Bisplit graphs satisfy the Chen-Chvátal conjecture

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In this paper, we give a lengthy proof of a small result! A graph is bisplit if its vertex set can be partitioned into three stable sets with two of them inducing a complete bipartite graph. We prove that these graphs satisfy the Chen-Chvátal conjecture: their metric space (in the usual sense) has a universal line (in an unusual sense) or at least as many lines as the number of vertices.

Keywords: bisplit graphs, Chen-Chvátal conjecture, distances

Given a set of n points in the Euclidean plane, they are all collinear or they define at least n distinct lines. This result is a corollary of Sylvester-Gallai Theorem (suggested by Sylvester (1893) in the late nineteenth century and proven by Gallai forty years later as reported by Erdős (1982)). Later, de Bruijn and Erdős (1948) proved a theorem on collections of subsets, which also implies that n points are either collinear or define at least n distinct lines.

The notion of line admits several generalizations, one of which is of interest for us in this paper. Namely, given a metric space (X, ρ), we say that an element b in X is *between* elements a and c if $\rho(a, b)+\rho(b, c) = \rho(a, c)$. More generally, we say that three elements of X are *collinear* if one of them is between the other two. In that setting, *the line generated by* a *and* b (denoted $\overline{ab}$) is the set $\{a, b\}$ completed by all elements collinear with a and b. Ten years ago, (Chen and Chvátal, 2008, Question 1) asked what has now become, by lack of counter-example, the Chen-Chvátal Conjecture.

Conjecture 1 (Chen and Chvátal (2008))**.** *Every finite metric space* (X, ρ) *where no line consists of the entire ground set* X *determines at least* $|X|$ *distinct lines.*

A line consisting of the entire ground set is called a *universal line*. Conjecture 1 remains unsettled when restricted to graph metrics (for connected graphs). Let us say that a graph G on n vertices *has the de Bruijn-Erdős property* if the metric space induced by G has a universal line or at least n distinct lines.

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In a paper gathering more coauthors than pages, Beaudou, Bondy, Chen, Chiniforooshan, Chudnovsky, Chvátal, Fraiman and Zwols (Beaudou et al., 2015, Theorem 1) proved that connected chordal graphs have the de Bruijn-Erdős property. Recently, Aboulker et al. (2018) improved this result by encompassing a larger family of graphs.

One may find out quite easily that connected graphs with a bridge have a universal line. As noted in (Beaudou et al., 2015, Section 3) connected bipartite graphs also have the de Bruijn-Erdős property (each line generated by both ends of an edge is universal).

A significant number of results have appeared concerning the asymptotic number of lines in a graph with no universal lines. A notable one is due to Aboulker, Chen, Huzhang, Kapadia and Supko. They prove (Aboulker et al., 2016, Theorem 7.4) that graphs with n vertices and diameter $d(n)$ have $\Omega((n/d(n))^{4/3})$ distinct lines or a universal line. This implies that any class of graphs with bounded diameter ultimately has the de Bruijn-Erdős property.

Thus, large graphs of diameter 2 have the de Bruijn-Erdős property. Chvátal filled the gap for small graphs of diameter 2 by proving the stronger result (Chvátal, 2014, Theorem 1) that every 1-2 metric space has the de Bruijn-Erdős property.

A connected graph G is *bisplit* if its vertex set can be partitioned into three stable sets X, Y and Z such that Y and Z induce a complete bipartite graph. This class of graphs has diameter bounded by 4. Thus they ultimately have the de Bruijn-Erdős property. Moreover, bisplit graphs are one step away from bipartite graphs (when Z or Y is empty). One may think that they could easily be tamed. It turns out that we could not find a short proof.

In this paper, we prove that bisplit graphs have the de Bruijn-Erdős property.

Theorem 2. *For any integer n greater than or equal to 2, all connected bisplit graphs on n vertices have a universal line or at least n distinct lines.*

This settles Problem 1 from Chvátal (2018). Chen and Chiniforooshan (see final note of Chvátal (2018) in the online version) provided a proof using computer enumeration for small cases. Our proof does not use computer enumeration.

1 Calculus 101

In this section, we state easy results that will be used in the flow of the proof of Theorem 2. We do not give the proof of the following lemma. It is straightforward.

