Self-complementing permutations of $k$-uniform hypergraphs

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A $k$-uniform hypergraph $H = (V; E)$ is said to be self-complementary whenever it is isomorphic with its complement $\overline{H} = (V; \binom{V}{k} - E)$. Every permutation $\sigma$ of the set $V$ such that $\sigma(e)$ is an edge of $\overline{H}$ if and only if $e \in E$ is called self-complementing. $2$-self-complementary hypergraphs are exactly self complementary graphs introduced independently by Ringel (1963) and Sachs (1962).

For any positive integer $n$ we denote by $\lambda(n)$ the unique integer such that $n = 2^\lambda(n)c$, where $c$ is odd.

In the paper we prove that a permutation $\sigma$ of $[1, n]$ with orbits $O_1, \ldots, O_m$ is a self-complementing permutation of a $k$-uniform hypergraph of order $n$ if and only if there is an integer $l \geq 0$ such that $k = a2^l + s$, $a$ is odd, $0 \leq s < 2^l$ and the following two conditions hold:

(i) $n = b2^l + 1 + r$, $r \in \{0, \ldots, 2^l - 1 + s\}$, and
(ii) $\sum_{i: \lambda(O_i) \leq l} |O_i| \leq r$.

For $k = 2$ this result is the very well known characterization of self-complementing permutation of graphs given by Ringel and Sachs.

Keywords: Self-complementing permutations, $k$-uniform hypergraphs

1 Introduction

Let $V$ be a set of $n$ elements. The set of all $k$-subsets of $V$ is denoted by $\binom{V}{k}$. A $k$-uniform hypergraph $H$ consists of a vertex-set $V(H)$ and an edge-set $E(H) \subseteq \binom{V(H)}{k}$. Two $k$-uniform hypergraphs $G$ and $H$ are isomorphic, if there is a bijection $\sigma : V(G) \to V(H)$ such that $e \in E(G)$ if and only if $\{\sigma(x) | x \in e\} \in E(H)$. The complement of a $k$-uniform hypergraph $H$ is the hypergraph $\overline{H}$ such that $V(\overline{H}) = V(H)$ and the edge set of which consists of all $k$-subsets of $V(H)$ not in $E(H)$ (in other words $E(\overline{H}) = \binom{V(H)}{k} - E$). A $k$-uniform hypergraph $H$ is called self-complementary (s-c for short) if it is isomorphic with its complement $\overline{H}$. Isomorphism of a $k$-uniform self-complementary hypergraph onto its complement is called self-complementing permutation (or s-c permutation).
The 2-uniform self-complementary hypergraphs are exactly self-complementary graphs. This class of graphs has been independently discovered by Ringel and Sachs who proved the following.

**Theorem 1 (Ringel (Ring63) and Sachs (Sac62))** Let \( n \) be a positive integer. A permutation \( \sigma \) of \( [1, n] \) is a self-complementing permutation of a self-complementary graph of order \( n \) if and only if all the orbits of \( \sigma \) have their cardinalities congruent to 0 (mod 4) except, possibly, one orbit of cardinality 1.

Observe that by Theorem 1 an s-c graph of order \( n \) exists if and only if \( n \equiv 0 \) or \( n \equiv 1 \) (mod 4) or, equivalently, whenever \( \binom{n}{2} \) is even. In (SW) we prove that a similar result is true for \( k \)-uniform hypergraphs.

**Theorem 2 (SW)** Let \( k \) and \( n \) be positive integers, \( k \leq n \). A \( k \)-uniform self-complementary hypergraph of order \( n \) exists if and only if \( \binom{n}{k} \) is even.

A simple criterion for evenness of \( \binom{n}{k} \) has been given in (Gla99) (and then rediscovered in (KHRM58)).

**Theorem 3 (Gla99 KHRM58)** Let \( k \) and \( n \) be positive integers, \( k = \sum_{i=0}^{+\infty} c_i 2^i \) and \( n = \sum_{i=0}^{+\infty} d_i 2^i \), where \( c_i, d_i \in \{0, 1\} \) for every \( i \). \( \binom{n}{k} \) is even if and only if there is \( i_0 \) such that \( c_{i_0} = 1 \) and \( d_{i_0} = 0 \).

