Discrete Mathematics and Theoretical Computer Science DMTCS vol. **11:1**, 2009, 117–124

Self-complementing permutations of k-uniform hypergraphs

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received January 20, 2008, *revised January 16, 2009*, *accepted January 23, 2009*.

A k-uniform hypergraph $H = (V; E)$ is said to be *self-complementary* whenever it is isomorphic with its complement $\overline{H} = (V; \binom{V}{k} - E)$. Every permutation σ of the set V such that $\sigma(e)$ is an edge of $\overline{H}$ if and only if $e \in E$ is called *self-complementing*. 2-self-comlementary hypergraphs are exactly self complementary graphs introduced independently by Ringel (1963) and Sachs (1962).
For any positive integer n we denote by $\lambda(n)$ the unique integer such that $n = 2^{\lambda(n)}c$, where c is odd.
In the paper we prove that a permutation σ of $[1, n]$ with orbits $O_1, \dots, O_m$ is a self-complementing permutation of a k-uniform hypergraph of order n if and only if there is an integer $l \geq 0$ such that $k = a2^l + s$, a is odd, $0 \leq s < 2^l$ and the following two conditions hold:

(i) $n = b2^{l+1} + r$, $r \in \{0, \dots, 2^l - 1 + s\}$, and

(ii) $\sum_{i:\lambda(|O_i|)\leq l} |O_i| \leq r$.

For $k = 2$ this result is the very well known characterization of self-complementing permutation of graphs given by Ringel and Sachs.

Keywords: Self-complementing permutations, k-uniform hypergraphs

1 Introduction

Let V be a set of n elements. The set of all k-subsets of V is denoted by $\binom{V}{k}$. A *k-uniform hypergraph* H consists of a *vertex-set* $V(H)$ and an *edge-set* $E(H) \subseteq \binom{V(H)}{k}$. Two k-uniform hypergraphs G and H are *isomorphic*, if there is a bijection $\sigma : V(G) \to V(H)$ such that $e \in E(G)$ if and only if $\{\sigma(x) | x \in e\} \in E(H)$. The complement of a k-uniform hypergraph H is the hypergraph $\overline{H}$ such that $V(\overline{H}) = V(H)$ and the edge set of which consists of all k-subsets of $V(H)$ not in $E(H)$ (in other words $E(\overline{H}) = \binom{V(H)}{k} - E$). A k-uniform hypergraph H is called *self-complementary* (*s-c* for short) if it is isomorphic with its complement $\overline{H}$. Isomorphism of a k-uniform self-complementary hypergraph onto its complement is called *self-complementing permutation* (or *s-c permutation*).

The 2-uniform self-complementary hypergraphs are exactly self-complementary graphs. This class of graphs has been independently discovered by Ringel and Sachs who proved the following.

Theorem 1 (Ringel (Rin63) and Sachs (Sac62)) *Let n be a positive integer. A permutation σ of $[1, n]$ is a self-complementing permutation of a self-complementary graph of order n if and only if all the orbits of σ have their cardinalities congruent to 0 (mod 4) except, possibly, one orbit of cardinality 1.*

Observe that by Theorem 1 an s-c graph of order n exists if and only if $n \equiv 0$ or $n \equiv 1$ (mod 4) or, equivalently, whenever $\binom{n}{2}$ is even. In (SW) we prove that a similar result is true for k-uniform hypergraphs.

Theorem 2 ((SW)) *Let k and n be positive integers, $k \leq n$. A k-uniform self-complementary hypergraph of order n exists if and only if $\binom{n}{k}$ is even.*

A simple criterion for evenness of $\binom{n}{k}$ has been given in (Gla99) (and then rediscovered in (KHRM58)).

Theorem 3 ((Gla99; KHRM58)) *Let k and n be positive integers, $k = \sum_{i=0}^{+\infty} c_i 2^i$ and $n = \sum_{i=0}^{+\infty} d_i 2^i$, where $c_i, d_i \in \{0, 1\}$ for every i. $\binom{n}{k}$ is even if and only if there is i_0 such that $c_{i_0} = 1$ and $d_{i_0} = 0$.*

Theorem 3 asserts that $\binom{n}{k}$ is even if and only if k has 1 in a certain binary place while n has 0 in the corresponding binary place. For example, $\binom{27}{13}$ is even since $13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$ and $27 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$ (so we have $c_2 = 1$ and $d_2 = 0$).

Except for Theorem 1 which is a characterization of the self-complementing permutations for graphs, there are already two published results characterizing the permutations of k-uniform s-c hypergraphs for $k > 2$. Namely, Kocay in (Koc92) (see also (Pal73)) and Szymański in (Szy05) have characterized the s-c permutations of s-c k-uniform hypergraphs for, respectively, $k = 3$ and $k = 4$. This work is a continuation of the work of (SW) and (Woj06). We generalize all the results mentioned above by giving a characterization of the s-c permutations of k-uniform hypergraphs for any integers k and n.