Lemma 3. *For any integer x, $\binom{x}{2} \geq x-1$. Besides, for any pair of positive integers x and y, $xy \geq x+y-1$. Moreover, if both x and y are greater than or equal to 2, then $xy \geq x+y$.*

The next lemma is a bit more tedious. While one might use any computer to have this answer, we give a formal proof hereafter. The reader is advised to skip the proof if the word *trinomial* does not sound thrilling enough.

Lemma 4. *Given two positive integers x and y,*

$$\binom{y}{2} + \binom{\lceil \frac{2x}{y} \rceil}{2} < x + y - 1 \tag{1}$$

if and only if $y = 2$ and x is in $\{1, 2\}$ or $y = 3$ and $x = 3$.

Proof: It is easy to check that the solutions provided satisfy the inequality. Let us now assume that we are given two integers x and y such that x is positive, y is at least 2 and they satisfy the inequality.

Note that

$$\binom{\lceil \frac{2x}{y} \rceil}{2} = \frac{\lceil \frac{2x}{y} \rceil^2 - \lceil \frac{2x}{y} \rceil}{2} \geq \frac{\frac{4x^2}{y^2} - \frac{2x}{y}}{2}.$$

Inequality (1) implies then that

$$\frac{y^2 - y}{2} + \frac{\frac{4x^2}{y^2} - \frac{2x}{y}}{2} - x - y + 1 < 0.$$

This inequality can be simplified and expressed as a trinomial on variable x.

$$4x^2 - 2(y^2 + y)x + (y^4 - 3y^3 + 2y^2) < 0. \quad (2)$$

Since the coefficient of x^2 is positive, (2) has solutions if and only if its discriminant is positive. This discriminant can be simplified and (2) has solutions if and only if,

$$-3y^2 + 14y - 7 \geq 0.$$

This trinomial is even easier than the first one. The value of y must be between 1 and 4. We may now check each case individually.

- If $y = 1$, Inequality (1) becomes $\binom{2x}{2} < x$ which has no integer solution.
- If $y = 2$, Inequality (1) reads $\binom{x}{2} < x$ and the only solutions are for x in $\{1, 2\}$.
- If $y = 3$, we may check the first values of x and see that the only solution is for $x = 3$.
- If $y = 4$, the solutions of the trinomial (2) are in the interval $[3, 7]$. One may check that no integral solution exists.

This concludes our proof of Lemma 4. □

2 Proof of Theorem 2

Let us consider a bisplit graph G. Among all valid partitions for G, we consider one that maximizes the size of $Y \cup Z$. Then, we specify a more precise partition of vertices:

- the set X is split into three sets: the set X_Y (respectively X_Z) made of vertices of X whose only neighbours are in Y (respectively Z) and the set X' made of vertices with neighbours in both Y and Z,
- the set Y (respectively Z) is split into two sets: the set Y_1 (respectively Z_1) made of vertices with at least one neighbour in X_Y (respectively X_Z) and the set Y_2 (respectively Z_2) made of vertices with no neighbour in X_Y (respectively X_Z).