Theorem 3 asserts that \( \binom{n}{k} \) is even if and only if \( k \) has 1 in a certain binary place while \( n \) has 0 in the corresponding binary place. For example, \( \binom{27}{13} \) is even since \( 13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \) and \( 27 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \) (so we have \( c_2 = 1 \) and \( d_2 = 0 \)).

Except for Theorem 1 which is a characterization of the self-complementing permutations for graphs, there are already two published results characterizing the permutations of \( k \)-uniform s-c hypergraphs for \( k > 2 \). Namely, Kocay in (Koc92) (see also (Pal73)) and Szymański in (Szy05) have characterized the s-c permutations of s-c \( k \)-uniform hypergraphs for, respectively, \( k = 3 \) and \( k = 4 \). This work is a continuation of the work of (SW) and (Woj06). We generalize all the results mentioned above by giving a characterization of the s-c permutations of \( k \)-uniform hypergraphs for any integers \( k \) and \( n \).

## 2 Result

Any positive integer \( n \) may be written in the form \( n = 2^l c \), where \( c \) is an odd integer. Moreover, \( l \) and \( c \) are uniquely determined. We write then \( \lambda(n) = l \). Note that in the binary expansion of \( n \), \( \lambda(n) \) is the index of the first 1-bit. For any set \( A \) we shall write \( \lambda(A) \) in place of \( \lambda(|A|) \), for short.

In the proof of our main result we shall need the following lemma proved in (Woj06).

**Lemma 1** Let \( k, m \) and \( n \) be positive integers, and let \( \sigma : V \rightarrow V \) be a permutation of a set \( V, |V| = n \), with orbits \( O_1, \ldots, O_m \). \( \sigma \) is a self-complementing permutation of a self-complementary \( k \)-uniform hypergraph, if and only if, for every \( p \in \{1, \ldots, k\} \) and for every decomposition

\[
k = k_1 + \ldots + k_p
\]

of \( k \) (\( k_j > 0 \) for \( j = 1, \ldots, p \), and for every subsequence of orbits

\[
O_{j_1}, \ldots, O_{j_p}
\]

such that \( k_j \leq |O_{j_j}| \) for \( j = 1, \ldots, p \), there is a subscript \( j_0 \in \{1, \ldots, p\} \) such that

\[
\lambda(k_{j_0}) < \lambda(|O_{j_0}|)
\]
Given any integer $l \geq 0$. If the binary expansion of $k$ is 1-bit in position $l$, then $k$ can be written in the form $k = a_l 2^l + s_l$, where $a_l$ is odd and $0 \leq s_l < 2^l$.

**Theorem 4** Let $k$ and $n$ be integers, $k \leq n$. A permutation $\sigma$ of $[1, n]$ with orbits $O_1, \ldots, O_m$ is a self-complementing permutation of a $k$-uniform hypergraph of order $n$ if and only if there is a nonnegative integer $l$ such that $k = a_l 2^l + s_l$, where $a_l$ is odd and $0 \leq s_l < 2^l$, and the following two conditions hold:

(i) $n = b_l 2^{l+1} + r_l$, $r_l \in \{0, \ldots, 2^l - 1 + s_l\}$, and

(ii) $\sum_{i: \lambda(O_i) \leq l} |O_i| \leq r_l$.

**Proof:**

**Sufficiency.** By contradiction. Let $n, k, l, a_l, b_l, s_l$ and $r_l$ be integers verifying the conditions of the theorem, let $\sigma$ be a permutation of $[1, n]$ with orbits $O_1, \ldots, O_m$ verifying (ii), and let us suppose that $\sigma$ is not a s-c permutation of any s-c hypergraph of order $n$. Then, by Lemma 1, there is a decomposition of $k = k_1 + \cdots + k_t$ and a subsequence of orbits $O_{i_1}, \ldots, O_{i_t}$ such that

$$0 < k_j \leq |O_{i_j}| \quad (1)$$

and

$$\lambda(k_j) \geq \lambda(O_{i_j}) \quad (2)$$

for $j = 1, \ldots, t$.

