Self-complementing permutations of k-uniform hypergraphs
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A \(k\)-uniform hypergraph \(H = (V; E)\) is said to be self-complementary whenever it is isomorphic with its complement \(\overline{H} = (V; \binom{V}{k} - E)\). Every permutation \(\sigma\) of the set \(V\) such that \(\sigma(e)\) is an edge of \(\overline{H}\) if and only if \(e \in E\) is called self-complementing. \(2\)-self-complementary hypergraphs are exactly self complementary graphs introduced independently by Ringel (1963) and Sachs (1962).

For any positive integer \(n\) we denote by \(\lambda(n)\) the unique integer such that \(n = 2^{\lambda(n)} c\), where \(c\) is odd. In the paper we prove that a permutation \(\sigma\) of \([1, n]\) with orbits \(O_1, \ldots, O_m\) is a self-complementing permutation of a \(k\)-uniform hypergraph of order \(n\) if and only if there is an integer \(l \geq 0\) such that \(k = a 2^l + s\), \(a\) is odd, \(0 \leq s < 2^l\) and the following two conditions hold:

(i) \(n = b 2^{l+1} + r,\) \(r \in \{0, \ldots, 2^l - 1 + s\}\), and

(ii) \(\sum_{i: \lambda(O_i) \leq l} |O_i| \leq r\).

For \(k = 2\) this result is the very well known characterization of self-complementing permutation of graphs given by Ringel and Sachs.

Keywords: Self-complementing permutations, \(k\)-uniform hypergraphs

1 Introduction

Let \(V\) be a set of \(n\) elements. The set of all \(k\)-subsets of \(V\) is denoted by \(\binom{V}{k}\). A \(k\)-uniform hypergraph \(H\) consists of a vertex-set \(V(H)\) and an edge-set \(E(H) \subseteq \binom{V(H)}{k}\). Two \(k\)-uniform hypergraphs \(G\) and \(H\) are isomorphic, if there is a bijection \(\sigma : V(G) \rightarrow V(H)\) such that \(e \in E(G)\) if and only if \(\{\sigma(x) | x \in e\} \in E(H)\). The complement of a \(k\)-uniform hypergraph \(H\) is the hypergraph \(\overline{H}\) such that \(V(\overline{H}) = V(H)\) and the edge set of which consists of all \(k\)-subsets of \(V\) of \(H\) not in \(E(H)\) (in other words \(E(\overline{H}) = \binom{V(H)}{k} - E\)). A \(k\)-uniform hypergraph \(H\) is called self-complementary (s-c for short) if it is isomorphic with its complement \(\overline{H}\). Isomorphism of a \(k\)-uniform self-complementary hypergraph onto its complement is called self-complementing permutation (or s-c permutation).
The 2-uniform self-complementary hypergraphs are exactly self-complementary graphs. This class of graphs has been independently discovered by Ringel and Sachs who proved the following.

**Theorem 1** (Ringel (Rin63) and Sachs (Sac62)) Let \( n \) be a positive integer. A permutation \( \sigma \) of \([1, n]\) is a self-complementing permutation of a self-complementary graph of order \( n \) if and only if all the orbits of \( \sigma \) have their cardinalities congruent to 0 (mod 4) except, possibly, one orbit of cardinality 1.

Observe that by Theorem 1 an s-c graph of order \( n \) exists if and only if \( n \equiv 0 \) or \( n \equiv 1 \) (mod 4) or, equivalently, whenever \( (n^2) \) is even. In (SW) we prove that a similar result is true for \( k \)-uniform hypergraphs.

**Theorem 2** (SW) Let \( k \) and \( n \) be positive integers, \( k \leq n \). A \( k \)-uniform self-complementary hypergraph of order \( n \) exists if and only if \( (\frac{n^2}{k}) \) is even.

A simple criterion for evenness of \( (\frac{n^2}{k}) \) has been given in (Gla99) (and then rediscovered in (KHRM58)).

