conditions of Theorem IX.9, the mean \( \mu_n \) and variance \( \sigma_n^2 \) are of the form
\[ \mu_n = \mathfrak{m} \left( \frac{\rho(1)}{\rho(u)} \right) n + O(1), \quad \sigma_n^2 = \mathfrak{v} \left( \frac{\rho(1)}{\rho(u)} \right) n + O(1), \]
where
\[ \mathfrak{m}(f) = \frac{f'(1)}{f(1)} \quad \text{and} \quad \mathfrak{v}(f) = \frac{f''(1)}{f(1)} + \frac{f'(1)}{f(1)} - \left( \frac{f'(1)}{f(1)} \right)^2. \]

These asymptotic expressions are easily checked to be in agreement with the exact results for the mean and variance found previously in Theorems 1 and 2.

**5 Size of the maximum ascent**

Given a composition \( a_1 a_2 \ldots a_k \) of \( n \) we shall study the size of the maximum ascent, that is, the parameter \( X \) where
\[ X := \max \{ a_{i+1} - a_i \mid 1 \leq i < k \text{ and } a_{i+1} \geq a_i \}. \]

We assign a value of 0 if no pair of consecutive integers satisfies this condition. The mean value of \( X \) is given by the expression
\[ \sum_{d=0}^{n} \mathbb{P}(X > d) = \sum_{d=0}^{n} (1 - \mathbb{P}(X \leq d)). \]
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Therefore for each fixed $d$ we need to compute the probability that a composition of $n$ has maximum ascent $X \leq d$.

We already know the generating function for compositions with no ascents of size $n$ has maximum ascent $X \leq d$. We will denote by $F_d(z)$ for this we use (2.3) with $d+1$ instead of $d$ and $v = 0$, giving

$$F_{d+1}(z) := \frac{1}{1 - \tau(z, 0)} = \frac{1}{1 - \sum_{i \geq 1} \frac{(-1)^i z^{i+(d+1)}}{(1-z)(1-z^2)\cdots(1-z^i)}}.$$  

We want the generating function of compositions with $X \leq d$. This is equivalent to the generating function of the compositions with no ascent of size $d+1$ or more, which is $F_{d+1}(z)$. We now need to study the dominant poles, $\rho_d$, of $F_{d+1}(z)$, that is, the dominant zeros of

$$1 - \sum_{i \geq 1} \frac{(-1)^i z^{i+(d+1)}}{(1-z)(1-z^2)\cdots(1-z^i)} = 0. \quad (5.1)$$

We follow the approach used by Gourdon and Prodinger in (5), which is also analogous to the one found in (10). Now for $|z| \leq \frac{3}{2}$, say, using the first two terms of the series above, the root $\rho_d$ above can be approximated by the smallest positive root of

$$1 - \frac{z}{1-z} + \frac{z^{d+3}}{(1-z)(1-z^2)} + O(z^{3d}) = 0,$$

since the omitted terms in (5.1) are $O(z^{3d})$. This error will be majorized by subsequent $O$ terms below. That is we want the root of

$$1 - 2z + \frac{z^{d+3}}{1-z^2} + O(z^{3d}) = 0. \quad (5.1)$$

The bootstrapping method gives a suitable approximation to $\rho_d$. Let $\rho_d := \frac{1}{2} + \varepsilon_d$, then

$$1 - 2\left(\frac{1}{2} + \varepsilon_d\right) + \frac{4}{3} 2^{-d-3} = O\left(\frac{3}{5}\right)^{3d} + O\left(\frac{d}{2^{2d}}\right),$$

from which we find

$$\varepsilon_d = \frac{2^{-d}}{12} + O\left(\frac{d}{2^{2d}}\right) \text{ as } d \to \infty.$$  

As $F_{d+1}(z)$ has a simple pole at $\rho_d$, and by means of Rouche’s Theorem, see IX.6.2 in (5), by comparing $|F_{d+1}(z)|$ with $|1 - \frac{z}{1-z}|$ on the circle $|z| = \frac{3}{4}$, we see that $\frac{1}{F_{d+1}(z)}$ has no other zeros in $|z| \leq 3/4$.

It follows that

$$[z^n] F_{d+1}(z) = \frac{1}{1 - \frac{z}{\rho_d}} + O\left(\left(\frac{4}{3}\right)^n\right) \text{ with } A_d = \frac{\frac{1}{\rho_d} \left[\frac{d\tau(z, 0)}{dz}\right]_{z=\rho_d}}{\left[\frac{d\tau(z, 0)}{dz}\right]_{z=\rho}} = \frac{2 + O(2^{-d})}{2}.$$
where from (2.4) as \( d \to \infty \)
\[
\frac{d\tau(z,0)}{dz} \bigg|_{z=\rho} = \frac{\partial}{\partial z} \sum_{i \geq 1} (-1)^{i-1} z^{i+(d+1)}(z) \bigg|_{z=\rho} = \left( \frac{z}{(1-z)^2} + O(z^d) \right) \bigg|_{z=\rho} = 4 + O(2^{-d}).
\]

Therefore as \( d \to \infty \), \( A_d = \frac{1}{2} + O(2^{-d}) \). Let us now restrict our attention to those \( d \) for which \( n^{-3} \leq 2^{-d} \leq \frac{\log n}{n} \). The probability that \( X \leq d \) is then approximated as \( n \to \infty \) by

\[
A_d \rho_d^n = A_d \left( \frac{1}{2} + \frac{2-d}{12} + O \left( \frac{d}{2^{2d}} \right) \right)^{-n} = 2^{n-1} \exp \left( -\frac{2-d}{6} n \right) \left( 1 + O \left( 2^{-d} + O(n \, d \, 2^{-2d}) \right) \right)
\]

\[
= 2^{n-1} \exp \left( -\frac{2-d}{6} n \right) \left( 1 + O \left( \log^3 n \right) \right).
\]