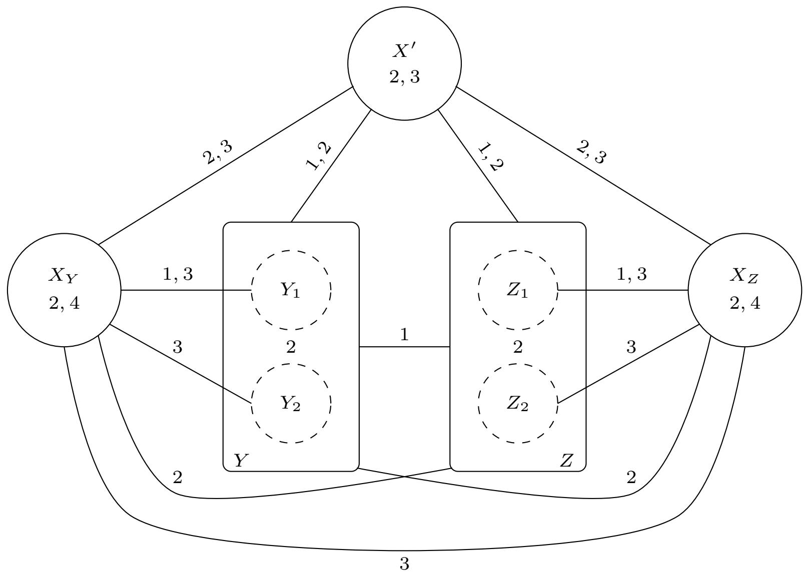

Fig. 1: Possible distances in a bisplit graph

Armed with this new partition, we may derive a sketch of all possible distances between vertices of a bisplit graph. Numbers on Figure 1 refer to the possible distance between two vertices in a same set (numbers in the circles), or between two vertices in two separate sets (numbers on edges between two sets). For example, if a and b are two vertices in X', they both have neighbours in Y and Z. If they have a common neighbour they are at distance 2 and if not, they must be at distance 3.

A vertex in X_Y adjacent to all the set Y could be put in the set Z from the start. Since we chose to maximize the size of $Y \cup Z$ among all valid partitions, we may assume that,

$$\text{no vertex in } X_Y \text{ is complete to } Y. \tag{3}$$

Moreover, if either X', Y or Z is empty, the whole graph is bipartite and thus satisfies the de Bruijn-Erdős property. From now on, we consider that

$$X', Y \text{ and } Z \text{ are not empty.} \tag{4}$$

Before diving into the proof, we warn the reader that we will extensively use a simple fact. For three points to be collinear in a metric space with distances ranging from 0 to 4, the triple of distances they define can only be one of the following: $(1, 1, 2), (1, 2, 3), (1, 3, 4)$ or $(2, 2, 4)$. They are the only cases when the triangle inequality is tight.

The proof is declined as a case study depending on the emptiness of sets X_Y and X_Z.

2.1 When both X_Y and X_Z are empty

Restricting to the case when X_Y and X_Z are empty amounts to considering bisplit graphs for which all vertices in X have at least one neighbour in Y and one in Z.

Proposition 5. *If a bisplit graph is such that all vertices in X have at least one neighbour in both Y and Z, then it has the de Bruijn-Erdős property.*

Proof: Let G be such a bisplit graph. By (4), sets X, Y and Z are not empty. If the distance between any two vertices is at most 2, then G has the de Bruijn-Erdős property (Chvátal proved that every 1-2 metric space has the de Bruijn-Erdős property (Chvátal, 2014, Theorem 1)). So there are at least two vertices a and b at distance 3 from each other. Both a and b must be vertices in X since all distances involving some vertex in Y or Z is 1 or 2 (see Figure 2). Moreover, Y and Z both have cardinality at least 2 (if Y was a singleton, a and b would have a common neighbour since they must have a neighbour in Y). Finally, vertex a cannot be complete neither to Y nor to Z (or it would be at distance 2 from b).

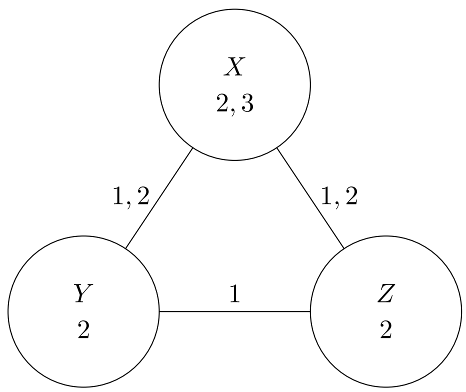

Fig. 2: Possible distances in a bisplit graph where X_Y and X_Z are empty

Concerning notation, we shall use $N(v)$ to represent the neighbourhood of a vertex a. If we want to restrict ourselves to the neighbours of a vertex a in a set S, we shall write $N_S(a)$. Moreover, for an integer i, $N^i(a)$ denotes the vertices which are at distance exactly i from a. Similarly, we may restrict to a specific set by using a subscript. Finally, $\overline{N_S(a)}$ denotes the non-neighbours of vertex a within set S.