**Theorem 3** (Gla99, KHRM58) Let \( k \) and \( n \) be positive integers, \( k = \sum_{i=0}^{+\infty} c_i 2^i \) and \( n = \sum_{i=0}^{+\infty} d_i 2^i \), where \( c_i, d_i \in \{0, 1\} \) for every \( i \). \((\frac{n^2}{k})\) is even if and only if there is \( i_0 \) such that \( c_{i_0} = 1 \) and \( d_{i_0} = 0 \).

**Theorem 3** asserts that \((\frac{n^2}{k})\) is even if and only if \( k \) has 1 in a certain binary place while \( n \) has 0 in the corresponding binary place. For example, \( (\frac{27}{3}) \) is even since \( 13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \) and \( 27 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \) (so we have \( c_2 = 1 \) and \( d_2 = 0 \)).

Except for Theorem 1 which is a characterization of the self-complementing permutations for graphs, there are already two published results characterizing the permutations of \( k \)-uniform s-c hypergraphs for \( k > 2 \). Namely, Kocay in (Koc92) (see also (Pal73)) and Szymański in (Szy05) have characterized the s-c permutations of \( k \)-uniform hypergraphs for, respectively, \( k = 3 \) and \( k = 4 \). This work is a continuation of the work of (SW) and (Woj06). We generalize all the results mentioned above by giving a characterization of the s-c permutations of \( k \)-uniform hypergraphs for any integers \( k \) and \( n \).

## 2 Result

Any positive integer \( n \) may be written in the form \( n = 2^l c \), where \( c \) is an odd integer. Moreover, \( l \) and \( c \) are uniquely determined. We write then \( \lambda(n) = l \). Note that in the binary expansion of \( n \), \( \lambda(n) \) is the index of the first 1-bit. For any set \( A \) we shall write \( \lambda(A) \) in place of \( \lambda(|A|) \), for short.

In the proof of our main result we shall need the following lemma proved in (Woj06).

**Lemma 1** Let \( k, m \) and \( n \) be positive integers, and let \( \sigma : V \to V \) be a permutation of a set \( V \), \( |V| = n \), with orbits \( O_1, \ldots, O_m \). \( \sigma \) is a self-complementing permutation of a self-complementary \( k \)-uniform hypergraph, if and only if, for every \( p \in \{1, \ldots, k\} \) and for every decomposition

\[
k = k_1 + \ldots + k_p
\]

of \( k \) \((k_j > 0 \text{ for } j = 1, \ldots, p)\), and for every subsequence of orbits

\[
O_{i_1}, \ldots, O_{i_p}
\]

such that \( k_j \leq |O_{i_j}| \) for \( j = 1, \ldots, p \), there is a subscript \( j_0 \in \{1, \ldots, p\} \) such that

\[
\lambda(k_{j_0}) < \lambda(O_{i_{j_0}})
\]
Given any integer \( l \geq 0 \). If the binary expansion of \( k \) is \( 1 \)-bit in position \( l \), then \( k \) can be written in the form \( k = a_l 2^l + s_l \), where \( a_l \) is odd and \( 0 \leq s_l < 2^l \).

**Theorem 4** Let \( k \) and \( n \) be integers, \( k \leq n \). A permutation \( \sigma \) of \([1, n]\) with orbits \( O_1, \ldots, O_m \) is a self-complementing permutation of a \( k \)-uniform hypergraph of order \( n \) if and only if there is a nonnegative integer \( l \) such that \( k = a_l 2^l + s_l \), where \( a_l \) is odd and \( 0 \leq s_l < 2^l \), and the following two conditions hold:

(i) \( n = b_l 2^{l+1} + r_l, \ r_l \in \{0, \ldots, 2^l - 1 + s_l\}, \) and

(ii) \( \sum_{i: \lambda(O_i) \leq l} |O_i| \leq r_l. \)

**Proof:**

**Sufficiency.** By contradiction. Let \( n, k, l, a_l, b_l, s_l \) and \( r_l \) be integers verifying the conditions of the theorem, let \( \sigma \) be a permutation of \([1, n]\) with orbits \( O_1, \ldots, O_m \) verifying (ii), and let us suppose that \( \sigma \) is not a s-c permutation of any \( k \)-uniform s-c hypergraph of order \( n \). Then, by Lemma \([1]\) there is a decomposition of \( k = k_1 + \cdots + k_t \) and a subsequence of orbits \( O_{i_1}, \ldots, O_{i_t} \) such that

\[
0 < k_j \leq |O_{i_j}|
\]

and

\[
\lambda(k_j) \geq \lambda(O_{i_j})
\]

for \( j = 1, \ldots, t \).

