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The Eulerian distribution on centrosymmetric involutions

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We present an extensive study of the Eulerian distribution on the set of centrosymmetric involutions, namely, involutions in S_n satisfying the property $\sigma(i) + \sigma(n + 1 - i) = n + 1$ for every $i = 1 \dots n$. We find some combinatorial properties for the generating polynomial of such distribution, together with an explicit formula for its coefficients. Afterwards, we carry out an analogous study for the subset of centrosymmetric involutions without fixed points.

Keywords: involution, permutation statistic, Eulerian distribution, hyperoctahedral group.

1 Introduction

The distribution of the descent statistic (classically known as *Eulerian distribution*) on peculiar subsets of permutations has been object of intensive studies in recent years. In particular, several authors examined the properties of the polynomial $I_n(x) = \sum_{j=0}^{n-1} i_{n,j} x^j$, where $i_{n,j}$ denotes the number of involutions on $[n] = \{1, 2, \dots, n\}$ with j descents. More specifically, in (13) V. Strehl proved that the coefficients of this polynomial are symmetric, and recently V.J. Guo and J. Zeng (5) showed that the polynomial $I_n(x)$ is unimodal. In a previous paper (2) the present authors proved that the polynomial $I_n(x)$ is not log-concave. The calculation involved in founding a counterexample to this property is based on a (not bijective) correspondence between involutions on $[n]$ with j descents and generalized involutions on length n on m symbols, with $m > j$. This correspondence yields an explicit formula for the coefficients $i_{n,j}$ of the polynomial $I_n(x)$.

In this paper we study the polynomial $S_n(x) = \sum_{j=0}^{n-1} s_{n,j} x^j$, where $s_{n,j}$ denotes the number of centrosymmetric involutions on $[n]$ with j ascents (and hence $n - 1 - j$ descents).
We say that $\sigma \in S_n$ is a *centrosymmetric permutation* if $\sigma(i)+\sigma(n+1-i) = n+1$ for every $i = 1, \dots, n$. We observe that the group consisting of centrosymmetric permutations in S_{2k} is isomorphic to the hyperoctahedral group B_k, namely, the group of permutations of the set $\{-k, \dots - 1, 1, \dots, k\}$ such that $\sigma(i) = -\sigma(-i)$.

Centrosymmetric involutions appear in many different contexts. For instance, it is well known (see (11) and (7) for more details) that an involution σ is centrosymmetric if and only if the Young tableau P corresponding to σ via the Robinson–Schensted algorithm is fixed under the Schützenberger map. Moreover, the permutation matrix associated with a centrosymmetric involution is both symmetric and centrosymmetric.
The descent statistic on the set of centrosymmetric involutions is equivalent to the statistic $fdes(\sigma)$ introduced in (1) for the hyperoctahedral group.

First of all, we exhibit an explicit formula and a recursive rule for the total number of centrosymmetric involutions on $[n]$. Then, following along the lines of (2), we obtain some enumerative results for the sequence $s_{n,j}$ by exploiting a map that associates a centrosymmetric involution with a suitable set of generalized involutions. In particular, we deduce an explicit formula for the integers $s_{n,j}$, which allows us to prove that the polynomials $I_n(x)$ and $S_n(x)$ share some properties, such as the symmetry of the coefficients and the non log-concavity.

The last section is devoted to the study of the Eulerian distribution on centrosymmetric involutions without fixed points. Also in this case, we find an explicit formula for the number $s^*_{n,j}$ of centrosymmetric involutions on $[n]$ without fixed points and j ascents. The main result of this section is the proof of the symmetry $s^*_{2k,d} = s^*_{2k,2k-d-2}$, based on the proof of the analogous result for all fixed point free involutions due to V.Strehl (13). Since Strehl's proof does not appear in the cited paper, we decided to describe it in full detail at the end of this paper.

2 Preliminary notions

In this section, we give some definitions and general results about involutions and generalized involutions.

We recall that the *descent set* of a permutation σ is defined as $Des(\sigma) = \{1 \leq i < n : \sigma(i) > \sigma(i+1)\}$. An analogous definition can be given for the *ascent set* $Asc(\sigma)$ of a permutation, by replacing "$\sigma(i) > \sigma(i+1)$" with "$\sigma(i) < \sigma(i+1)$".

Given a Ferrers diagram λ, a *(dual) semistandard tableau* of shape λ over the alphabet $[m]$ is an array obtained by placing into each box of the diagram λ an integer in $[m]$ so that the entries are strictly increasing by rows and weakly increasing by columns.

There are many different bijections between pairs of tableaux of the same shape and suitable two-lines arrays, that are based on the classical Robinson–Schensted-Knuth procedure (see (6)). These procedures differ in the definition of the insertion rule and the characterization of the two-line array involved. In the following, we will use the variation described below that concerns pairs of dual semistandard tableaux.

A *generalized permutation* is defined to be a biword:

$$\alpha = \binom{x}{y} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

such that:

- $x_i \leq x_{i+1}$,
- $x_i = x_{i+1} \Longrightarrow y_i \geq y_{i+1}$.

The integer n is called the *length* of the generalized permutation α. The word $x = x_1 \cdots x_n$ is called the *x-content* of α and, similarly, The word $y = y_1 \cdots y_n$ is called the *y-content* of α.

A *generalized involution* will be a generalized permutation α satisfying the further condition that each pair $\binom{a}{b}$ occurs as many times as the pair $\binom{b}{a}$. In this case, the x-content and the y-content of α coincide and we will call each of them the *content* of α.

We associate a generalized permutation with a pair (P, Q) of dual semistandard tableaux as follows. The tableau P is obtained from the empty tableau by inserting the sequence $y_1, \ldots, y_n$ in accordance with the following variation of the row-insertion procedure. Suppose that we inserted the symbols $y_1, \ldots, y_{i-1}$, obtaining the tableau P_{i-1}. Then, we insert y_i in the tableau P_{i-1}, getting the tableau P_i as follows

- if y_i is strictly greater than each symbol in the first row of P_{i-1}, add y_i in a new box to the end of the first row;
- otherwise, find the left-most entry in the first row of P_{i-1} that is larger than or equal to y_i. Put the symbol y_i in the box of this entry, and remove the entry. Take this entry and repeat the process on the second row.

The tableau Q is obtained, as usual, filling the box b_i created at the i-th step with the integer x_i.

The above procedure appears in (3), Section 3 (see also (8), subsect. 4.4). It is easy to check that, when we apply this procedure to a generalized involution, we obtain a pair (P, Q), where $P = Q$.

We recall that the *standardization* map Π associates with a generalized permutation

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

the permutation

$$\Pi(\alpha) = \begin{pmatrix} 1 & 2 & \cdots & n \\ y'_1 & y'_2 & \cdots & y'_n \end{pmatrix},$$

where $y'_i = 1$ if y_i is the least symbol occurring in the word y, $y'_j = 2$ if y_j is the second least symbol in y and so on. In the case $y_i = y_j$, with $i > j$, we consider y_i to be less then y_j.
Note that the polarization of a generalized involution is an involution, since the above given rule for handling the case $y_i = y_j$ is coherent with the rule for the ordering of biletters in a generalized permutation.