Let F_X denote the set of all lines generated by a and another vertex x in $X \setminus \{a\}$. For every such vertex x, the intersection of $\overline{ax}$ with X is always restricted to the generators a and x (all tight triple of distances must have a 1 or a 4). Then F_X is a set of $|X| - 1$ distinct lines.

$$\text{For all } x \text{ in } X \setminus \{a\}, \overline{ax} = \{a, x\} \cup \begin{cases} N(a) \cap N(x) & \text{if } ax = 2 \\ N(a) \cup N(x) & \text{if } ax = 3. \end{cases} \qquad (L_X)$$

Let F_Y denote the set of all lines generated by a and a vertex in Y which is not adjacent to a.

$$\text{For all } y \text{ in } \overline{N_Y(a)}, \overline{ay} = \{a\} \cup (N^3_X(a) \cap N_X(y)) \cup \{y\} \cup N_Z(a). \qquad (L_Y)$$

Since the intersection of such a line with Y is restricted to the singleton containing the other generator, all those lines must be distinct.

Let F'_Y be the set of all lines generated by a pair of vertices in Y which are adjacent to a.

$$\text{For all } y \text{ and } y' \text{ in } N_Y(a), \overline{yy'} = N_X(y) \cap N_X(y') \cup \{y, y'\} \cup Z. \qquad (L'_Y)$$

Since the intersection of such a line with Y is exactly the pair of generators, all those lines are also distinct. Moreover, they are distinct from lines in F_Y since the latter intersect Y on a singleton. As a consequence, $F_Y \cup F'_Y$ is a set of

$$\binom{d_Y(a)}{2} + |Y| - d_Y(a)$$

distinct lines. Note that this quantity is always greater than or equal to $|Y| - 1$.

We define F_Z and F'_Z similarly.

No intersection. We first prove that F_X does not intersect the other families of lines. For a contradiction, suppose that F_X intersects F_Y. Then there are two vertices x in $X \setminus \{a\}$ and y in $Y \cap \overline{N(a)}$ such that $\overline{ax} = \overline{ay}$. Let us focus on the intersection with Y. It must be exactly $\{y\}$. If $d(a, x) = 2$ then $\{y\} = N(a) \cap N(x)$ which is impossible since ay is not an edge in G. If $d(a, x) = 3$ then $\{y\} = N(a) \cup N(x)$ but a and x have no common neighbour and at least one neighbour each in Y. This is also a contradiction. Now suppose that F_X intersects F'_Y. Then there are three vertices x in $X \setminus \{a\}$ and y and y' in $N_Y(a)$ such that $\overline{ax} = \overline{yy'}$. This line must contain the whole set Z. Since a is not complete to Z, $N(a) \cap N(x)$ cannot contain Z. From this and statement (L_X), we derive that a and x must be at distance 3. But then $\{y, y'\}$ must be the union of $N_Y(a)$ and $N_Y(x)$. Since both y and y' are neighbours of a and as x must have a neighbour in Y (by our initial hypothesis), a and x must have a common neighbour which is a contradiction.

Let us now prove that F_Y does not intersect F_Z. For a contradiction, assume that there are two vertices y in $Y \cap \overline{N(a)}$ and z in $Z \cap \overline{N(a)}$ such that $\overline{ay} = \overline{az}$. By looking at the intersection with Z, this would mean that z is in $N(a)$ which is a contradiction. We keep going and prove that F_Y does not intersect F'_Z. Lines in F'_Z contain the whole set Y but lines in F_Y contain only non-neighbour of a. Since a has at least a neighbour in Y these lines cannot be equal.

Finally, let us prove that F'_Y does not intersect F'_Z. Once again, if two such lines were equal, they would contain the whole sets Y and Z which should be of cardinality 2 but then a would be complete to both of them which is impossible.

In the end, we may sum all those lines together. We obtain at least

$$|X| - 1 + |Y| - 1 + |Z| - 1$$

distinct lines. They all contains vertex a.

Reaching for the last three lines. If X has cardinality 4 or more, we may consider all the lines generated by a pair of vertices in $X \setminus \{a\}$. Those line do not contain a and are distinct from each other. There are at least three such lines.

If X has cardinality at most 3, any line generated by a pair of vertices in X contains a or b (recall that b is a vertex at distance 3 from a). We shall distinguish two extra lines. Vertex b must have a neighbour y_b in Y and a neighbour z_b in Z by our hypothesis. Similarly, vertex a has a neighbour y_a in Y and z_a in Z. Since a and b are at distance 3, those four vertices must be distinct.