Since $a_l$ is odd, we have $k \equiv 2^l + s_l \pmod{2^{l+1}}$. By (2), $\sum_{j: \lambda(O_{i_j}) > l} k_j \equiv 0 \pmod{2^{l+1}}$. Therefore

$$k = \sum_{j=1}^t k_j = \sum_{j: \lambda(O_{i_j}) > l} k_j + \sum_{j: \lambda(O_{i_j}) \leq l} k_j \equiv \sum_{j: \lambda(O_{i_j}) \leq l} k_j \quad (\text{mod}\ 2^{l+1})$$

Hence, by (1), (i) and (ii) we have $\sum_{j: \lambda(O_{i_j}) \leq l} k_j \leq \sum_{j: \lambda(O_{i_j}) \leq l} |O_{i_j}| \leq 2^{l+1}$, and therefore

$$2^l + s_l = \sum_{j: \lambda(O_{i_j}) \leq l} k_j \leq \sum_{j: \lambda(O_{i_j}) \leq l} |O_{i_j}| \leq r_l < 2^l + s_l$$

a contradiction.

**Necessity.** Let $1 \leq k \leq n$ and let $\sigma$ be a permutation of the set $[1, n]$ with orbits $O_1, \ldots, O_m$. Let us suppose that for every integer $l$ such that $k = a_l 2^l + s_l$, where $a_l$ is odd positive integer, $0 \leq s_l < 2^l$, and $n = b_l 2^{l+1} + r_l$, $0 \leq r_l < 2^{l+1}$ we have either

$$r_l \in \{2^l + s_l, \ldots, 2^{l+1} - 1\}$$

or

$$r_l \in \{0, \ldots, 2^l - 1 + s_l\}$$

and

$$\sum_{i: \lambda(O_i) \leq l} |O_i| > r_l$$

We shall prove that $\sigma$ is not a s-c permutation of any s-c $k$-uniform hypergraph of order $n$. For this purpose we shall give two claims.
Claim 1 For every nonnegative integer \( l \) such that \( k = a_l 2^l + s_l \), where \( a_l \) is odd and \( 0 \leq s_l < 2^l \), we have

\[
\sum_{i: \lambda(O_i) \leq l} |O_i| \geq 2^l + s_l
\]

Proof of Claim 1 Let us write \( \sum_{i: \lambda(O_i) \leq l} |O_i| \) and \( \sum_{i: \lambda(O_i) > l} |O_i| \) in their binary forms:

\[
\sum_{i: \lambda(O_i) \leq l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j
\]

\[
\sum_{i: \lambda(O_i) > l} |O_i| = \sum_{j=0}^{\infty} f_j 2^j
\]

where \( e_j, f_j \in \{0, 1\} \) for every \( j \). Observe that \( f_j = 0 \) for \( j = 0, \ldots, l \) and therefore

\[
\sum_{j=0}^{l} e_j 2^j = r_l
\]

We shall consider two cases.

Case 1. \( r_l \in \{0, \ldots, 2^l + s_l - 1\} \) and \( \sum_{i: \lambda(O_i) \leq l} |O_i| > r_l \).

We have \( n \geq 2^l + 1 \) (otherwise \( r_l = n = \sum_{i: \lambda(O_i) \leq l} |O_i| \)).

Since \( \sum_{j=0}^{\infty} e_j 2^j > r_l \), and by (3), we obtain \( \sum_{j=0}^{\infty} e_j 2^j \geq 2^l + 1 > 2^l + s_l \).

Case 2. \( r_l \in \{2^l + s_l, \ldots, 2^{l+1} - 1\} \).

