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Tropical Vertex-Disjoint Cycles of a Vertex-Colored Digraph: Barter Exchange with Multiple Items Per Agent

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In a barter exchange market, agents bring items and seek to exchange their items with one another. Agents may agree to a k-way exchange involving a cycle of k agents. A barter exchange market can be represented by a digraph where the vertices represent items and the edges out of a vertex indicate the items that an agent is willing to accept in exchange for that item. It is known that the problem of finding a set of vertex-disjoint cycles with the maximum total number of vertices (MAX-SIZE-EXCHANGE) can be solved in polynomial time. We consider a barter exchange where each agent may bring multiple items, and items of the same agent are represented by vertices with the same color. A set of cycles is said to be tropical if for every color there is a cycle that contains a vertex of that color. We show that the problem of determining whether there exists a tropical set of vertex-disjoint cycles in a digraph (TROPICAL-EXCHANGE) is NP-complete and APX-hard. This is equivalent to determining whether it is possible to arrange an exchange of items among agents such that every agent trades away at least one item. TROPICAL-MAX-SIZE-EXCHANGE is a similar problem, where the goal is to find a set of vertex-disjoint cycles that contains the maximum number of vertices and also contains all of the colors in the graph. We show that this problem is likewise NP-complete and APX-hard. For the restricted case where there are at most two vertices of each color (corresponding to a restriction that each agent may bring at most two items), both problems remain NP-hard but are in APX. Finally, we consider MAX-SIZE-TROPICAL-EXCHANGE, where the set of cycles must primarily include as many colors as possible and secondarily include as many vertices as possible. We show that this problem is NP-hard.

Keywords: vertex-disjoint cycles, vertex-colored digraph, barter exchange, kidney exchange, assignment problem

1 Introduction

Consider a barter exchange where many different agents each bring multiple items to trade. For each item at the barter exchange, the agent lists which other items may be accepted in trade for that item. Based on the lists, an algorithm determines which items are traded for which other items. Instead of restricting exchanges to one-to-one trades, k-way trades are also permitted, where k is limited only by the number of items at the swap meet. If k-way trades are permitted, the traders potentially give items to and receive items from different agents.

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This barter exchange can be represented as a directed graph where each item brought to the exchange is a vertex and there are edges from each item to those items that may be accepted in trade for that item. The problem of determining the maximum possible number of items that can be simultaneously exchanged is equivalent to finding a set of vertex-disjoint cycles that maximizes the total number of vertices in the cycles. (Throughout this paper, all cycles are assumed to be simple cycles.)

This problem is equivalent to the Assignment Problem, which can be solved in polynomial time. It has also been called MAX-SIZE-EXCHANGE (Biro, Manlove, and Rizzi, 2009). The decision and optimization versions of the problem are defined as follows.

Problem 1a. EXCHANGE-x

Input: A directed graph G and an integer x

Question: Is there a set of vertex-disjoint cycles that includes at least x vertices?

Problem 1b. MAX-SIZE-EXCHANGE

Input: A directed graph G

Output: A set of vertex-disjoint cycles that includes as many vertices as possible

If there are many items, it is very likely that there are multiple solutions that tie for the maximum number of items traded. How should the tie be broken? As a secondary criterion, it may be desirable to choose, from among those item-maximizing solutions, a solution that maximizes the number of agents who trade away at least one item. This can be accomplished by converting the previously described directed graph into a vertex-colored digraph, with each participant having a unique color. The problem, then, is equivalent to finding a set of vertex-disjoint cycles with the maximum total number of vertices in the cycles, and in the case of ties, finding a solution that secondarily contains the maximum number of colors. A subset of vertices from a vertex-colored graph is called *tropical* if it includes every color in the vertex-colored graph (Angles d'Auriac et al., 2016a). We define the decision and optimization versions of the TROPICAL-MAX-SIZE-EXCHANGE as follows.

Problem 2a. TROPICAL-MAX-SIZE-EXCHANGE-d (TMaxEx-d)

Input: A vertex-colored digraph G

Question: Is there a set of vertex-disjoint cycles S such that no other set of vertex-disjoint cycles contains more vertices in the cycles, and for every color in the graph there is a cycle in S that contains a vertex of that color?