2 Result

Any positive integer n may be writen in the form $n = 2^l c$, where c is an odd integer. Moreover, l and c are uniquely determined. We write then $\lambda(n) = l$. Note that in the binary expansion of n, $\lambda(n)$ is the index of the first $1-$bit. For any set A we shall write $\lambda(A)$ in place of $\lambda(|A|)$, for short.

In the proof of our main result we shall need the following lemma proved in (Woj06).

Lemma 1 *Let k, m and n be positive integers, and let $\sigma : V \to V$ be a permutation of a set V, $|V| = n$, with orbits $O_1, \ldots, O_m$. σ is a self-complementing permutation of a self-complementary k-uniform hypergraph, if and only if, for every $p \in \{1, \ldots, k\}$ and for every decomposition*

$$k = k_1 + \ldots + k_p$$

of k ($k_j > 0$ for $j = 1, \ldots, p$), and for every subsequence of orbits

$$O_{i_1}, \ldots, O_{i_p}$$

such that $k_j \leq |O_{i_j}|$ for $j = 1, \ldots, p$, there is a subscript $j_0 \in \{1, \ldots, p\}$ such that

$$\lambda(k_{j_0}) < \lambda(O_{i_{j_0}})$$

Given any integer $l \geq 0$. If the binary expansion of k is $1-$bit in position l, then k can be written in the form $k = a_l 2^l + s_l$, where a_l is odd and $0 \leq s_l < 2^l$.

Theorem 4 *Let k and n be integers, $k \leq n$. A permutation σ of $[1, n]$ with orbits $O_1, \ldots, O_m$ is a self-complementing permutation of a k-uniform hypergraph of order n if and only if there is a nonnegative integer l such that $k = a_l 2^l + s_l$, where a_l is odd and $0 \leq s_l < 2^l$, and the following two conditions hold:*

(i) $n = b_l 2^{l+1} + r_l$, $r_l \in \{0, \ldots, 2^l - 1 + s_l\}$, *and*

(ii) $\sum_{i:\lambda(O_i)\leq l} |O_i| \leq r_l$.

Proof:

Sufficiency. By contradiction. Let n, k, l, a_l, b_l, s_l and r_l be integers verifying the conditions of the theorem, let σ be a permutation of $[1, n]$ with orbits $O_1, \ldots, O_m$ verifying (ii), and let us suppose that σ is not a s-c permutation of any k-uniform s-c hypergraph of order n. Then, by Lemma 1, there is a decomposition of $k = k_1 + \cdots + k_t$ and a subsequence of orbits $O_{i_1}, \ldots, O_{i_t}$ such that

$$0 < k_j \leq |O_{i_j}| \tag{1}$$

and

$$\lambda(k_j) \geq \lambda(O_{i_j}) \tag{2}$$

for $j = 1, \ldots, t$.
Since a_l is odd, we have $k \equiv 2^l + s_l \pmod{2^{l+1}}$. By (2), $\sum_{j:\ \lambda(O_{i_j})>l} k_j \equiv 0 \pmod{2^{l+1}}$. Therefore

$$k = \sum_{j=1}^{t} k_j = \sum_{j:\ \lambda(O_{i_j})>l} k_j + \sum_{j:\ \lambda(O_{i_j})\leq l} k_j \equiv \sum_{j:\ \lambda(O_{i_j})\leq l} k_j \pmod{2^{l+1}}$$

Hence, and by (1), (i) and (ii) we have $\sum_{j:\lambda(O_{i_j})\leq l} k_j \leq \sum_{j:\lambda(O_{i_j})\leq l} |O_{i_j}| < 2^{l+1}$, and therefore

$$2^l + s_l = \sum_{j:\ \lambda(O_{i_j})\leq l} k_j \leq \sum_{j:\ \lambda(O_{i_j})\leq l} |O_{i_j}| \leq r_l < 2^l + s_l$$

a contradiction.

Necessity. Let $1 \leq k \leq n$ and let σ be a permutation of the set $[1, n]$ with orbits $O_1, \ldots, O_m$. Let us suppose that for every integer l such that $k = a_l 2^l + s_l$, where a_l is odd positive integer, $0 \leq s_l < 2^l$, and $n = b_l 2^{l+1} + r_l$, $0 \leq r_l < 2^{l+1}$ we have either

$$r_l \in \{2^l + s_l, \ldots, 2^{l+1} - 1\}$$

or

$$r_l \in \{0, \ldots, 2^l - 1 + s_l\} \qquad \text{and} \quad \sum_{i:\ \lambda(O_i)\leq l} |O_i| > r_l$$

We shall prove that σ is not a s-c permutation of any s-c k-uniform hypergraph of order n. For this purpose we shall give two claims.