Since \( a_l \) is odd, we have \( k \equiv 2^l + s_l \pmod{2^{l+1}} \). By (2), \( \sum_{j: \lambda(O_{i_j}) > l} k_j \equiv 0 \pmod{2^{l+1}} \). Therefore

\[
k = \sum_{j=1}^{t} k_j = \sum_{j: \lambda(O_{i_j}) > l} k_j + \sum_{j: \lambda(O_{i_j}) \leq l} k_j \equiv \sum_{j: \lambda(O_{i_j}) \leq l} k_j \pmod{2^{l+1}}
\]

Hence, and by (1), (i) and (ii) we have \( \sum_{j: \lambda(O_{i_j}) \leq l} k_j \leq \sum_{j: \lambda(O_{i_j}) \leq l} |O_{i_j}| < 2^{l+1} \), and therefore

\[
2^l + s_l = \sum_{j: \lambda(O_{i_j}) \leq l} k_j \leq \sum_{j: \lambda(O_{i_j}) \leq l} |O_{i_j}| \leq r_l < 2^l + s_l
\]

a contradiction.

**Necessity.** Let \( 1 \leq k \leq n \) and let \( \sigma \) be a permutation of the set \([1, n]\) with orbits \( O_1, \ldots, O_m \). Let us suppose that for every integer \( l \) such that \( k = a_l 2^l + s_l \), where \( a_l \) is odd positive integer, \( 0 \leq s_l < 2^l \), and \( n = b_l 2^{l+1} + r_l, \ 0 \leq r_l < 2^{l+1} \) we have either

\[
r_l \in \{2^l + s_l, \ldots, 2^{l+1} - 1\}
\]

or

\[
r_l \in \{0, \ldots, 2^l - 1 + s_l\} \quad \text{and} \quad \sum_{i: \lambda(O_i) \leq l} |O_i| > r_l
\]

We shall prove that \( \sigma \) is not a s-c permutation of any s-c \( k \)-uniform hypergraph of order \( n \). For this purpose we shall give two claims.
Claim 1. For every nonnegative integer \( l \) such that \( k = a_l 2^l + s_l \), where \( a_l \) is odd and \( 0 \leq s_l < 2^l \), we have
\[
\sum_{i: \lambda(O_i) \leq l} |O_i| \geq 2^l + s_l
\]

Proof of Claim 1. Let us write \( \sum_{i: \lambda(O_i) \leq l} |O_i| \) and \( \sum_{i: \lambda(O_i) > l} |O_i| \) in their binary forms:
\[
\sum_{i: \lambda(O_i) \leq l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j
\]
\[
\sum_{i: \lambda(O_i) > l} |O_i| = \sum_{j=0}^{\infty} f_j 2^j
\]
where \( e_j, f_j \in \{0, 1\} \) for every \( j \). Observe that \( f_j = 0 \) for \( j = 0, \ldots, l \) and therefore
\[
\sum_{j=0}^{l} e_j 2^j = r_l
\]
We shall consider two cases.

Case 1. \( r_l \in \{0, \ldots, 2^l + s_l - 1\} \) and \( \sum_{i: \lambda(O_i) \leq l} |O_i| > r_l \).

We have \( n \geq 2^{l+1} \) (otherwise \( r_l = n = \sum_{i: \lambda(O_i) \leq l} |O_i| \)).

Since \( \sum_{j=0}^{\infty} e_j 2^j > r_l \), and by (3), we obtain \( \sum_{j=0}^{\infty} e_j 2^j \geq 2^{l+1} > 2^l + s_l \).