Now if X has cardinality 2, the line $\overline{y_a z_b}$ is universal. And if X has cardinality 3, the third vertex and b generate one line that does not go through a. Moreover we may consider lines $\overline{y_a y_b}$ and $\overline{z_a z_b}$. These lines do not contain neither a nor b. Thus they are different from all the lines described above. The only issue comes if they are equal. In that case, Y and Z must have cardinality 2 and we know almost everything about the graph. Let c be the third vertex in X. It must have at least one neighbour in Y. Without loss of generality, we may assume it is y_a. Then for $\overline{y_a z_b}$ not to be universal, c must be a neighbour of z_b. For the same reason, c is either adjacent to both y_b and z_a or to none of them. This leads to two graphs (see Figure 3).

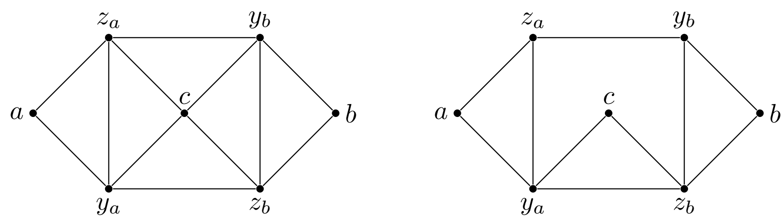

Fig. 3: The last two suspects.

It is then straightforward to check that these two graphs have more than seven lines. This concludes our proof of Proposition 5. □

2.2 When X_Y or X_Z is non-empty

In Proposition 5, we proved that when X_Y and X_Z are empty, the graph satisfies the de Bruijn-Erdős property. For the other cases, we may assume without loss of generality that

$$X_Y \text{ is not empty.} \tag{5}$$

As a direct consequence of (5), we may assume that

$$Y_1 \text{ has cardinality at least 2} \tag{6}$$

because otherwise, the graph is not connected (if Y_1 is empty) or has a bridge (if Y_1 is a singleton) which generates a universal line. Moreover, if Y has cardinality exactly 2, since every vertex in X_Y has degree at least 2, it is complete to Y which contradicts (3). Therefore, we may consider that

$$Y \text{ has cardinality at least 3.} \tag{7}$$

The remainder of our proof relies on a careful choice of families of distinct lines. For any two sets A and B of vertices, we define F_{AB} to be the set of lines generated by any two vertices a in A and b in B.

Fact A. The set $F_{X'X'}$ is made of $\binom{|X'|}{2}$ distinct lines each of which intersect X' on exactly two vertices. Moreover, those lines do not intersect neither X_Y nor X_Z.

Proof: Fact A is obtained by checking the possible distances in the graph (see Figure 1). □

Fact B. The set F_{Y_1Y} is made of $\binom{|Y_1|}{2} + |Y_1||Y_2|$ distinct lines each of which intersects Y on exactly two vertices (the generators). Moreover, those lines do intersect X_Y while they do not intersect X_Z.

Proof: Fact B is also obtained through a straightforward analysis of Figure 1. Moreover, by looking at the intersection with X_Y, we may derive from Facts A and B that $F_{X'X'}$ and F_{Y_1Y} are disjoint. □

Fact C. The set $F_{X_Y Z}$ contains at least $|Z|$ lines which intersect Z on a singleton (the generator) and which contain the whole set Y. Moreover the intersection of such lines with X_Y is not empty.

Proof: Fact C is obtained by pinning one vertex x in X_Y and studying lines $\overline{xz}$ for every vertex z in Z. The intersection of these lines with X_Y guarantees that they are distinct from lines in $F_{X'X'}$. By (7) and Fact B, they are also distinct from lines in $F_{Y_1 Y}$. □

Fact D. If G has no universal line, and X_Y is not empty, then F_{YZ} has cardinality at least 2. Moreover, every line in F_{YZ} contains all X_Y, Y and Z.