For example, the standardization of the generalized involution

$$\alpha = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 4 & 6 \\ 4 & 3 & 2 & 1 & 6 & 4 & 1 & 4 \end{pmatrix}$$

is the involution

$$\Pi(\alpha) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 3 & 2 & 8 & 6 & 1 & 5 \end{pmatrix}.$$

Note that the map Π is not injective, since, for any given involution σ, there are infinitely many generalized involutions whose standardization is σ. For example, the generalized involution

$$\beta = \begin{pmatrix} 1 & 1 & 1 & 3 & 4 & 4 & 5 & 6 \\ 5 & 3 & 1 & 1 & 6 & 4 & 1 & 4 \end{pmatrix}$$

has the same standardization as α in the previous example (i.e., $\Pi(\alpha) = \Pi(\beta)$) .

We will denote by $\text{Gen}_m(\sigma)$ the set of generalized involutions, with symbols taken from $[m]$, whose standardization is σ. Observe that two generalized involutions in $\text{Gen}_m(\sigma)$ can not have the same content. For this reason the set $\text{Gen}_m(\sigma)$ corresponds bijectively with the set of contents of its elements.
We will say that a content x is *compatible with* σ if there exists a generalized involution in $\text{Gen}_m(\sigma)$ whose content is x, for some m.
It is easy to check that a content $x = x_1 \cdots x_n$ is compatible with an involution σ if and only if

$$i \in Asc(\sigma) \Longrightarrow x_i < x_{i+1}.$$

The key tool in the present paper is the interplay between involutions and generalized involutions. For this reason we need to evaluate the cardinality of the set $\text{Gen}_m(\sigma)$, for any given involution σ. It turns out that this cardinality depends only on the number of ascents of σ. In fact, denoting by $\mathscr{I}_n$ the set of involutions in S_n, we have the following result, formerly stated in (2):

Proposition 1 *Let $\sigma \in \mathscr{I}_n$ be an involution with t ascents. Then,*

$$|Gen_m(\sigma)| = \binom{n+m-t-1}{n}. \tag{1}$$

Proof: Choose an involution $\sigma \in \mathscr{I}_n$ with t ascents. As we remarked above, the set $\text{Gen}_m(\sigma)$ corresponds bijectively to the set of contents $x = x_1 \ldots x_n$ with $1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq m$, where the inequalities are strict at places corresponding to the ascents of σ. Every such content is uniquely determined by the sequence $\delta := \delta_0\delta_1 \ldots \delta_n$, with

$$\delta_0 = x_1 - 1, \quad \delta_1 = x_2 - x_1, \quad \ldots, \quad \delta_n = m - x_n$$

which is a (weak) composition of the integer $m-1$ such that its i-th component δ_i is at least one whenever σ has an ascent at the i-th position. For this reason we can consider the word $\delta' = \delta'_0, \delta'_1, \ldots, \delta'_n$ defined as follows:

$$\delta'_i = \begin{cases} \delta_i - 1 & \text{if } \sigma \text{ has an ascent at the i-th position} \\ \delta_i & \text{otherwise} \end{cases},$$

which is a composition of the integer $m - t - 1$ in $n + 1$ parts. This gives the assertion. $\square$

3 Centrosymmetric involutions

We say that a permutation $\sigma \in S_n$ is *centrosymmetric* whenever $\sigma(i) + \sigma(n+1-i) = n+1$ for every $i = 1, \ldots, n$. For instance, the permutation $\sigma = 3\,5\,1\,6\,2\,4$ is centrosymmetric, while $\sigma' = 3\ 4\ 6\ 2\ 1\ 5$ is not centrosymmetric because, for instance, $\sigma'(1) + \sigma'(6) = 8 \neq 7$.

In other terms, a permutation $\sigma \in S_n$ is centrosymmetric if and only if it commutes with the involution $\psi \in S_n$ defined by $\psi(i) = n+1-i$.

When $n = 2k$, the set $\mathscr{C}_{2k}$ of centrosymmetric permutations on $2k$ objects is a subgroup of S_{2k} isomorphic to the hyperoctahedral group B_k via the map $\Theta : \mathscr{C}_{2k} \to B_k$ that associates a permutation $\sigma \in \mathscr{C}_{2k}$ with the signed permutation $\Theta(\sigma)$ defined by:

$$\Theta(\sigma)(i) = \begin{cases} \sigma(k+i) - k & \text{if } \sigma(k+i) > k \\ \sigma(k+i) - k - 1 & \text{otherwise.} \end{cases}$$

For example, if $\sigma = 2\ 1\ 6\ 5\ 4\ 3\ 8\ 7$, then we have $\Theta(\sigma) = -1 -2\ 4\ 3$.

If $n = 2k+1$, for every $\sigma \in \mathscr{C}_{2k+1}$, we must have $\sigma(k+1) = k+1$ by definition. Hence, a permutation in $\mathscr{C}_{2k+1}$ is associated to the unique permutation in $\mathscr{C}_{2k}$ obtained by deleting the central symbol.

We denote by $\mathscr{S}_n$ the subset of $\mathscr{C}_n$ consisting of centrosymmetric involutions. Note that the classical *complementation* and *reversal* maps

$$c : \sigma \mapsto \psi\sigma \qquad r : \sigma \mapsto \sigma\psi$$

coincide when restricted to the set $\mathscr{S}_n$. These two bijections map centrosymmetric involutions to centrosymmetric involutions, while the whole set of involutions in not closed under such maps.

It is well known (see, e.g., (6) and (11)) that centrosymmetric involutions correspond via the Robinson–Schensted bijection to standard Young tableaux that are fixed under the Schützenberger map (evacuation). In fact, this map can be described in terms of involutions as

$$\sigma \mapsto ev(\sigma) = \psi\sigma\psi.$$

We now introduce some notation concerning the cycle decomposition of centrosymmetric involutions. Recall that σ is an involution if and only if its disjoint cycle decomposition consists uniquely of fixed points and transpositions. We will say that the transposition $\tau = (i,j)$ *divides* σ, in symbols $\tau|\,\sigma$, whenever τ appears in the cycle decomposition of σ. We will say that (i,j) is a *smooth transposition* of S_n if $i \neq n+1-j$. In this language, we have:

Proposition 2 *An involution $\sigma \in \mathscr{I}_n$ is centrosymmetric if and only if the following two conditions hold:*

$$\sigma(i) = i \iff \sigma(n+1-i) = n+1-i, \tag{2}$$

$$(i,j)|\,\sigma \iff (n+1-i, n+1-j)|\,\sigma. \tag{3}$$

□

Note that Proposition 2 implies that, whenever a transposition τ divides an involution σ, this forces four values of σ if τ is smooth, and two values otherwise.