So after dividing by \( 2^{n-1} \) we have for \( n \to \infty \) and \( n^{-3} \leq 2^{-d} \leq \frac{\log n}{n} \),

\[
\mathbb{P}(X \leq d) = \exp \left( \frac{-n}{2^{d} 6} \right) \left( 1 + O \left( \log^3 n \right) \right).
\]  (5.2)

Turning now to smaller values of \( d \geq 1 \), that is, \( d \) such that \( 2^{-d} > \frac{\log n}{n} \), a similar computation shows that (5.2) remains valid in this range, although now the probabilities \( \mathbb{P}(X \leq d) \) are small, since for such \( d \), \( \exp(\frac{-n}{2^{d} 6}) = O(\frac{1}{n}) \) as \( n \to \infty \). Finally we must consider larger values of \( d \leq n \) that is, \( d \) for which \( n^{-3} > 2^{-d} \), or equivalently, \( d \geq 3 \log_2 n \). In this range we find that

\[
\mathbb{P}(X \leq d) = 2^{n-1} \exp \left( -\frac{2-d}{6} n \right) \left( 1 + O \left( \frac{1}{n^2} \right) \right).
\]  (5.3)

In view of the \( O \) estimates in (5.2) and (5.3) we may deduce that the mean value of \( X \) satisfies

\[
E_{\text{max}}(n) := \sum_{d=0}^{n} \left( 1 - \mathbb{P}(X \leq d) \right) = \left( \sum_{d=0}^{n} \left( 1 - \exp \left( \frac{-n}{2^{d} 6} \right) \right) \right) \left( 1 + O \left( \frac{\log^4 n}{n} \right) \right) \text{ as } n \to \infty.
\]

We now use the Mellin transform, (see (3)), to estimate the function

\[
f(t) = \sum_{d \geq 0} \left( 1 - \exp \left( -\frac{t}{2^{d} 6} \right) \right) \text{ as } t \to \infty.
\]

The Mellin transform of \( f(t) \) is

\[
f^*(s) = -\sum_{d \geq 0} \left( 2^{d} 6 \right)^s \Gamma(s) = -\frac{\Gamma(s) 6^s}{1 - 2^s} \text{ for } -1 < \Re s < 0.
\]

Next we apply the Mellin inversion formula to recover \( f(t) \):

\[
f(t) = -\frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \Gamma(s) \frac{6^s}{1 - 2^s} t^{-s} ds.
\]
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We then shift the line of integration to the right and collect the (negative) residues at $s = 2k\pi i$ where $k \in \mathbb{Z}$ and $L = \log 2$.

$$f(t) = \sum_{k \geq 0} \text{Res} \Gamma(s) \left. \frac{6^s}{1 - 2^s} t^{-s} \right|_{s = \frac{2k\pi i}{L}} + \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \Gamma(s) \frac{6^s}{1 - 2^s} t^{-s} ds .$$

To evaluate the residue at the double pole $s = 0$, we need the expansion of the terms in the integrand to two terms as $s \to 0$, (here $\gamma$ denotes Euler’s constant),

$$\Gamma(s) \sim \frac{1}{s} - \gamma ,$$
$$6^s \sim 1 + s \log 6 ,$$
$$t^{-s} \sim 1 - s \log t ,$$
$$\frac{1}{1 - 2^s} \sim \frac{1}{2} - \frac{1}{sL} .$$

Hence the (negative) residue at $s = 0$ is

$$[s^{-1}]\left\{ \left( \frac{1}{2} - \frac{1}{sL} \right) (1 + s \log 6) \left( \frac{1}{s} - \gamma \right) (1 - s \log t) \right\} = \log_2 t - \frac{1}{2} - \log_2 3 + \frac{\gamma}{L} .$$

For the residue at $s = \frac{2k\pi i}{L}$ for $k \neq 0$, let $\varepsilon = s - \frac{2k\pi i}{L}$ we have

$$[\varepsilon^{-1}] \Gamma \left( \frac{2k\pi i}{L} \right) \left( \frac{t}{6} \right)^{-\frac{2k\pi i}{L}} = - \frac{1}{L} \Gamma \left( \frac{2k\pi i}{L} \right) \left( \frac{t}{6} \right)^{-\frac{2k\pi i}{L}} .$$

The remainder integral $\frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty}$ is of smaller order, so we have found

**Theorem 4** The mean value of the size of the greatest ascent in the compositions of $n$ satisfies as $n \to \infty$,

$$\mathbb{E}_{\text{max}}(n) \sim \log_2 n - \frac{1}{2} - \log_2 3 + \frac{\gamma}{L} - \delta \left( \log_2 \frac{n}{6} \right)$$

where $\delta(x)$ is a continuous periodic function of mean zero, period one and small amplitude with Fourier series

$$\delta(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma \left( -\frac{2k\pi i}{L} \right) e^{2k\pi ix} .$$

Computations show that $\delta(x) < 1.7 \times 10^{-6}$, as a result of the fast decrease of the gamma function with imaginary argument.

The diagram below shows the plot of $\delta(\log_2 \frac{n}{6})$ for $1 \leq n \leq 100$. 

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Fig. 1: $\delta\left(\log_2 \frac{n}{6}\right)$ for $1 \leq n \leq 100$

References