Proof: Let z be a vertex in Z (it is not empty otherwise G is bipartite). For any vertex y in Y, the line $\overline{yz}$ includes all vertices in X_Y, Y and Z. Moreover, in X' it includes all vertices that are at distinct distances from y and z. These are exactly the vertices in the symmetric difference of $N(y)$ and $N(z)$ in X'. Suppose that F_{YZ} has cardinality 1, then the symmetric difference between $N(y)$ and $N(z)$ is the same for all y in Y. This implies that all vertices of Y have the same neighbourhood in X'. In other words, every vertex of X' is adjacent to all vertices of Y or to none of them. But our definition of X' states that all its vertices have at least one neighbour in Y. We may conclude that Y is complete to X'. Now consider a vertex x in X_Y. It has at least one neighbour y in Y_1 and the line $\overline{xy}$ is universal which is a contradiction. □

2.2.1 When both X_Y and X_Z are non-empty.

Let us now suppose that X_Y and X_Z are non-empty. We prove that we find many lines.

Proposition 6. *Given a bisplit graph G such that X_Y and X_Z are non-empty, the metric space induced by G satisfies the de Bruijn-Erdős property.*

Proof: In addition to the three families of lines described in facts A, B and C, we shall find three more families, namely F_{YX_Z}, $F_{Z_1 Z}$ and $F_{X_Y X_Z}$. First notice that since X_Z is non-empty, in the same manner as (5) leads to (6) and (7), we may assume that

$$Z_1 \text{ has cardinality at least 2 and } Z \text{ has cardinality at least 3.} \tag{8}$$

Moreover, using the same arguments as for facts B and C, the set $F_{Z_1 Z}$ defines $\binom{|Z_1|}{2} + |Z_1||Z_2|$ lines which are distinct from lines in $F_{X'X'}$ and the set $F_{X_Z Y}$ contains at least $|Y|$ lines which are distinct from lines in $F_{X'X'}$ or in $F_{Z_1 Z}$. We may observe additionally that any line in these new families has a non-empty intersection with X_Z. Then they are all distinct from lines in $F_{Y_1 Y}$. They cannot be equal to a line in $F_{X_Y Z}$ since they either intersect Y on a singleton, or Z on exactly two vertices.

In the end, we also add the lines in $F_{X_Y X_Z}$. Such lines intersect both X_Y and X_Z on singletons (the generators) and have at least two elements in Y_1 and in Z. They are all distinct so they define $|X_Y||X_Z|$ lines. All of them are distinct from the lines described above. Table 1 gives a quick overview of the considered families and the reason why they are distinguished from one another.

Now we can sum all those lines. Since Y_1 and Z_1 have cardinality at least 2, families $F_{Y_1 Y}$ and $F_{Z_1 Z}$ each provides at least one line. Moreover, $\binom{|X'|}{2}$ is lower bounded by $|X'| - 1$ and $|X_Y||X_Z|$ is lower bounded by $|X_Y| + |X_Z| - 1$. In the end we have at least

$$|X'| + |X_Y| + |X_Z| + |Y| + |Z| \text{ lines,}$$

which is the order of the graph. This concludes the proof of Proposition 6. □

Family	Generators	Intersection of $\overline{ab}$ with X'	X_Y	$Y_1 \cup Y_2$	$Z_1 \cup Z_2$	X_Z	Number of lines
$F_{X'X'}$	$a \in X', b \in X'$	$\{a, b\}$	$\emptyset$			$\emptyset$	$\binom{\lvert X'\rvert}{2}$
F_{Y_1Y}	$a \in Y_1, b \in Y$		≥ 1	$\{a, b\}$	$Z_1 \cup Z_2$	$\emptyset$	$\binom{\lvert Y_1\rvert}{2} + \lvert Y_1\rvert\lvert Y_2\rvert$
F_{X_YZ}	$a \in X_Y, b \in Z$		≥ 1	$Y_1 \cup Y_2$	$\{b\}$		$\geq \lvert Z\rvert$
F_{Z_1Z}	$a \in Z_1, b \in Z$		$\emptyset$	$Y_1 \cup Y_2$	$\{a, b\}$	≥ 1	$\binom{\lvert Z_1\rvert}{2} + \lvert Z_1\rvert\lvert Z_2\rvert$
F_{X_ZY}	$a \in X_Z, b \in Y$			$\{b\}$	$Z_1 \cup Z_2$	≥ 1	$\geq \lvert Y\rvert$
$F_{X_YX_Z}$	$a \in X_Y, b \in X_Z$		$\{a\}$	≥ 2	≥ 2	$\{b\}$	$\geq \lvert X_Y\rvert\lvert X_Z\rvert$

Tab. 1: Families of lines and their intersections

2.2.2 When X_Y is non-empty and X_Z is empty.

We now consider the case when X_Z is empty. In this situation, the set Z_1 is also empty. Thus, the last three families of lines in Table 1 cannot be used anymore. We introduce a new family $F_{X_YY_2}$ and through a straightforward analysis of distances (see Figure 1) we obtain Table 2.