Claim 1 *For every nonnegative integer l such that $k = a_l 2^l + s_l$, where a_l is odd and $0 \le s_l < 2^l$, we have*

$$\sum_{i:\ \lambda(O_i)\le l} |O_i| \ge 2^l + s_l$$

Proof of Claim 1. Let us write $\sum_{i:\lambda(O_i)\le l} |O_i|$ and $\sum_{i:\lambda(O_i)>l} |O_i|$ in their binary forms:

$$\sum_{i:\lambda(O_i)\le l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j$$

$$\sum_{i:\lambda(O_i)> l} |O_i| = \sum_{j=0}^{\infty} f_j 2^j$$

where $e_j, f_j \in \{0,1\}$ for every j. Observe that $f_j = 0$ for $j = 0, \dots, l$ and therefore

$$\sum_{j=0}^{l} e_j 2^j = r_l \tag{3}$$

We shall consider two cases.

Case 1. $r_l \in \{0, \dots, 2^l + s_l - 1\}$ and $\sum_{i:\lambda(O_i)\le l} |O_i| > r_l$.
We have $n \ge 2^{l+1}$ (otherwise $r_l = n = \sum_{i:\lambda(O_i)\le l} |O_i|$).
Since $\sum_{j=0}^{\infty} e_j 2^j > r_l$, and by (3), we obtain $\sum_{j=0}^{\infty} e_j 2^j \ge 2^{l+1} > 2^l + s_l$.

Case 2. $r_l \in \{2^l + s_l, \dots, 2^{l+1} - 1\}$.
We have $\sum_{i:\lambda(O_i)\le l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j \ge \sum_{j=0}^{l} e_j 2^j = r_l \ge 2^l + s_l$, and the claim is proved. □

Claim 2 *Let $\alpha_1, \dots, \alpha_q$ and $\lambda_1, \dots, \lambda_q$ be integers such that $0 < \alpha_i$, $0 \le \lambda_i \le \lambda(\alpha_i)$ and $\lambda_i \le l$ for $i = 1, \dots, q$ and $\sum_{i=1}^{q} \alpha_i \ge 2^l$. Then there are $\beta_1, \dots, \beta_q$ such that for every $i = 1, \dots, q$*

$$0 \le \beta_i \le \alpha_i \tag{4}$$

and

$$\text{either } \beta_i = 0 \text{ or } \lambda(\beta_i) \ge \lambda_i \tag{5}$$

and

$$\sum_{i=1}^{q} \beta_i = 2^l \tag{6}$$

Proof of Claim 2. The existence of $\beta_1, \dots, \beta_q$ verifying (4)-(5) and $\sum_{i=1}^{q} \beta_i \le 2^l$ is very easy. Indeed, it is immediate that $\beta_1 = 2^{\lambda_1}, \beta_2 = \dots \beta_q = 0$ is a sequence with the desired properties.
So let us suppose that $\beta_1, \dots, \beta_q$ is a sequence verifying (4)-(5) and $\sum_{i=1}^{q} \beta_i \le 2^l$ such that $\sum_{i=1}^{q} \beta_i$ is maximal. If $\sum_{i=1}^{q} \beta_i = 2^l$ then the proof is complete. So let us suppose that $\sum_{i=1}^{q} \beta_i < 2^l$. Then there is $i_0 \in \{1, \dots, q\}$ such that $\beta_{i_0} < \alpha_{i_0}$. Observe that $\beta_{i_0} + 2^{\lambda_{i_0}} \le \alpha_{i_0}$. The sequence $\overline{\beta}_1, \dots, \overline{\beta}_q$ defined by $\overline{\beta}_{i_0} = \beta_{i_0} + 2^{\lambda_{i_0}}$ and $\overline{\beta}_i = \beta_i$ for $i \ne i_0$ also verifies (4)-(5) and $\sum_{i=1}^{q} \overline{\beta}_i \le 2^l$, which contradicts the maximality of the sum $\sum_{i=1}^{q} \beta_i$, and the claim is proved. □

We shall use our claims to construct a decomposition of k in the form $k = k_1 + \dots + k_m$ such that

(1) $k_1, \ldots, k_m$ are nonnegative integers,

(2) $k_i \leq |O_i|$ for $i = 1, \ldots, m$, and

(3) $\lambda(k_i) \geq \lambda(O_i)$ whenever $k_i > 0$

By Lemma 1, this will imply that σ is not a s-c permutation of any k-uniform s-c hypergraph. Let us write k in its binary form:

$$k = 2^{l_t} + 2^{l_{t-1}} + \ldots + 2^{l_1} + 2^{l_0}$$

where $l_0 < l_1 < \ldots < l_t$.