Denote by s_n the cardinality of the set $\mathscr{S}_n$. By previous considerations, $s_{2k} = s_{2k+1}$. Hence, we restrict our attention to the even case.
The sequence s_{2k} satisfies the following well known recurrence (see, e.g., (10)):

Theorem 3 *We have:*

$$s_{2k} = 2s_{2k-2} + (2k-2)s_{2k-4} \tag{4}$$

□

The characterization given in Proposition 2 allows us to give an explicit formula for the integers s_{2k}:

Theorem 4 *The number of centrosymmetric involutions on $2k$ symbols is*

$$s_{2k} = \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(2k)!!}{(k-2h)!h!2^{2h}}.$$

Proof: Fix an integer $h \leq \lfloor \frac{k}{2} \rfloor$. We count the number of involutions in $\mathscr{S}_{2k}$ with exactly $2h$ smooth transpositions. Choose a word $w = w_1 \cdots w_k$ consisting of k different letters taken from the alphabet $[2k]$ such that w does not contain simultaneously an integers i and its complement $2k+1-i$. We have $(2k)(2k-2)\cdots(2) = (2k)!!$ choices for such a word. This word corresponds to a unique centrosymmetric involution τ with $2h$ smooth transpositions defined by the following conditions:

$$\tau(w_1) = w_2, \quad \ldots \quad , \tau(w_{2h-1}) = w_{2h};$$

$$\tau(w_{2h+j}) = \begin{cases} w_{2h+j} & \text{if } w_{2h+j} \leq k \\ 2k+1-w_{2h+j} & \text{otherwise} \end{cases},$$

with $0 < j \leq k-2h$. It is easily checked that the involution τ arises from $(k-2h)!h!2^h2^h$ different words w. This completes the proof. □

4 Centrosymmetric generalized involutions

We recall that the Schützenberger map can be extended to semistandard tableaux (see, for instance, (12)). In terms of generalized involution, this map can be described as follows: the image of the generalized involution

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

over the alphabet $[m]$ is the generalized involution

$$ev(\alpha) = \begin{pmatrix} m+1-x_n & m+1-x_{n-1} & \cdots & m+1-x_1 \\ m+1-y_n & m+1-y_{n-1} & \cdots & m+1-y_1 \end{pmatrix}.$$

A generalized involution α is said to be *centrosymmetric* whenever $\alpha = ev(\alpha)$, namely,

$$x_i + x_{n+1-i} = y_i + y_{n+1-i} = m + 1$$

for every $i = 1 \ldots n$.

From now on, extending the previous notation, we will write $\binom{i}{j}|\alpha$ with multiplicity s whenever the pair $\binom{i}{j}$ appears exactly s times in the generalized involution α. Also in this case, we will say that $\binom{i}{j}$ is a smooth transposition if $i \neq m + 1 - j$ and $i \neq j$. Obviously, if $\binom{i}{j}$ is a smooth transposition of the generalized involution α, also $\binom{j}{i}$ is a smooth transposition of α (with the same multiplicity). Moreover:

Proposition 5 *A generalized involution α is centrosymmetric if and only if, whenever $\binom{i}{j}|\alpha$ with multiplicity s, we have also $\binom{m+1-j}{m+1-i}|\alpha$ with the same multiplicity.*

□

Proposition 5 yields a further characterization of centrosymmetric generalized involutions, which will be useful in the following sections.

Proposition 6 *A generalized involution α is centrosymmetric if and only if it satisfies the following properties:*

- *the content $x = x_1 \ldots x_n$ of α is symmetric, namely $x_i + x_{n+1-i} = m + 1$,*
- *the standardization $\Pi(\alpha)$ of α is a centrosymmetric involution.*

□

We denote by $c_{n,m}$ the number of generalized involutions of length n over the alphabet $[m]$.

Setting $n = 2k + 1$, straightforward considerations lead to the following properties:

- if $m = 2h$, $c_{2k+1,m} = 0$;
- if $m = 2h + 1$, the central pair $\binom{x_{k+1}}{y_{k+1}}$ of every centrosymmetric generalized involution of length n over the alphabet $[m]$ is necessarily the pair $\binom{h+1}{h+1}$. This implies that $c_{2k+1,m} = c_{2k,m}$.

Hence, the values of the sequences $c_{2k+1,m}$ can derived from the sequences $c_{2k,m}$. For this reason we restrict our considerations to the even case.

Theorem 7 *The number of centrosymmetric generalized involutions of length $2k$ over $[m]$ is:*

$$c_{2k,m} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{m}{2} - \lfloor \frac{m}{2} \rfloor}{2} + j - 1}{j} \binom{m + k - 2j - 1}{k - 2j}. \tag{5}$$

Proof: Fix $h \leq \lfloor \frac{k}{2} \rfloor$. We count centrosymmetric generalized involutions of length $2k$ over $[m]$ symbols with exactly $2h$ smooth transpositions which, in the present case, can or can not be different. The set A of all possible smooth transposition has cardinality

$$\binom{m}{2} - \left\lfloor \frac{m}{2} \right\rfloor.$$

Observe that, given a generalized involution α and a smooth transposition $\tau = \binom{i}{j}$, we have that $\tau | \alpha$ with multiplicity s if and only if $\tau' | \alpha$ with the same multiplicity, where $\tau' = \binom{m+1-j}{m+1-i}$. It is evident that τ can be chosen in

$$\frac{\binom{m}{2} - \lfloor \frac{m}{2} \rfloor}{2}$$

ways. Every such choice determines four pairs of α. The remaining $2k - 4h$ pairs can be chosen to be either fixed points or non-smooth transpositions. This completes the proof. □

Hence, the column generating function of the array $c_{2k,m}$ is

$$\sum_{k \geq 0} c_{2k,m} x^k = \frac{1}{(1-x)^m (1-x^2)^{\frac{\binom{m}{2} - \lfloor \frac{m}{2} \rfloor}{2}}}.$$

Note that the expression of this generating function is closely similar to the classical expression of the analogous generating function of generalized involutions $a_{n,m}$ or, equivalently, of semistandard tableaux (see (9)):

$$\sum_{n \geq 0} a_{n,m} x^n = \frac{1}{(1-x)^m (1-x^2)^{\binom{m}{2}}}.$$

5 The Eulerian distribution on centrosymmetric involutions

We now study the distribution of the ascent statistic on the set of centrosymmetric involutions. The combinatorial relations between involutions and generalized involutions pointed out in the previous section will play a crucial role for this analysis.

The distribution of the ascent statistic on the set of involutions behaves properly with respect to the action of the Schützenberger map. In fact:

Proposition 8 *For every involution $\sigma \in \mathscr{I}_{2k}$, we have:*

$$|Asc(\sigma)| = |Asc(ev(\sigma))|.$$

Moreover, the ascent sets $Asc(\sigma)$ and $Asc(ev(\sigma))$ are mirror symmetric, i.e. σ has an ascent at position i if and only if $ev(\sigma)$ has an ascent at position $2k - i$.