Family	Generators	Intersection of $\overline{ab}$ with X'	X_Y	Y_1	Y_2	Z	Number of lines
$F_{X'X'}$	$a \in X', b \in X'$	$\{a, b\}$	$\emptyset$				$\binom{\lvert X'\rvert}{2}$
F_{Y_1Y}	$a \in Y_1, b \in Y$		≥ 1	$\{a, b\}$		Z	$\binom{\lvert Y_1\rvert}{2} + \lvert Y_1\rvert\lvert Y_2\rvert$
F_{X_YZ}	$a \in X_Y, b \in Z$		≥ 1	$Y_1 \cup Y_2$		$\{b\}$	$\geq \lvert Z\rvert$
$F_{X_YY_2}$	$a \in X_Y, b \in Y_2$		$\{a\}$	≥ 2	$\{b\}$	Z	$\lvert X_Y\rvert\lvert Y_2\rvert$

Tab. 2: Families of lines when X_Z is empty

The first three families are distinguished by facts A, B and C. Lines in $F_{X_YY_2}$ are also different from lines in $F_{X'X'}$ and F_{Y_1Y}. Moreover, such a line could be equal to a line in F_{X_YZ} in a very specific case only. Indeed, suppose that a, b, c, d are vertices in X_Y, Y_2, X_Y and Z respectively, such that $\overline{ab} = \overline{cd}$. First note that a must equal c (consider intersection with X_Y) so $\overline{ab} = \overline{ad}$. Moreover, in Y_1, the line $\overline{ab}$ contains only the neighbours of a. Since line $\overline{ad}$ contains the whole set Y, vertex a must be complete to Y_1; and Y_2 must be a singleton. Thus, Y_2 is exactly $\{b\}$. Observing the intersections of these lines with Z, we need Z to be the singleton $\{d\}$. All vertices of X' have a neighbour in Z so d is complete to X'. As a consequence $\overline{ad}$ contains all vertices of X' that are at distance 3 from a. But since $\overline{ab}$ intersects X' only on vertices at distance 2 from a, this implies that no element of X' is at distance 3 from a. Since b has degree at least 2 (or the graph has a bridge and thus a universal line), there must be a vertex x in X' that is a neighbour of b. In return, vertex x is in $\overline{ab}$ but it cannot be in $\overline{ac}$ since it is at distance 2 from a. This yields a contradiction. Thus all four families of lines are disjoint.

If Y_2 has two or more elements. Then we may sum all lines described in Table 2. We get at least

$$\binom{|X'|}{2} + \binom{|Y_1|}{2} + |Y_1||Y_2| + |Z| + |X_Y||Y_2| \text{ lines.}$$

Note that $\binom{x}{2}$ is always bounded below by $x-1$ and xy is bounded below by $x+y-1$ when both x and y are positive integers and by $x+y$ if both are at least 2 (recall that $|Y_1|$ is at least 2). By applying these common properties (stated in Section 1), we bound the number of lines by

$$|X'|+|Y_1|+|Y_2|+|X_Y|+|Z|+(|Y_1|+|Y_2|-3).$$

And by (7), we may conclude that graph G has sufficiently many lines.

If Y_2 is a singleton. In that case, our four families bring at least

$$|X'|+|Y_1|+|Y_2|+|X_Y|+|Z|+(|Y_1|+|Y_2|-4) \text{ lines.}$$

By (7), we only miss one line to reach our goal.

• If Z is a singleton $\{z\}$, then every vertex in X' is adjacent to z. Thus, they are all at distance 2 from one another and all the lines in $F_{X'X'}$ contain z. Then z is an element in all our lines but there must be a line that does not go trough z (or there is a universal line). Then G satisfies the de Bruijn-Erdős property.