Problem 2b. TROPICAL-MAX-SIZE-EXCHANGE-o (TMaxEx-o)

Input: A vertex-colored digraph G

Output: A set of vertex-disjoint cycles that includes as many colors as possible, subject to the restriction that there does not exist another set of vertex-disjoint cycles that has more vertices

If the goal is instead to simply maximize the number of agents who trade away at least one item (with no regard for the total number of items traded away), that is equivalent to the problem of finding a set of vertex-disjoint cycles that collectively contain the maximum number of colors. To this end, we define the decision and optimization versions of TROPICAL-EXCHANGE as follows.

Problem 3a. TROPICAL-EXCHANGE-d (TEx-d)

Input: A vertex-colored digraph G

Question: Is there a set of vertex-disjoint cycles such that for every color there is a cycle that contains a vertex of that color?

Problem 3b. TROPICAL-EXCHANGE-o (TEx-o)

Input: A vertex-colored digraph G.

Output: A set of vertex-disjoint cycles such that the total number of vertex colors in the cycles is maximized

We show that both TMaxEx-d and TEx-d are NP-complete via a reduction from CNFSAT. We show that both TMaxEx-o and TEx-o are APX-hard.

We also consider restricted cases where each agent is permitted to bring at most j items to the barter exchange. We then maximize the number of agents who get to trade an item:

Problem 4. j-PER-COLOR-TROPICAL-EXCHANGE (jPC-TEx)

Input: A vertex-colored digraph G with at most j vertices of each color

Output: A set of vertex-disjoint cycles such that the total number of vertex colors in the cycles is maximized

Problem 5. j-PER-COLOR-TROPICAL-MAX-SIZE-EXCHANGE (jPC-TMaxEx)

Input: A vertex-colored digraph G with at most j vertices of each color

Output: A set of vertex-disjoint cycles that includes as many colors as possible, subject to the restriction that no other set of vertex-disjoint cycles has more vertices

These problems can be trivially solved in polynomial time if $j = 1$, but even if $j = 2$, a reduction from MAX-2-SAT shows that these problems are NP-hard. They are, however, in APX for any fixed j.

Finally, we define MAX-SIZE-TROPICAL-EXCHANGE, which reverses the criteria of TMaxEx-o. In TMaxEx-o, the first criterion is maximizing the total number of vertices in the cycles, and the second criterion is maximizing the total number of colors. MAX-SIZE-TROPICAL-EXCHANGE first maximizes the total number of colors in the cycles and secondarily maximizes the total number of vertices. We show that it is NP-hard and APX-hard. We define the decision and optimization versions of MAX-SIZE-TROPICAL-EXCHANGE as follows.

Problem 6a. MAX-SIZE-TROPICAL-EXCHANGE-d (MaxTEx-d)

Input: A vertex-colored digraph G and an integer x.

Question: Is there a set of vertex-disjoint cycles S with a total of at least x vertices in the cycles such that no other set of vertex-disjoint cycles contains more colors in the cycles?

Problem 6b. MAX-SIZE-TROPICAL-EXCHANGE-o (MaxTEx-o)

Input: A vertex-colored digraph G.

Output: A set of vertex-disjoint cycles S such that no other set of vertex-disjoint cycles contains more colors, and no other set with the same number of colors contains more vertices.

Barter exchanges as described in this paper regularly occur online. For example, since 2006 an average of 10 barter exchanges per month have been organized at boardgamegeek.com, with a total of over 500,000 items offered for trade. When participants are informed of the results, they exchange items either in person or by mailing items to each other. Usually, a participant will give an item to one person and receive an item from a different person. Maximizing the number of items traded is the primary goal, and is accomplished using a solution to the Assignment Problem. Maximizing the number of users that trade at least one item is the secondary goal, and this is addressed by solving the Assignment Problem multiple times and after a set number of iterations selecting the solution that yields the greatest number of users trading. However, there is no guarantee that the chosen solution actually maximizes the number of users who trade at least one item. This paper demonstrates that, in general, the problem of maximizing the number of users trading cannot be solved in polynomial time unless $P = NP$. Other websites, such as barterquest.com, swapdom.com, netcycle.com, and rehash.com, have offered similar services in a more general context.