Proof: Suppose that σ has an ascent at position i, namely, $\sigma(i) < \sigma(i+1)$. Then,

$$ev(\sigma)(2k-i) = ev(\sigma)(2k+1-(i+1)) = 2k+1-\sigma(i+1) <$$

$$< 2k+1-\sigma(i) = ev(\sigma)(2k+1-i),$$

as desired. □

For example, let

$$\sigma = \begin{pmatrix} \mathbf{1} & \mathbf{2} & 3 & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 \\ \mathbf{5} & \mathbf{7} & 8 & 6 & \mathbf{1} & 4 & \mathbf{2} & 3 \end{pmatrix},$$

where, from now on, the bold-faced numbers denote the ascent positions. Then,

$$ev(\sigma) = \begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & 5 & \mathbf{6} & \mathbf{7} & 8 \\ \mathbf{6} & 7 & \mathbf{5} & 8 & 3 & \mathbf{1} & \mathbf{2} & 4 \end{pmatrix}.$$

The sets $Asc(\sigma) = \{1, 2, 5, 7\}$ and $Asc(ev(\sigma)) = \{1, 3, 6, 7\}$ are mirror symmetric (as defined before).

In particular, if σ is a centrosymmetric involution on $2k$ objects, then its ascent set must be mirror symmetric with respect to the k-th entry.

We are now interested in finding an explicit formula for the number $s_{2k,d}$ of centrosymmetric involutions with d ascents. First of all, we have:

Proposition 9 *The sequence $s_{2k,d}$ is symmetric, namely,*

$$s_{2k,i} = s_{2k,2k-1-i}.$$

Proof: Given a centrosymmetric involution σ, it is easily checked that the permutation $\psi\sigma$ satisfies the following properties:

- $\psi\sigma$ is an involution;
- $\psi\sigma$ is centrosymmetric;
- $\psi\sigma$ has a descent at position i whenever σ has an ascent at the same position.

Hence, the function $\sigma \mapsto \psi\sigma$ maps an involution with i ascents into an involution with $2k - 1 - i$ ascents.

□

For example, let

$$\sigma = \begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} & 7 & 8 \\ 1 & \mathbf{7} & 5 & \mathbf{6} & 3 & \mathbf{4} & 2 & 8 \end{pmatrix}.$$

Then,

$$\psi\sigma = \begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 \\ \mathbf{8} & 2 & \mathbf{4} & 3 & \mathbf{6} & 5 & \mathbf{7} & 1 \end{pmatrix}.$$

The preceding result shows that the integer $s_{2k,d}$ counts simultaneously the involutions in $\mathscr{S}_{2k}$ with d descents and those with d ascents.

Now we want to express the number $c_{2k,m}$ of centrosymmetric generalized involutions of length $2k$ over $[m]$ in terms of the sequence $s_{2k,d}$ by exploiting the combinatorial relations between involutions and generalized involutions. As in the general case (Proposition 1), it turns out that the number of centrosymmetric generalized involutions on m symbols whose standardization is a given involution σ, namely, belonging to the set $Gen_m(\sigma)$, depends only on the number of ascents of σ. In fact:

Theorem 10 *We have:*

$$c_{2k,m} = \sum_{j=0}^{m-1} \binom{k + \lfloor \frac{j}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} s_{2k,m-1-j}. \tag{6}$$

Proof: Let $\sigma \in \mathscr{I}_{2k}$ be a centrosymmetric involution with t ascents. We want to determine the number of centrosymmetric elements in the set $Gen_m(\sigma)$. Combining Proposition 6 and the arguments used in the proof of Proposition 1, we deduce that such elements correspond bijectively to those compositions δ' of the integer $m-1-t$ into $2k+1$ parts which are centrosymmetric, namely, such that $\delta'_{k-i} = \delta'_{k+i}$. Each one of these compositions δ' is determined as soon as we:

- choose an integer $h \leq \lfloor \frac{m-1-t}{2} \rfloor$,
- choose a composition $\eta = \eta_0 \eta_2 \dots \eta_{k-1}$ of h into k parts;

The composition $\delta' = \delta'_0, \delta'_1, \dots, \delta'_{2k}$ is now obtained as follows:

$$\delta' = \eta_0, \eta_1, \dots, \eta_{k-1}, \delta'_k, \eta_{k-1}, \dots, \eta_1 \eta_0,$$

where $\delta'_{k+1} = m-1-t-2h$.

Hence, the number of such compositions is

$$\sum_{h=0}^{\lfloor \frac{m-1-t}{2} \rfloor} \binom{h+k-1}{k-1} = \binom{\lfloor \frac{m-1-t}{2} \rfloor + k}{k} = \binom{\lfloor \frac{m-1-t}{2} \rfloor + k}{\lfloor \frac{m-1-t}{2} \rfloor}.$$

By setting $j = m-1-t$, we get the assertion. $\square$

Identity (6) yields the following expression for the row generating function of the array $c_{2k,m}$:

$$\sum_{m \geq 1} c_{2k,m} x^m = \sum_{m \geq 1} \sum_{j=0}^{m-1} \binom{k + \lfloor \frac{j}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} s_{2k,m-1-j} x^m = \frac{x+x^2}{(1-x^2)^{k+1}} \sum_{j \geq 0} s_{2k,j} x^j.$$

We now exploit the described combinatorial relation between involutions and generalized involutions to determine an explicit formula for $s_{2k,d}$.

Theorem 11 *The number of centrosymmetric involutions of length* $2k$ *with* d *ascents is:*

$$s_{2k,d} = \sum_{j=1}^{d+1} (-1)^{\lfloor \frac{d-j}{2} + 1 \rfloor} \binom{k}{\lfloor \frac{d+1-j}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{j}{2} - \lfloor \frac{j}{2} \rfloor}{2} + i - 1}{i} \binom{j+k-2i-1}{k-2i}. \tag{7}$$

Proof: Formula (6) shows that, for every fixed integer k, the vector $(c_{2k,1},\dots,c_{2k,2k})$ can be expressed as the product of a lower triangular matrix $M(k)$ and the vector $(s_{2k,0},\dots,s_{2k,2k-1})$, where

$$M(k)_{ij}=\binom{k+\lfloor\frac{i-j}{2}\rfloor}{\lfloor\frac{i-j}{2}\rfloor}.$$

for $i\geq j$. The matrix $M(k)$ is invertible, since it is lower triangular with unitary diagonal. The matrix $M(k)^{-1}$ can be easily computed and used to express the vector $(s_{2k,0},\dots,s_{2k,2k-1})$ in terms of $(c_{2k,1},\dots,c_{2k,2k})$ as follows:

$$s_{2k,d}=\sum_{j=1}^{d+1}(-1)^{\lfloor\frac{d-j}{2}+1\rfloor}\binom{k}{\lfloor\frac{d+1-j}{2}\rfloor}c_{2k,j}. \tag{8}$$