Case 2. \( r_l \in \{2^l + s_l, \ldots, 2^{l+1} - 1\} \).
We have \( \sum_{i: \lambda(O_i) \leq l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j \geq \sum_{j=0}^{l} e_j 2^j = r_l \geq 2^l + s_l \), and the claim is proved. \( \square \)

Claim 2. Let \( \alpha_1, \ldots, \alpha_q \) and \( \lambda_1, \ldots, \lambda_q \) be integers such that \( 0 < \alpha_i, 0 \leq \lambda_i \leq \lambda(\alpha_i) \) and \( \lambda_i \leq 1 \) for \( i = 1, \ldots, q \) and \( \sum_{i=1}^{q} \alpha_i \geq 2^l \). Then there are \( \beta_1, \ldots, \beta_q \) such that for every \( i = 1, \ldots, q \)
\[
0 \leq \beta_i \leq \alpha_i
\]
and
\[
either \beta_i = 0 or \lambda(\beta_i) \geq \lambda_i
\]
and
\[
\sum_{i=1}^{q} \beta_i = 2^l \tag{6}
\]

Proof of Claim 2. The existence of \( \beta_1, \ldots, \beta_q \) verifying (4)-(5) and \( \sum_{i=1}^{q} \beta_i \leq 2^l \) is very easy. Indeed, it is immediate that \( \beta_1 = 2^{\lambda_1}, \beta_2 = \ldots, \beta_q = 0 \) is a sequence with the desired properties.

So let us suppose that \( \beta_1, \ldots, \beta_q \) is a sequence verifying (4)-(5) and \( \sum_{i=1}^{q} \beta_i < 2^l \) such that \( \sum_{i=1}^{q} \beta_i \) is maximal. If \( \sum_{i=1}^{q} \beta_i = 2^l \) then the proof is complete. So let us suppose that \( \sum_{i=1}^{q} \beta_i < 2^l \). Then there is \( i_0 \in \{1, \ldots, q\} \) such that \( \beta_{i_0} < \alpha_{i_0} \). Observe that \( \beta_{i_0} + 2^{\lambda_{i_0}} \leq \alpha_{i_0} \). The sequence \( \overline{\beta}_1, \ldots, \overline{\beta}_q \) defined by \( \overline{\beta}_{i_0} = \beta_{i_0} + 2^{\lambda_{i_0}} \) and \( \overline{\beta}_i = \beta_i \) for \( i \neq i_0 \) also verifies (4)-(5) and \( \sum_{i=1}^{q} \overline{\beta}_i \leq 2^l \), which contradicts the maximality of the sum \( \sum_{i=1}^{q} \beta_i \), and the claim is proved. \( \square \)

We shall use our claims to construct a decomposition of \( k \) in the form \( k = k_1 + \ldots + k_m \) such that
Self-complementing permutations of \( k \)-uniform hypergraphs

(1) \( k_1, \ldots, k_m \) are nonnegative integers,

(2) \( k_i \leq |O_i| \) for \( i = 1, \ldots, m \), and

(3) \( \lambda(k_i) \geq \lambda(O_i) \) whenever \( k_i > 0 \)

By Lemma 1, this will imply that \( \sigma \) is not a \( s \)-c permutation of any \( k \)-uniform \( s \)-c hypergraph.

Let us write \( k \) in its binary form:

\[ k = 2^{l_t} + 2^{l_{t-1}} + \ldots + 2^{l_1} + 2^{l_0} \]

where \( l_0 < l_1 < \ldots < l_t \).

By Claim 1, \( \sum_{i: \lambda(O_i) \leq l_0} |O_i| \geq 2^{l_0} \). Hence, and by Claim 2, there are nonnegative integers \( k_1^{(0)}, k_2^{(0)}, \ldots, k_m^{(0)} \) such that \( k_i^{(0)} = 0 \) for \( i \) such that \( \lambda(O_i) > l_0 \) and

\[ k_i^{(0)} \leq |O_i| \text{ for } i = 1, \ldots, m \]

\[ \lambda(k_i^{(0)}) \geq \lambda(O_i) \text{ whenever } k_i^{(0)} > 0 \]

and

\[ \sum_{i=1}^{m} k_i^{(0)} = 2^{l_0} \]

Note that, for \( i = 1, \ldots, m \), we have \( \lambda(|O_i| - k_i^{(0)}) \geq \lambda(O_i) \).