2 Related Work

The Assignment Problem is the minimum weight bipartite perfect matching problem. It is a well-known fundamental optimization problem that can be solved in polynomial time. The polynomial bound was first shown by Jacobi (see Jacobi, 2016), but was not widely known until Kuhn's publication of the Hungarian Algorithm (Kuhn, 1955). More efficient algorithms for the Assignment Problem have been discovered since then (Daitch and Spielman, 2008) (Lee and Sidford, 2014). It is known that MAX-SIZE-EXCHANGE can be solved by reducing it to the Assignment Problem (Abraham et al., 2007) (Biro et al., 2009).

MAX-SIZE-EXCHANGE and related problems have been studied extensively in the context of kidney exchange programs. In kidney exchange programs, it is impractical to allow cycles of unbounded length because it is important for all surgeries to happen at the same time. As a result, most work related to kidney exchanges addresses the MAX SIZE $\leq k$-WAY EXCHANGE problem, which is APX-complete when $k \geq 3$ (Abraham et al., 2007) (Biro et al., 2009). Work on the kidney exchange problem includes investigation of approximability (Biró and Cechlárová, 2007), techniques to mitigate failure after matches have been determined (Dickerson, Procaccia, and Sandholm, 2013), dynamic exchanges (Ünver, 2010), and incentive-compatible mechanisms to ensure that hospitals publish all incompatible pairs (Ashlagi et al., 2015). The problems discussed in this paper differ from the kidney exchange problem because we are not concerned with the length of the cycles, and because of the introduction of colored vertices.

We explore problems that are similar to the Assignment Problem and kidney exchange problem but based on vertex-colored graphs. Vertex-colored graphs have also been called tropical graphs (Foucaud et al., 2016). As mentioned earlier, a subset of vertices from a vertex-colored graph is called tropical if it includes every color in the vertex-colored graph. Tropical connected subgraphs, tropical dominating sets, and tropical homomorphisms have been studied (Angles d'Auriac et al., 2016a) (Angles d'Auriac et al., 2016b) (Foucaud et al., 2016). In this paper, we are concerned with sets of vertex-disjoint cycles. We say that a set of cycles is tropical if every color in the vertex-colored graph is represented by a vertex in at least one cycle.

We study the total number of vertices and colors in a set of vertex-disjoint cycles, but not the total number of cycles in the set. Cycle packing is concerned with the total number of cycles, and it has been studied extensively for both edge-disjoint and vertex-disjoint cycles (Bodlaender et al., 2009) (Krivelevich et al., 2007) (Pedroso, 2014).

There is other work on graphs and colors that is not directly related to our work, but we provide representative citations to help readers delineate the differences. For instance, Fellows et al. (2011) and Dondi, Fertin, and Vialette (2011) explored pattern-matching in vertex-colored graphs. Coudert et al. (2007) explored complexity and approximability properties of edge-colored graphs. Our work assumes that the color of a vertex is an inherent property of the graph. Graph-coloring is a separate topic where colors are added to a graph such that adjacent nodes have different colors. The Four-Color Theorem is a famous result from graph-coloring (Appel and Haken, 1977) (Appel, Haken, and Koch, 1977). Coloring graphs are graphs where each node represents a possible coloring of another graph (Beier, Fierson, Haas, Russell, and Shavo, 2016). More recent graph-coloring work includes the search for rainbow connections (Chartrand et al., 2008) (Li and Sun, 2012) (Li and Shi, 2013). Colorful paths are paths that are both rainbow paths (all vertices in the path have different colors) and tropical paths (all colors in the graph are represented in the path) (Akbari et al., 2011).

3 CNFSAT as a Vertex-Colored Graph

Consider an instance of CNFSAT with q clauses. Let x_1, ..., x_n be the n variables in the expression. For each variable, create a vertex. Give all of these vertices the same color, and refer to them as the“variable vertices.” On each vertex, create two edges that loop back to the same vertex. Label one edge TRUE and one edge FALSE on each vertex.