Combining Formulae (5) and (8), we derive (7). □

Formula (7) allows to check that the polynomials $S_{2k}(x)=\sum_{j=0}^{2k-1}s_{2k,j}x^j$ are not, in general, log-concave, since we have, for example:

$$s_{6,0}\cdot s_{6,2}=37>36=s_{6,1}^2.$$

In the case of centrosymmetric involutions on an odd number of symbols, the Eulerian distribution can be computed as follows:

Proposition 12

$$s_{2k+1,d}=\begin{cases}1 & \text{if } d=0\\ 0 & \text{if } d \text{ is odd}\\ s_{2k,d-1}+s_{2k,d} & \text{otherwise}\end{cases}$$

Proof: We recall that an involution σ in $\mathscr{S}_{2k+1}$ is obtained from a unique involution σ' in $\mathscr{S}_{2k}$ by adding the fixed point $k+1$ and renormalizing the remaining symbols. In doing this, either the central ascent of σ' is changed into two ascents of σ if σ' has an odd number of ascents, or the number of ascents remains unchanged. □

The first values of $s_{n,d}$ are shown in the following table:

n/d	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	0	1							
4	1	2	2	1						
5	1	0	4	0	1					
6	1	3	6	6	3	1				
7	1	0	9	0	9	0	1			
8	1	4	13	20	20	13	4	1		
9	1	0	17	0	40	0	17	0	1	
10	1	5	23	49	78	78	49	23	5	1

These first values lead to the belief that the polynomials $S_{2k}(x)$ are unimodal for every $k \in \mathbb{N}$. It would be interesting to find a combinatorial proof of this property.

6 Centrosymmetric involutions without fixed points

In this section we extend the study of the Eulerian distribution to the set of centrosymmetric involutions on $[n]$ without fixed points. Obviously, such involutions exist only if n is even.
Denote by $\mathscr{S}^*_{2k}$ the set of centrosymmetric involutions on $2k$ objects without fixed points and by s^*_{2k} the cardinality of $\mathscr{S}^*_{2k}$. Then:

Theorem 13 *We have:*

$$s^*_{2k} = \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2h)!h!}, \tag{9}$$

and

$$s^*_{2k} = s^*_{2k-2} + (2n-2)s^*_{2k-4}. \tag{10}$$

Proof: We count centrosymmetric involutions without fixed points with exactly $2h$ smooth transpositions, $2h \leq k$. Choose a word $w = w_1 \cdots w_k$ consisting of k different letters taken from the alphabet $1, \ldots, 2k$ such that w does not contain simultaneously the integers i and $2k+1-i$. We have $(2k)!!$ choices for such a word. This word corresponds to a unique centrosymmetric involution τ without fixed points with $2h$ smooth transpositions defined by the following conditions:

$$\tau(w_1) = w_2, \quad \ldots, \quad \tau(w_{2h-1}) = w_{2h},$$

$$\tau(w_{2h+j}) = 2k+1-w_{2h+j}, \quad \text{for } 0 < j \leq k-2h.$$

It is easily checked that the involution τ arises from $(k-2h)!h!2^h 2^h 2^{k-2h}$. This gives Formula (9).

Let now $\sigma \in \mathscr{S}^*_{2k}$. If $\sigma(1) = 2k$, and hence $\sigma(2k) = 1$, the restriction of σ to the set $\{2, \ldots, 2k-1\}$ is a centrosymmetric involution on $2k-2$ symbols without fixed points. If $\sigma(1) = j$, with $j < 2k$, the symbol 1 is involved in a smooth transposition, hence we must have $\sigma(j) = 1$, $\sigma(2k+1-j) = 2k$ and $\sigma(2k) = 2k+1-j$. Then, the restriction of σ to the set $\{2, \ldots, 2k-1\} \setminus \{j, 2k+1-j\}$ is a centrosymmetric involution on $2k-4$ symbols without fixed points. Observing that there are $2k-2$ possible choices for the integer j, we get (10). □

We point out that the proof of Theorems 13 suggests constructive rules for generating centrosymmetric involutions without fixed points.

Denote by $s^*_{2k,d}$ the number of involutions in $\mathscr{S}^*_{2k}$ with d ascents. Once more, in order to find an explicit formula for the integers $s^*_{2k,d}$, we need to set up a connection between centrosymmetric involutions without fixed points and a suitable set of generalized involutions.

First of all, let α be a generalized involution. We say that the integer a is a *fixed point of multiplicity r* if

$$x_i = y_i = x_{i+1} = y_{i+1} = \cdots = x_{i+r-1} = y_{i+r-1} = a.$$

It is easy to see that each fixed point of the standardization $\Pi(\alpha)$ corresponds to a fixed point a of α of odd multiplicity. Hence we must consider the set of centrosymmetric generalized involutions with fixed points of even multiplicity. Denote by $c^*_{2k,m}$ the number of such involutions of length $2k$ over the alphabet $[m]$. Then:

Theorem 14 *We have:*

$$c^*_{2k,m} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{m}{2}+\lfloor \frac{m}{2} \rfloor}{2} + j - 1}{j} \binom{\lceil \frac{m}{2} \rceil + k - 2j - 1}{k - 2j}. \qquad (11)$$

Proof: Fix $j \leq \lfloor \frac{k}{2} \rfloor$. We count the number of centrosymmetric generalized involutions of length $2k$ on the alphabet $[m]$ containing only fixed points of even multiplicity, such that exactly $4j$ of its pairs are either smooth transpositions or non central fixed points. A *non central fixed point* is an occurrence of a pair $\binom{i}{i}$ in α, with $i \neq \frac{m+1}{2}$. Observe that, if $\binom{i}{h}$ is a smooth transposition of the generalized involution α, then α must contain the four pairs $\binom{i}{h}$, $\binom{h}{i}$, $\binom{m+1-h}{m+1-i}$, and $\binom{m+1-i}{m+1-h}$. Moreover, if $\binom{i}{i}$ is a non central fixed point of α, then α must contain two occurrences of the pair $\binom{i}{i}$ (in order to be fixed point free), and two occurrences of the pair $\binom{m+1-i}{m+1-i}$. Hence, if we want α to contain exactly $4j$ among smooth transpositions and non central fixed points, it is sufficient to choose j (not necessarily distinct) pairs $\binom{i}{h}$ with $i \leq h$ and $i < \frac{m+1}{2}$. There are

$$\frac{\binom{m}{2} + \lfloor \frac{m}{2} \rfloor}{2}$$

such pairs.
The remaining pairs must be chosen to be either central fixed points or non smooth transpositions. This completes the proof. □

Repeating the same arguments as in the proofs of Theorems 10 and 11, we obtain the following result:

Theorem 15 *We have:*

$$c^*_{2k,m} = \sum_{j=0}^{m-1} \binom{k + \lfloor \frac{j}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} s^*_{2k,m-1-j}. \qquad (12)$$