We have \( \sum_{i: \lambda(O_i) \leq l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j \geq \sum_{j=0}^{l} e_j 2^j = r_l \geq 2^l + s_l \), and the claim is proved.

Claim 2 Let \( \alpha_1, \ldots, \alpha_q \) and \( \lambda_1, \ldots, \lambda_q \) be integers such that \( 0 < \alpha_i, 0 \leq \lambda_i \leq \lambda(\alpha_i) \) and \( \lambda_i \leq 1 \) for \( i = 1, \ldots, q \) and \( \sum_{i=1}^{q} \alpha_i \geq 2^l \). Then there are \( \beta_1, \ldots, \beta_q \) such that for every \( i = 1, \ldots, q \)

\[
0 \leq \beta_i \leq \alpha_i
\]

and

\[
\text{either } \beta_i = 0 \text{ or } \lambda(\beta_i) \geq \lambda_i
\]

and

\[
\sum_{i=1}^{q} \beta_i = 2^l
\]

Proof of Claim 2 The existence of \( \beta_1, \ldots, \beta_q \) verifying (4)-(5) and \( \sum_{i=1}^{q} \beta_i \leq 2^l \) is very easy. Indeed, it is immediate that \( \beta_1 = 2^{\lambda_1}, \beta_2 = \ldots, \beta_q = 0 \) is a sequence with the desired properties.

So let us suppose that \( \beta_1, \ldots, \beta_q \) is a sequence verifying (4)-(5) and \( \sum_{i=1}^{q} \beta_i \leq 2^l \) such that \( \sum_{i=1}^{q} \beta_i \) is maximal. If \( \sum_{i=1}^{q} \beta_i = 2^l \) then the proof is complete. So let us suppose that \( \sum_{i=1}^{q} \beta_i < 2^l \). Then there is \( i_0 \in \{1, \ldots, q\} \) such that \( \beta_{i_0} < \alpha_{i_0} \). Observe that \( \beta_{i_0} + 2^{\lambda_{i_0}} \leq \alpha_{i_0} \). The sequence \( \beta_1, \ldots, \beta_q \) defined by \( \beta_{i_0} = \beta_{i_0} + 2^{\lambda_{i_0}} \) and \( \beta_i = \beta_i \) for \( i \neq i_0 \) also verifies (4)-(5) and \( \sum_{i=1}^{q} \beta_i \) \( \leq 2^l \), which contradicts the maximality of the sum \( \sum_{i=1}^{q} \beta_i \), and the claim is proved.

We shall use our claims to construct a decomposition of \( k \) in the form \( k = k_1 + \ldots + k_m \) such that
Self-complementing permutations of $k$-uniform hypergraphs

1. $k_1, \ldots, k_m$ are nonnegative integers,
2. $k_i \leq |O_i|$ for $i = 1, \ldots, m$, and
3. $\lambda(k_i) \geq \lambda(O_i)$ whenever $k_i > 0$

By Lemma 1, this will imply that $\sigma$ is not a s-c permutation of any $k$-uniform s-c hypergraph.

Let us write $k$ in its binary form:

$$k = 2^{l_0} + 2^{l_{t-1}} + \ldots + 2^{l_1} + 2^{l_0}$$

where $l_0 < l_1 < \ldots < l_t$.

By Claim 1, $\sum_{i: \lambda(O_i) \leq l_0} |O_i| \geq 2^{l_0}$. Hence, and by Claim 2, there are nonnegative integers $k_1^{(0)}, k_2^{(0)}, \ldots, k_m^{(0)}$ such that $k_i^{(0)} = 0$ for $i$ such that $\lambda(O_i) > l_0$ and

$$k_i^{(0)} \leq |O_i| \text{ for } i = 1, \ldots, m$$

$$\lambda(k_i^{(0)}) \geq \lambda(O_i) \text{ whenever } k_i^{(0)} > 0$$

and

$$\sum_{i=1}^m k_i^{(0)} = 2^{l_0}$$

Note that, for $i = 1, \ldots, m$, we have $\lambda(|O_i| - k_i^{(0)}) \geq \lambda(O_i)$.