Assign each clause of the CNFSAT instance a separate color: c_1, c_2, ..., c_q. For each literal in the clause, create a vertex that has the clause's color. Refer to these vertices as “literal vertices.” For negative literals, put the corresponding vertex on the FALSE loop of the variable vertex for that variable by removing one edge from that loop and adding the new literal vertex along with two edges so that the FALSE loop remains a single cycle from the variable vertex back to itself. Similarly, for positive literals put the literal vertex on the TRUE loop of the variable vertex for that variable.

Figure 1 depicts an example graph for a given CNFSAT instance.

4 CNFSAT reduces to TROPICAL-EXCHANGE-d

Constructing a graph for a CNFSAT instance can clearly be accomplished in polynomial time. Each variable only adds one vertex and two edges to the graph, and each clause only adds a vertex and an edge for each literal in the clause.

Based on the way that the graph is constructed, there can be at most n vertex-disjoint cycles. Each variable vertex is part of two cycles (the TRUE loop and the FALSE loop), so a set of vertex-disjoint cycles can only contain one or the other. There are no other cycles in the graph. The cycle that is chosen for the set of vertex-disjoint cycles corresponds to the truth value of that variable in the expression. If we have a polynomial time algorithm for TROPICAL-EXCHANGE-d (TEx-d), then solving TEx-d will also solve CNFSAT in polynomial time. This is because each clause's color will be in a cycle if and only if a variable

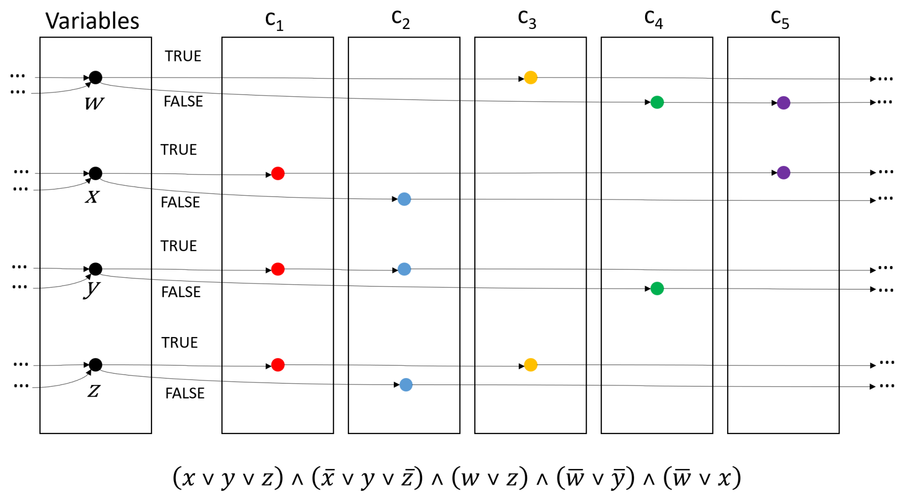


Fig. 1: CNFSAT instance represented as a vertex-colored directed graph.

is set to a truth value that satisfies that clause. If all colors can be in the set of vertex-disjoint cycles, then all clauses and thus the entire expression is satisfiable. Since CNFSAT is NP-complete, TEx-d is NP-hard. TEx-d is also in NP (it is trivial to verify that all colors are in the solution), so TEx-d is NP-complete.

Theorem 1 *TEx-d is NP-complete.*

5 CNFSAT reduces to TROPICAL-MAX-SIZE-EXCHANGE

EXCHANGE-x can be solved in polynomial time, but TROPICAL-EXCHANGE-d is NP-complete. What about TROPICAL-MAX-SIZE-EXCHANGE-d (TMaxEx-d)? That is, if we restrict the search space to only those sets of cycles that maximize the total number of vertices in the cycles, then does that make it easier to determine whether it is possible for a set of vertex-disjoint cycles to cover all of the colors in the graph?

The previous reduction does not work for TMaxEx-d due to the fact that solving TMaxEx-d for the constructed graph does not solve CNFSAT. Because TMaxEx-d first maximizes the total number of vertices in the cycles, it may be the case that the CNFSAT instance is satisfiable even though the solution to TMaxEx-d for the graph does not include all of the colors. That would be the case if the only way to include all of the colors in the graph in the cycles is to accept a lower total number of vertices in the cycles.