Hence:

$$s^*_{2k,d} = \sum_{j=1}^{d+1} (-1)^{\lfloor \frac{d-j}{2} + 1 \rfloor} \binom{k}{\lfloor \frac{d+1-j}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{j}{2}+\lfloor \frac{j}{2} \rfloor}{2} + i - 1}{i} \binom{\lceil \frac{j}{2} \rceil + k - 2i - 1}{k - 2i}. \qquad (13)$$

□

The next table contains the first values of the sequences $s^*_{2k,d}$:

$2k/d$	0	1	2	3	4	5	6	7	8	9
0	1									
2	1									
4	1	1	1							
6	1	1	3	1	1					
8	1	2	7	5	7	2	1			
10	1	2	12	12	27	12	12	2	1	

This table shows that the polynomial $S^*_{2k}(x)$ is not in general unimodal, and hence not log-concave. These values suggest that the Eulerian distribution on centrosymmetric involutions without fixed points has the following symmetry:

$$s^*_{2k,d} = s^*_{2k,2k-2-d}. \tag{14}$$

The analogous symmetry of the Eulerian distribution on the whole set of centrosymmetric involutions has been proved exploiting the map $\sigma \mapsto \psi\sigma$ (see Proposition 9). In this case, this approach fails, since this map does not preserve fixed point freeness. The last part of this section is devoted to the proof of such symmetry.

Denote by I^*_{2k} the set of fixed point free involutions in S_{2k} and by $i^*_{2k,d}$ the number of fixed point free involutions in S_{2k} with d ascents. The symmetry of the ascent distribution on involutions without fixed point

$$i^*_{2k,d} = i^*_{2k,2k-d-2}. \tag{15}$$

is a well known result due to Strehl (13). In fact, (13) is the summary of a talk given at the first session of Séminaire Lotharingien de Combinatoire and does not contain any proof. For this reason, we describe in full detail Strehl's argumentations, communicated to us by the author.
Strehl's proof of Identity (15) is based on a bijection θ that maps an involution in I^*_{2k} with d ascents into an involution in I^*_{2k} with $2k-d-2$ ascents. Identity (14) will be proved as soon as we show that the bijection θ preserves centrosymmetry.

In order to define the map θ we need some preliminaries. We recall that a permutation σ has an *excedence* at position i whenever $\sigma(i) > i$. Denote by R^σ the set of all excendences and by C^σ the set of all non excedences of σ. Of course, if $\sigma \in I^*_{2k}$, both these two sets have cardinality k.
We associate with a given $\sigma \in I^*_{2k}$ a path $\mathscr{D}_\sigma$ defined as follows: the i-th step of $\mathscr{D}_\sigma$ is an up-step if $i \in R^\sigma$, a down-step otherwise. Since $\sigma \in I^*_{2k}$, such a path is easily seen to be a Dyck path. Observe that the map $\sigma \mapsto \mathscr{D}_\sigma$ is not injective. We denote by $I^{*\mathscr{D}}_{2k}$ the set of all involutions in I^*_{2k} with a given associated Dyck path $\mathscr{D}$. The Dyck path $\mathscr{D}$ can be written as:

$$\mathscr{D} = U^{r_1}D^{c_1}U^{r_2}D^{c_2}\dots U^{r_p}D^{c_p}.$$

The set of up-steps U^{r_i} will be called the *i-th rise* of the Dyck path $\mathscr{D}$ and, similarly, the set of down-steps D^{c_j} the *j-th fall* of $\mathscr{D}$. We associate with $\mathscr{D}$ two families of sets $R_1,\dots,R_p,C_1,\dots,C_p$, as follows: the integer x belongs to R_i (respectively C_j) whenever the x-th step of $\mathscr{D}$ belongs to the i-th rise (resp. j-th fall) of $\mathscr{D}$. Obviously, we have:

$$R = \bigcup_{1\le i\le p} R_i \qquad C = \bigcup_{1\le j\le p} C_j,$$

$$|R_i| = r_i \qquad |C_j| = c_j.$$

If $\sigma \in I^{*\mathscr{D}}_{2k}$, we denote by ρ^σ_i the restriction of the permutation σ to the set R_i. Clearly, we have:

$$\rho^\sigma_i : R_i \to \bigcup_{i\le j\le p} C_j.$$

Similarly, we denote by γ_j^σ the restriction of the permutation σ to the set C_j:

$$\gamma_j^\sigma : C_j \to \bigcup_{1\leq i\leq j} R_i.$$

Note that the involution σ is completely determined by the Dyck path $\mathscr{D}$ and the maps ρ_i^σ. In the following, if σ is clear from the context, we will omit the symbol σ in the previous notation.

We consider the further (possibly empty) sets

$$R_{ij} = \sigma(C_j) \cap R_i,$$

$$C_{ij} = \sigma(R_i) \cap C_j,$$

and denote by

$$\rho_{ij} : R_{ij} \to C_{ij},$$

$$\gamma_{ij} : C_{ij} \to R_{ij},$$

the restrictions of the maps ρ_i and γ_j to the domains indicated. Note that these maps are bijections.

For example, consider the involution in I_{16}^*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 11 & 5 & 9 & 10 & 2 & 8 & 16 & 6 & 3 & 4 & 1 & 13 & 12 & 15 & 14 & 7 \end{pmatrix}$$

The Dyck path associated with σ is the path in Figure 1, and we have:

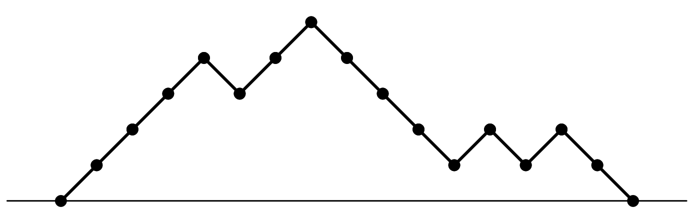

Fig. 1: The Dyck path associated with σ.

$$R = \{1, 2, 3, 4, 6, 7, 12, 14\} \qquad C = \{5, 8, 9, 10, 11, 13, 15, 16\},$$

$$R_1 = \{1, 2, 3, 4\} \quad R_2 = \{6, 7\} \quad R_3 = \{12\} \quad R_4 = \{14\},$$

$$C_1 = \{5\} \quad C_2 = \{8, 9, 10, 11\} \quad C_3 = \{13\} \quad C_4 = \{15, 16\},$$

and the non empty R_{ij} and C_{ij} associated with σ are:

$$R_{1,1} = \{2\} \quad R_{1,2} = \{1, 3, 4\} \quad R_{2,2} = \{6\} \quad R_{2,4} = \{7\} \quad R_{3,3} = \{12\} \quad R_{4,4} = \{14\},$$

$$C_{1,1} = \{5\} \quad C_{1,2} = \{9, 10, 11\} \quad C_{2,2} = \{8\} \quad C_{2,4} = \{16\} \quad C_{3,3} = \{13\} \quad C_{4,4} = \{15\}.$$