Let us suppose that we have already constructed $k_1^{(j)}, \ldots, k_m^{(j)}$, $(j \leq t)$, such that $k_i^{(j)} = 0$ for $i$ such that $\lambda(O_i) > l_j$ and

$$k_i^{(j)} \leq |O_i| \text{ for } i = 1, \ldots, m$$

$$\lambda(k_i^{(j)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j)} > 0$$

and

$$\sum_{i=0}^m k_i^{(j)} = 2^{l_j} + 2^{l_{j-1}} + \ldots + 2^{l_0}$$

and

$$\lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i)$$

If $j = t$, then we have already found a desired decomposition of $k$. If $j < t$, then, by Claim 1, we have $\sum_{i: \lambda(O_i) \leq l_{j+1}} (|O_i| - k_i^{(j)}) \geq 2^{l_{j+1}}$.

$\lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i)$ for every $i \in \{1, \ldots, m\}$ such that $|O_i| - k_i^{(j)} > 0$. Hence, and by Claim 2, there are $\beta_1, \ldots, \beta_m$ such that $\beta_i = 0$ for $i$ such that $\lambda(O_i) > l_{j+1}$ and

$$0 \leq \beta_i \leq |O_i| - k_i^{(j)} \text{ for } i = 1, \ldots, m$$

$$\lambda(O_i) \leq \lambda(\beta_i) \text{ for } i = 1, \ldots, m \text{ whenever } \beta_i \neq 0$$

$$\sum_{i=1}^m \beta_i = 2^{l_{j+1}}$$
Thus we may define for every $i = 1, \ldots, m$

$$k_i^{(j+1)} = k_i^{(j)} + \beta_i$$

to obtain the sequence $(k_1^{(j+1)}, \ldots, k_m^{(j+1)})$ verifying for every $i \in \{1, \ldots, m\}$

$$k_i^{(j+1)} = 0 \text{ for } i \text{ such that } \lambda(O_i) > l_{j+1}$$

$$k_i^{(j+1)} \leq |O_i|$$

$$\lambda(k_i^{(j+1)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j+1)} > 0$$

and

$$\sum_{i=1}^{m} k_i^{(j+1)} = 2^{l_{j+1}} + 2^{l_j} + \ldots + 2^{l_0}$$

It is clear that $k = \sum_{i=1}^{m} k_i^{(t)}$ and the proof of Theorem 4 is complete.

Theorem 4 implies very easily the following theorem first proved by Kocay.

**Corollary 1 (Kocay (Koc92))**  \(\sigma\) is a self-complementing permutation of a self-complementary 3-uniform hypergraph if and only if either all the orbits of \(\sigma\) have even cardinalities, or else, it has 1 or 2 fixed points and the all remaining orbits of \(\sigma\) have their cardinalities being multiples of 4.

For $k = 2^l$ Theorem 4 may be written as follows.

**Corollary 2** Let $l$ and $n$ be nonnegative integers, $2^l < n$, and let $0 \leq r < 2^{l+1}$ be such that $n \equiv r \pmod{2^{l+1}}$. A permutation $\sigma$ of $[1, n]$ with orbits $O_1, \ldots, O_m$ is a self-complementing permutation of a $2^l$-uniform self-complementary hypergraph if and only if

(i) $r \in \{0, \ldots, 2^l - 1\}$ and

(ii) $\sum_{v: \lambda(O_i) \leq l} |O_i| \leq r$.

Theorem 2 for $l = 1$ (i.e. for graphs) is exactly Theorem 1 and for $l = 2$ the following theorem proved by Szymański in [Szy05].

**Corollary 3** A permutation $\sigma$ is self-complementing permutation of a 4-uniform hypergraph of order $n$ if and only if $n \equiv r \pmod{8}$ with $r = 0, 1, 2$ or 3, and the sum of the cardinalities of orbits which are not multiples of 8 is at most 3.

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Self-complementing permutations of \(k\)-uniform hypergraphs

References