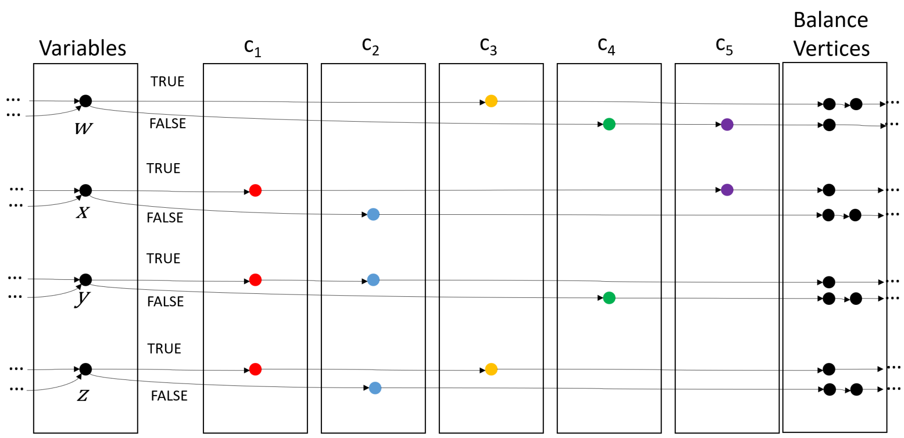
We can adjust the graph by adding "balance vertices" that all have the same color (the "balance color"). First, we identify the cycle with the most vertices. In the case of a tie, we choose any of the tied cycles. We add one balance vertex to this cycle and then add balance vertices to all other cycles so that every cycle has the same number of vertices. These balance vertices accomplish two tasks:

- The number of vertices in each cycle has no impact on the solution produced by TMaxEx-d. With the addition of the balance vertices, every cycle has the same number of vertices, so every possible combination of vertex-disjoint cycles has the same total number of vertices, assuming one cycle per variable vertex. As a result, the secondary criterion (number of colors in the set of cycles) determines the solution.

- The balance color has no impact. Because every cycle has a balance vertex, the balance color is guaranteed to be in the set of vertex-disjoint cycles, and the balance color will not have an impact on the final solution.

If all possible combinations of vertex-disjoint cycles have the same number of total vertices, then by the same logic described for the previous reduction, the solution to TMaxEx-d for the constructed graph will also solve CNFSAT.

Figure 2 depicts an example graph for a given CNFSAT instance.



$$(x \vee y \vee z) \wedge (\bar{x} \vee y \vee \bar{z}) \wedge (w \vee z) \wedge (\bar{w} \vee \bar{y}) \wedge (\bar{w} \vee x)$$

Fig. 2: CNFSAT graph with the addition of Balance Vertices.

TMaxEx-d is easily shown to be in NP. A proposed solution to an instance of TMaxEx-d can be verified by first solving MAX-SIZE-EXCHANGE-o for the same graph, which can be accomplished in polynomial time. The next step is verifying that the proposed solution to TMaxEx-d and the solution produced by MAX-SIZE-EXCHANGE-o have the same number of vertices in their respective sets of vertex-disjoint cycles. Assuming they do, it is simple to verify that all colors appear in the proposed solution. Therefore, TMaxEx-d is NP-complete.

Theorem 2 *TMaxEx-d is NP-complete.*

6 TROPICAL-EXCHANGE and TROPICAL-MAX-SIZE-EXCHANGE are APX-hard

MAX-3-SAT3 is a well-known APX-complete problem that restricts CNFSAT to at most three literals per clause, and each literal can appear in only three clauses. Additionally, it is an optimization problem instead of a decision problem. Section 4 contains a reduction from CNFSAT to TROPICAL-EXCHANGE. A reduction from MAX-3-SAT3 to TEx-o works like the reduction in Section 4. We show that such a reduction is an L-reduction, so we can conclude that TEx-o is APX-hard.