Now define three bijections $\theta_1, \theta_2, \theta_3 : I^*_{2k} \to I^*_{2k}$, as follows:

- if $\sigma \in I^*_{2k}$, its image $\theta_1(\sigma)$ is the unique element of I^*_{2k} associated with the Dyck path $\mathscr{D}_\sigma$ and such that:
$$\rho_i^{\theta_1(\sigma)} = \rho_i^\sigma \circ \mathrm{rev}_{R_i},$$
where, given a linearly ordered set $A = \{a_1, \dots, a_n\}$, $\mathrm{rev}_A : A \to A$ is the correspondence that maps the element a_s into a_{n+1-s}.
- if $\sigma \in I^*_{2k}$, its image $\theta_2(\sigma)$ is the unique element of I^*_{2k} associated with the Dyck path $\mathscr{D}_\sigma$ and such that:
$$\gamma_j^{\theta_2(\sigma)} = \gamma_j^\sigma \circ \mathrm{rev}_{C_j},$$
- the definition of the map θ_3 is more complicated than the previous two and requires some preliminary notions.

  Given a permutation τ corresponding via the Robinson–Schensted algorithm to the pair (P, Q) of standard Young tableaux, denote by $\delta(\tau)$ the unique permutation associated with the pair of transposed tableaux $(P^{\mathrm{T}}, Q^{\mathrm{T}})$. The map $\tau \mapsto \delta(\tau)$ will be called the *transposition map*.

  For every $\sigma \in I^*_{2k}$, its image $\theta_3(\sigma)$ is the unique element of I^*_{2k} such that:
$$R_{ij}^{\theta_3(\sigma)} = R_{ij}^\sigma,$$
$$C_{ij}^{\theta_3(\sigma)} = C_{ij}^\sigma,$$
$$\rho_{ij}^{\theta_3(\sigma)} = \delta(\rho_{ij}^\sigma),$$
  for every $1 \le i, j \le p$, where p is the number of rises of the Dyck path $\mathscr{D}_\sigma$.

Loosely speaking, θ_1 acts by reversing the maps ρ_i, θ_2 acts by reversing the maps γ_j, and θ_3 acts by applying the δ-procedure to all the "local" maps ρ_{ij}.
It is immediately checked that each one of these bijections maps the set I^*_{2k} into itself.

We can also describe the maps θ_1 and θ_2 in algebraic language as follows: given $\sigma \in I^*_{2k}$, for every set R_i we define the *reversal map* α_i to be the permutation in S_{2k} that fixes all the integers $x \notin R_i$ and reverses the symbols appearing in R_i. Similarly, for every set C_j we denote by β_j the permutation that fixes all the integers $x \notin C_j$ and reverses the symbols appearing in C_j. Define α and β to be the compositions of the permutations α_i and β_j, respectively. The following commutation properties of the reversal maps can be easily verified:
$$\alpha_i\alpha_h = \alpha_h\alpha_i, \quad \beta_j\beta_l = \beta_l\beta_j, \quad \alpha_i\beta_j = \beta_j\alpha_i.$$
Hence, we do not need to specify the order of such permutations in the compositions α and β. With this notation, we can readily deduce that
$$\theta_1(\sigma) = \alpha\sigma\alpha, \tag{16}$$

$$\theta_2(\sigma) = \beta\sigma\beta. \tag{17}$$

Finally, we define the bijection θ to be the composition

$$\theta = \theta_3 \circ \theta_2 \circ \theta_1.$$

We are now in position to state Strehl's theorem:

Theorem 16 *The bijection θ maps involutions in I_{2k}^* with t ascents into involutions with $2k - 2 - t$ ascents.*

Proof: First of all, a given ascent of σ at position $x \in R_i$ (respectively $x \in C_j$) will be called

- *small ascent* if $x+1 \in R_i$ (resp. $x+1 \in C_j$) and and there exists h such that $\sigma(x), \sigma(x+1) \in C_h$ (resp. $\sigma(x), \sigma(x+1) \in R_h$).
- *large ascent*, otherwise.

The notion of small descent and large descent are defined analogously.

The maps θ_1, θ_2 and θ_3 act on the ascents of σ as follows:

[θ_1]

- If $R_i = \{z, z+1, \ldots, z+r_i-1\}$ and $z+x$ is an ascent (descent) of ρ_i^σ, then $z+r_i-2-x$ is a descent (ascent) in $\theta_1(\sigma)$.
- If $y \in C_j$ is a large ascent (descent) in γ_j^σ, then y is a large ascent (descent) in $\theta_1(\sigma)$.
- If $y \in C_j$ is a small ascent (descent) in γ_j^σ, then y is a small descent (ascent) in $\theta_1(\sigma)$;

[θ_2]

- If $C_j = \{z, z+1, \ldots, z+c_j-1\}$ and $z+x$ is an ascent (descent) of γ_j^σ, then $z+r_j-2-x$ is a descent (ascent) in $\theta_2(\sigma)$.
- If $y \in R_i$ is a large ascent (descent) in ρ_i^σ, then y is a large ascent (descent) in $\theta_2(\sigma)$.
- If $y \in R_i$ is a small ascent (descent) in ρ_i^σ, then y is a small descent (ascent) in $\theta_2(\sigma)$;

[θ_3]

- Large ascents and descents of ρ_i^σ and γ_j^σ remain unaffected by the map θ_3.
- The properties of the Robinson–Schensted algorithm imply that, if $x \in R_i$ is a small ascent (descent) in ρ_i^σ, then x is a small descent (ascent) in $\theta_3(\sigma)$.
- Similarly, if $y \in C_j$ is a small ascent (descent) in γ_j^σ, then y is a small descent (ascent) in $\theta_3(\sigma)$.

Combining now the previous remarks, we deduce that:

- if $R_i = \{z, z+1, \ldots, z+r_i-1\}$ and $z+x$ is an ascent (descent) of ρ_i^σ, then $z+r_i-2-x$ is a descent (ascent) in $\theta(\sigma)$.
- if $C_j = \{z, z+1, \ldots, z+c_j-1\}$ and $z+x$ is an ascent (descent) of γ_j^σ, then $z+r_j-2-x$ is a descent (ascent) in $\theta(\sigma)$.
- ascents and descents of σ that are not ascents or descents of any ρ_i^σ or γ_j^σ remain unaffected by the map θ.