By Claim 1, $\sum_{i:\lambda(O_i)\leq l_0} |O_i| \geq 2^{l_0}$. Hence, and by Claim 2, there are nonnegative integers $k_1^{(0)}, k_2^{(0)}, \ldots, k_m^{(0)}$ such that $k_i^{(0)} = 0$ for i such that $\lambda(O_i) > l_0$ and

$$k_i^{(0)} \leq |O_i| \text{ for } i = 1, \ldots, m$$

$$\lambda(k_i^{(0)}) \geq \lambda(O_i) \text{ whenever } k_i^{(0)} > 0$$

and

$$\sum_{i=1}^{m} k_i^{(0)} = 2^{l_0}$$

Note that, for $i = 1, \ldots, m$, we have $\lambda(|O_i| - k_i^{(0)}) \geq \lambda(O_i)$.

Let us suppose that we have already constructed $k_1^{(j)}, \ldots, k_m^{(j)}$, ($j \leq t$), such that $k_i^{(j)} = 0$ for i such that $\lambda(0_i) > l_i$ and

$$k_i^{(j)} \leq |O_i| \text{ for } i = 1, \ldots, m$$

$$\lambda(k_i^{(j)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j)} > 0$$

$$\sum_{i=0}^{m} k_i^{(j)} = 2^{l_j} + 2^{l_{j-1}} + \cdots + 2^{l_0}$$

and

$$\lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i)$$

If $j = t$, then we have already found a desired decomposition of k. If $j < t$, then, by Claim 1, we have $\sum_{i:\lambda(O_i)\leq l_{j+1}} (|O_i| - k_i^{(j)}) \geq 2^{l_{j+1}}$.

$\lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i)$ for every $i \in \{1, \ldots, m\}$ such that $|O_i| - k_i^{(j)} > 0$. Hence, and by Claim 2, there are $\beta_1, \ldots, \beta_m$ such that $\beta_i = 0$ for i such that $\lambda(O_i) > l_{j+1}$ and

$$0 \leq \beta_i \leq |O_i| - k_i^{(j)} \text{ for } i = 1, \ldots, m$$

$$\lambda(O_i) \leq \lambda(\beta_i) \text{ for } i = 1, \ldots, m \text{ whenever } \beta_i \neq 0$$

$$\sum_{i=1}^{m} \beta_i = 2^{l_{j+1}}$$

Thus we may define for every $i = 1, \ldots, m$

$$k_i^{(j+1)} = k_i^{(j)} + \beta_i$$

to obtain the sequence $(k_1^{(j+1)}, \ldots, k_m^{(j+1)})$ verifying for every $i \in \{1, \ldots, m\}$

$$k_i^{(j+1)} = 0 \text{ for } i \text{ such that } \lambda(O_i) > l_{j+1}$$

$$k_i^{(j+1)} \leq |O_i|$$

$$\lambda(k_i^{(j+1)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j+1)} > 0$$

and

$$\sum_{i=1}^{m} k_i^{(j+1)} = 2^{l_{j+1}} + 2^{l_j} + \cdots + 2^{l_0}$$

It is clear that $k = \sum_{i=1}^{m} k_i^{(t)}$ and the proof of Theorem 4 is complete. □

Theorem 4 implies very easily the following theorem first proved by Kocay.

Corollary 1 (Kocay (Koc92)) *σ is a self-complementing permutation of a self-complementary 3-uniform hypergraph if and only if either all the orbits of σ have even cardinalities, or else, it has 1 or 2 fixed points and the all remaining orbits of σ have their cardinalities being multiples of 4.*

For $k = 2^l$ Theorem 4 may be written as follows.

Corollary 2 *Let l and n be nonnegative integers, $2^l < n$, and let $0 \leq r < 2^{l+1}$ be such that $n \equiv r$ (mod 2^{l+1}). A permutation σ of $[1, n]$ with orbits $O_1, \ldots O_m$ is a self-complementing permutation of a 2^l-uniform self-complementary hypergraph if and only if*

(i) $r \in \{0, \ldots, 2^l - 1\}$ *and*

(ii) $\sum_{i:\lambda(O_i) \leq l} |O_i| \leq r$.

Theorem 2 for $l = 1$ (i.e. for graphs) is exactly Theorem 1, and for $l = 2$ the following theorem proved by Szymański in (Szy05).

Corollary 3 *A permutation σ is self-complementing permutation of a 4-uniform hypergraph of order n if and only if $n \equiv r($ mod $8)$ with $r = 0, 1, 2$ or 3, and the sum of the cardinalities of orbits which are not multiples of 8 is at most 3.*

Acknowledgements

The research was partially supported by AGH local grant No 11 420 04.

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