Some additional definitions from Crescenzi (1997) and Papadimitriou and Yannakakis (1991) are needed. A reduction from problem A to problem B consists of a function f that maps an instance of problem A to an instance of problem B, and a function g that maps a solution of problem B to a solution of problem A. The set of instances of problem A is I_A. The set of feasible solutions of an instance $x \in I_A$ is $sol(x)$. The measure of the quality of a solution $y \in sol(x)$ of an instance $x \in I_A$ is denoted by $m(x, y)$. If a solution y is optimal, then $m(x, y) = opt(x)$. The absolute error $E(x, y) = |opt(x) - m(x, y)|$.

A reduction is said to be an L-reduction if it meets the following criteria.

1. For any instance of A and corresponding instance of B, the optimal solution of B is within a constant factor of the optimal solution of A. More formally: $\forall x \in I_A, opt_B(f(x)) \le \alpha \cdot opt_A(x)$

2. For any x that is an instance of problem A and any y that is the solution to corresponding instance of problem B, the absolute error of the proposed solution to A is within a constant factor of the absolute error of y. More formally: $\forall x \in I_A \forall y \in sol_B(f(x)), E_A(x, g(x, y)) \le \beta E_B(f(x), y)$

For our purposes, problem A is MAX-3-SAT3 and problem B is TEx-o. For TEx-o and TMaxEx-o, $m(x, y)$ measures the number of colors in the solution. The reduction in Section 4 satisfies criterion 1 because the number of colors in the graph is equal to the number of clauses in the expression plus one. Similarly, the number of colors in the solution to TEx-o is equal to one plus the number of satisfied clauses in the solution to MAX-3-SAT3. For each instance of MAX-3-SAT3 it is necessary to find a value for α that satisfies criterion 1. If, for example, $\alpha = 3$, then the criterion is satisfied for all instances.

The reduction satisfies criterion 2 because the absolute errors for the two problems are equal. The number of clauses not satisfied is exactly equal to the number of colors that do not appear in the proposed solution to TEx-o. For each instance it is necessary to find a value for β that satisfies criterion 2. If $\beta = 1$, criterion 2 is satisfied for all instances.

Therefore, we conclude that it is an L-reduction and MAX-3-SAT3 $\le_L$ TEx-o. Because an L-reduction preserves membership in PTAS (Crescenzi, 1997, Proposition 7), we can say that TEx-o is APX-hard and there is no PTAS for TEx-o. A similar argument can be made about the reduction in Section 5. That allows us to conclude that MAX-3-SAT3 $\le_L$ TMaxEx-o, and that TMaxEx-o is APX-hard and has no PTAS.

7 TROPICAL-EXCHANGE with only 2 vertices per color

We now consider the problems jPC-TEx and jPC-TMaxEx, which are special cases of TROPICAL-EXCHANGE-o and TROPICAL-MAX-SIZE-EXCHANGE-o, respectively. For these problems, there can be at most j vertices of each color. In terms of a barter exchange, this corresponds to an organized exchange where each person is permitted to bring at most j items. We will show that these restricted cases are in APX and, unless $j = 1$, they are NP-hard.

There is a trivial algorithm to show that both 2PC-TEx and 2PC-TMaxEx are in APX. Simply solve MAX-SIZE-EXCHANGE for the given graph, completely ignoring vertex colors. Because there are only two vertices in the graph for each color, the solution to MAX-SIZE-EXCHANGE is guaranteed to have at least half as many colors as the maximum number of colors possible in a solution to 2PC-TEx or 2PC-TMaxEx.

Therefore it is a 2-approximation. Similarly, a j-approximation can be attained for any j, so jPC-TEx and jPC-TMaxEx are in APX for all j.

Consider an instance of the NP-hard problem MAX-2-SAT. Build a graph in polynomial time as described in Section 3, but instead of making all variable vertices the same color, assign a unique color to each variable vertex. This graph has at most two vertices of any given color, so it is an instance of 2PC-TEx. Refer to Figure 3, but ignore the balance vertices for now. A solution of 2PC-TEx will include as many colors as possible. If a clause's color is included in the solution, that corresponds to a satisfied clause, so MAX-2-SAT reduces to 2PC-TEx, which means 2PC-TEx is NP-hard, just as MAX-2SAT is NP-hard.