In conclusion, consider an involution $\sigma \in I_{2k}^*$ with d ascents. Observe that $p-1$ of these ascents correspond to positions x such that $x \in C_j$ and $x+1 \in R_{j+1}$ and are unaffected by θ. Similarly, the p descents corresponding to positions y such that $y \in R_i$ and $y+1 \in C_i$ remain unchanged. We now focus on the remaining $(2k-1)-(2p-1) = 2k-2p$ positions. The involution σ has $d-(p-1)$ ascents within these positions, and hence $(2k-2p)-(d-(p-1)) = 2k-p-d-1$ descents. Previous argumentations show that these descents are transformed into ascents of $\theta(\sigma)$ in suitable positions. This implies that the involution $\theta(\sigma)$ has

$$(2k-p-d-1)+(p-1) = 2k-d-2$$

ascents, as required. □

For example, if σ is the involution in I_{16}^* of the previous example, then:

$$\theta_1(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 10 & 9 & 5 & 11 & 3 & 16 & 8 & 7 & 2 & 1 & 4 & 13 & 12 & 15 & 14 & 6 \end{pmatrix}$$

$$\pi = \theta_2(\theta_1(\sigma)) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 9 & 10 & 5 & 8 & 3 & 15 & 11 & 4 & 1 & 2 & 7 & 13 & 12 & 16 & 6 & 14 \end{pmatrix}$$

Then, we have:

$$R_{1,1}^\pi = \{3\} \quad R_{1,2}^\pi = \{1,2,4\} \quad R_{2,2}^\pi = \{7\} \quad R_{2,4}^\pi = \{6\} \quad R_{3,3}^\pi = \{12\} \quad R_{4,4}^\pi = \{14\},$$

$$C_{1,1}^\pi = \{5\} \quad C_{1,2}^\pi = \{8,9,10\} \quad C_{2,2}^\pi = \{11\} \quad C_{2,4}^\pi = \{15\} \quad C_{3,3}^\pi = \{13\} \quad C_{4,4}^\pi = \{16\}.$$

The map θ_3 leaves unchanged every R_{ij} except $R_{1,2}$:

$$\theta(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 10 & 8 & 5 & 9 & 3 & 15 & 11 & 2 & 4 & 1 & 7 & 13 & 12 & 16 & 6 & 14 \end{pmatrix}.$$

Observe that, since σ has 7 ascents, then $\theta(\sigma)$ has $16-2-7=7$ ascents, as expected.

We now switch to the centrosymmetric case. The following properties can be readily verified:

Proposition 17 *If σ is a centrosymmetric involution in I_{2k}^*, then:*

- *for every $1 \leq i \leq p$, if $R_i = \{x_1, \ldots, x_{r_i}\}$, then $C_{p+1-i} = \{2k+1-x_{r_i}, \ldots, 2k+1-x_1\}$ and, hence, $c_{p+1-i} = r_i$,*
- *for every $1 \leq i, j \leq p$, if $R_{ij} = \{x_1, \ldots, x_h\}$, then $C_{p+1-i,p+1-j} = \{2k+1-x_h, \ldots, 2k+1-x_1\}$,*

- *the Dyck path associated with σ is symmetric.*

□

The maps $\theta_2 \circ \theta_1$ and θ_3 preserve centrosymmetry. In fact:

Lemma 18 *If $\sigma \in I^*_{2k}$ is centrosymmetric, then $(\theta_2 \circ \theta_1)(\sigma)$ is centrosymmetric.*

Proof: Consider a centrosymmetric element $\sigma \in I^*_{2k}$. Proposition 17 implies that the reversal maps α_i and β_j associated with σ are related as follows:

$$\psi\beta_j\psi = \alpha_{p+1-j},$$

where $\psi \in S_n$ is defined by $\psi(i) = n + 1 - i$, and hence

$$\psi\beta\psi = \alpha.$$

Then, recalling Identities (16) and (17), we have:

$$\psi((\theta_2 \circ \theta_1)(\sigma))\psi = \psi\beta\alpha\sigma\alpha\beta\psi =$$

$$= (\psi\beta\psi)(\psi\alpha\psi)(\psi\sigma\psi)(\psi\alpha\psi)(\psi\beta\psi) =$$

$$\alpha\beta\sigma\beta\alpha = (\theta_2 \circ \theta_1)(\sigma).$$

Hence, the permutation $(\theta_2 \circ \theta_1)(\sigma)$ is centrosymmetric. □

Lemma 19 *If $\sigma \in I^*_{2k}$ is centrosymmetric, then $\theta_3(\sigma)$ is centrosymmetric.*

Proof: Note that the map θ_3 consist in applying the transposition map δ defined above to each subset R_{ij}, and adjusting the entries in the corresponding C_{ij} in order to get an involution. We denote by δ_{ij} the transposition map δ referred to the set R_{ij}, and by $\hat{\delta}_{ij}$ the unique permutation in S_{2k} such that

$$\delta_{ij}(\sigma) = \hat{\delta}_{ij}\sigma.$$

Clearly, the permutation $\hat{\delta}_{ij}$ fixes all the elements of the interval $[2k]$ belonging neither to R_{ij} nor to C_{ij}. We denote by $\hat{\delta}$ the composition of such permutations (we do not need to specify the order in the composition, since the permutations $\hat{\delta}_{ij}$ commute). We have:

$$\theta_3(\sigma) = \hat{\delta}\sigma.$$

Let now σ be a centrosymmetric element of I^*_{2k}. Then

$$\psi\theta_3(\sigma)\psi = \psi\hat{\delta}\sigma\psi =$$

$$= (\psi\hat{\delta}\psi)(\psi\sigma\psi) = (\psi\hat{\delta}\psi)\sigma.$$

Since σ is centrosymmetric, for every pair i, j we have

$$\psi\hat{\delta}_{ij}\psi = \hat{\delta}_{p+1-j,p+1-i},$$

namely, the conjugate with respect to ψ of the transposition map on R_{ij} is the transposition map on $R_{p+1-j,p+1-i}$. Hence, since the permutations $\hat{\delta}_{ij}$ commute, we have

$$\psi\hat{\delta}\psi = \hat{\delta}.$$

This yields the assertion. □

These two lemmas immediately imply the following theorem:

Theorem 20 *The map θ preserves centrosymmetry.*

□

As a consequence, we have:

Theorem 21 *For every positive integer k, we have:*

$$s^*_{2k,d} = s^*_{2k,2k-2-d}.$$

□

For example, consider the centrosymmetric fixed point free involution

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 6 & 10 & 5 & 11 & 3 & 1 & 12 & 15 & 17 & 2 & 4 & 7 & 18 & 16 & 8 & 14 & 9 & 13 \end{pmatrix}$$

with 10 ascents. Then,

$$\theta_1(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 11 & 5 & 10 & 6 & 2 & 4 & 17 & 15 & 12 & 3 & 1 & 9 & 16 & 18 & 8 & 13 & 7 & 14 \end{pmatrix},$$

$$(\theta_2 \circ \theta_1)(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 11 & 6 & 12 & 5 & 4 & 2 & 16 & 18 & 10 & 9 & 1 & 3 & 17 & 15 & 14 & 7 & 13 & 8 \end{pmatrix}$$

$$\theta(\sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 12 & 5 & 11 & 6 & 2 & 4 & 18 & 16 & 10 & 9 & 3 & 1 & 15 & 17 & 13 & 8 & 14 & 7 \end{pmatrix},$$

that is centrosymmetric with $18 - 2 - 10 = 6$ ascents.

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