Let us suppose that we have already constructed \( k_1^{(j)}, \ldots, k_m^{(j)} \), \( (j \leq t) \), such that \( k_i^{(j)} = 0 \) for \( i \) such that \( \lambda(O_i) > l_i \) and

\[ k_i^{(j)} \leq |O_i| \text{ for } i = 1, \ldots, m \]

\[ \lambda(k_i^{(j)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j)} > 0 \]

\[ \sum_{i=0}^{m} k_i^{(j)} = 2^{l_j} + 2^{l_{j-1}} + \ldots + 2^{l_0} \]

and

\[ \lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i) \text{ for } i = 1, \ldots, m \]

If \( j = t \), then we have already found a desired decomposition of \( k \). If \( j < t \), then, by Claim 1, we have \( \sum_{i: \lambda(O_i) \leq l_{j+1}} (|O_i| - k_i^{(j)}) \geq 2^{l_{j+1}} \).

\( \lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i) \) for every \( i \in \{1, \ldots, m\} \) such that \( |O_i| - k_i^{(j)} > 0 \). Hence, and by Claim 2, there are \( \beta_1, \ldots, \beta_m \) such that \( \beta_i = 0 \) for \( i \) such that \( \lambda(O_i) > l_{j+1} \) and

\[ 0 \leq \beta_i \leq |O_i| - k_i^{(j)} \text{ for } i = 1, \ldots, m \]

\[ \lambda(O_i) \leq \lambda(\beta_i) \text{ for } i = 1, \ldots, m \text{ whenever } \beta_i \neq 0 \]

\[ \sum_{i=1}^{m} \beta_i = 2^{l_{j+1}} \]
Thus we may define for every \( i = 1, \ldots, m \)
\[
k_i^{(j+1)} = k_i^{(j)} + \beta_i
\]
to obtain the sequence \((k_1^{(j+1)}, \ldots, k_m^{(j+1)})\) verifying for every \( i \in \{1, \ldots, m\} \)
\[
k_i^{(j+1)} = 0 \text{ for } i \text{ such that } \lambda(O_i) > l_{j+1}
\]
\[
k_i^{(j+1)} \leq |O_i|
\]
\[
\lambda(k_i^{(j+1)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j+1)} > 0
\]
and
\[
\sum_{i=1}^{m} k_i^{(j+1)} = 2^{l_{j+1}} + 2^{l_j} + \ldots + 2^{l_0}
\]
It is clear that \( k = \sum_{i=1}^{m} k_i^{(t)} \) and the proof of Theorem 4 is complete. \(\square\)

Theorem 4 implies very easily the following theorem first proved by Kocay.

**Corollary 1 (Kocay (Koc92))** \(\sigma\) is a self-complementing permutation of a self-complementary 3-uniform hypergraph if and only if either all the orbits of \(\sigma\) have even cardinalities, or else, it has 1 or 2 fixed points and the all remaining orbits of \(\sigma\) have their cardinalities being multiples of 4.

For \( k = 2^l \) Theorem 4 may be written as follows.

**Corollary 2** Let \( l \) and \( n \) be nonnegative integers, \( 2^l < n \), and let \( 0 \leq r < 2^{l+1} \) be such that \( n \equiv r \pmod{2^{l+1}} \). A permutation \( \sigma \) of \([1, n]\) with orbits \( O_1, \ldots, O_m \) is a self-complementing permutation of a \( 2^l \)-uniform self-complementary hypergraph if and only if

(i) \( r \in \{0, \ldots, 2^l - 1\} \) and

(ii) \( \sum_{i: \lambda(O_i) \leq l} |O_i| \leq r \).

Theorem 2 for \( l = 1 \) (i.e. for graphs) is exactly Theorem 1 and for \( l = 2 \) the following theorem proved by Szymański in (Szy05).

**Corollary 3** A permutation \( \sigma \) is self-complementing permutation of a 4-uniform hypergraph of order \( n \) if and only if \( n \equiv r \pmod{8} \) with \( r = 0, 1, 2 \) or 3, and the sum of the cardinalities of orbits which are not multiples of 8 is at most 3.

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Self-complementing permutations of $k$-uniform hypergraphs

References