Similarly, 2PC-TMaxEx can be shown to be NP-hard with a reduction from MAX-2-SAT. As before, create a graph to represent the MAX-2-SAT instance, giving each variable vertex a unique color. Add balance vertices so that every cycle contains the same number of vertices. Give each balance vertex its own unique color. Finally, add an additional cycle that includes a vertex for each of the balance vertex colors. Each balance vertex color appears only twice: once in a TRUE or FALSE cycle and once in the balance vertex cycle. Due to the balance vertex cycle, every balance vertex color is guaranteed to appear in the solution, so those colors do not affect any decisions regarding whether to take a TRUE cycle or a FALSE cycle. As a result, every combination of TRUE and FALSE cycles has the same number of vertices in the cycles, and a solution to 2PC-TMaxEx will simply maximize the number of colors in the cycles. Maximizing the number of colors in the solution corresponds to maximizing the number of satisfied clauses. Thus, 2PC-TMaxEx is also NP-hard. Figure 3 depicts the reduction of an instance of MAX-2-SAT to 2PC-TMaxEx.

Because 2PC-TEx and 2PC-TMaxEx are NP-hard, jPC-TEx and jPC-TMaxEx are also NP-hard for all $j > 2$.

Theorem 3 *jPC-TEx and jPC-TMaxEx are NP-hard.*

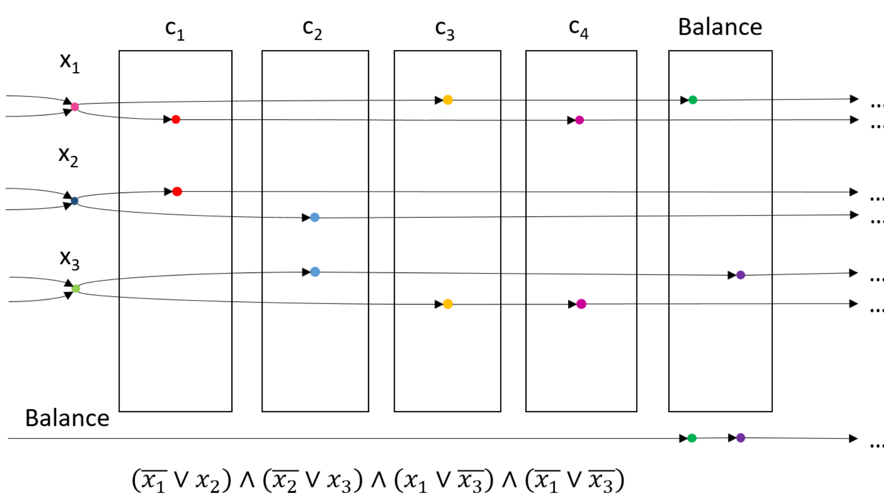


Fig. 3: MAX-2-SAT instance reduced to 2PC-TMaxEx.

8 MAX-SIZE-TROPICAL-EXCHANGE is NP-Hard

For completeness, we consider MAX-SIZE-TROPICAL-EXCHANGE-o (MaxTEx-o), where the primary criterion is maximizing the total number of colors in the set of vertex-disjoint cycles, and the secondary criterion is maximizing the number of vertices. This simply reverses the criteria of TMaxEx-o. It is easy to see that a solution to MaxTEx-o simultaneously solves TEx-o. One only needs to observe the number of colors in the solution to MaxTEx-o. Thus, MaxTEx-o is NP-hard because TEx-o is NP-hard. Whether MaxTEx-d is in NP remains an open question.

9 Conclusion

In this paper, we have defined and analyzed problems that have practical application in the area of algorithmically arranged barter exchanges. We have shown that TROPICAL-EXCHANGE-d and TROPICAL-MAX-SIZE-EXCHANGE-d are NP-complete and that TROPICAL-EXCHANGE-o and TROPICAL-MAX-SIZE-EXCHANGE-o are APX-hard. We have also shown that MAX-SIZE-TROPICAL-EXCHANGE-o is NP-hard. When instances of TROPICAL-EXCHANGE-o and TROPICAL-MAX-SIZE-EXCHANGE-o are restricted to j vertices per color (jPC-TEx and jPC-TMaxEx, respectively), the optimization problems remain NP-hard if $j > 1$, but are in APX.

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