• If Z and X_Y both have size at least 2, then no line of our families contains X_Y, Y and Z. But we may easily consider the line generated by the end vertices of any edge between Y and Z and see that it contains all X_Y, Y and Z. It is then different from all considered lines and we have sufficiently many lines.

• If Z has size at least 2 and X_Y is a singleton, we consider only the three families of lines $F_{X'X'}$, F_{Y_1Y} and F_{X_YZ}. By usual inequalities they provide at least

$$|X'|+|Y_1|+|Z|+(|Y_1|-2) \text{ lines.}$$

Since Y_1 has cardinality at least 2, we only need two more lines to reach the order of G. Note that in our three families, no line contains all X_Y, Y and Z. By Fact D we may add two lines from F_{YZ}.

If Y_2 is empty. In this last case, we shall exhibit one last family of lines. For this, let y_0 be a vertex in Y with maximum degree in X_Y. Since every vertex in X_Y has degree at least 2 and all neighbours must be in Y, we know that y_0 has at least $\lceil \frac{2|X_Y|}{|Y|} \rceil$ neighbours in X_Y.

Now let X_0 denote a largest set of vertices in X_Y which are at distance 2 from each other. The set X_0 has size at least $\lceil \frac{2|X_Y|}{|Y|} \rceil$. Now, lines in $F_{X_0X_0}$ intersect X_0 in exactly two vertices (the generators). Moreover, they do not intersect Z. The families of lines considered are shown in Table 3. Those lines are all distinct except for the two last rows if Z is a singleton.

Note that since $|Y|$ is at least 3 and $|X_0| \geq \lceil \frac{2|X_Y|}{|Y|} \rceil$, Lemma 4 tells us that whenever $|X_Y|$ or $|Y|$ is not 3, we have:

$$\binom{|X_0|}{2}+\binom{|Y|}{2} \geq |X_Y|+|Y|-1. \tag{9}$$

Moreover, if X_Y and Y both have size exactly 3, all vertices of X_Y are at distance 2 from each other. Indeed, either a vertex of Y has degree 3 in X_Y, or all of them must have degree 2 (they must be incident to at least six edges coming from X_Y) and as a consequence all vertices of X_Y have degree 2 (X_Y and Y induce a cycle of length 6). In both cases, we deduce that all vertices of X_Y are at distance 2 from each other. Thus, we may always consider that $|X_0|$ has value 3. Therefore (9) remains true.

Family	Generators	Intersection of $\overline{ab}$ with X'	X_0	X_Y	Y	Z	Number of lines
$F_{X'X'}$	$a \in X', b \in X'$	$\{a,b\}$	$\emptyset$	$\emptyset$			$\binom{\vert X'\vert}{2}$
F_{YY}	$a \in Y, b \in Y$			≥ 1	$\{a,b\}$	Z	$\binom{\vert Y\vert}{2}$
$F_{X_0Y_0}$	$a \in X_0, b \in Y_0$		$\{a,b\}$		≥ 1	$\emptyset$	$\binom{\vert X_0\vert}{2}$
F_{YZ}	$a \in Y, b \in Z$			X_Y	Y	Z	≥ 2
F_{X_YZ}	$a \in X_Y, b \in Z$			≥ 1	Y	$\{b\}$	$\geq \vert Z\vert$

Tab. 3: Families of lines when X_Z and Y_2 are empty

When Z is not a singleton. If Z is not a singleton, we may sum all rows of Table 3. By (9) we have a lower bound of

$$|X'| + |Z| + |X_Y| + |Y| \text{ lines.}$$

Thus G satisfies the de Bruijn-Erdős property.

When Z is a singleton. If Z has size 1, we do not count the last row of Table 3. Then we miss only one line. To find it, just observe that there must be a vertex x' in X' by (4). This vertex has a neighbour y in Y which is equal to Y_1 since Y_2 is empty. This guarantees that there is a vertex x in X_Y at distance 2 from x'. Analyzing the distances in G, we derive that line $\overline{xx'}$ intersects X' exactly on x', X_0 on at most one vertex, and does not intersect Z. Thus, this line is different from the first four rows in Table 3. This completes the number of lines to reach the order of G.

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