Semitotal domination in trees

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In this paper, we study a parameter that is squeezed between arguably the two important domination parameters, namely the domination number, $\gamma(G)$, and the total domination number, $\gamma_t(G)$. A set $S$ of vertices in $G$ is a semitotal dominating set of $G$ if it is a dominating set of $G$ and every vertex in $S$ is within distance 2 of another vertex of $S$. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of $G$. We observe that $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. In this paper, we give a lower bound for the semitotal domination number of trees and we characterize the extremal trees. In addition, we characterize trees with equal domination and semitotal domination numbers.

Keywords: domination, semitotal domination, tree

1 Introduction

Let $G = (V, E)$ be a graph without isolated vertices with vertex set $V$ of order $n(G) = |V|$ and edge set $E$ of size $m(G) = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is $d(v) = |N(v)|$. For two vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ between $u$ and $v$ is the length of a shortest $(u, v)$-path in $G$. The maximum distance among all pairs of vertices of $G$ is the diameter of a graph $G$ which is denoted by $\text{diam}(G)$. A leaf of $G$ is a vertex of degree 1 and a support vertex of $G$ is a vertex adjacent to a leaf. Denote the sets of leaves and support vertices of $G$ by $L(T)$ and $S(T)$, respectively. Let $l(T) = |L(T)|$ and $s(T) = |S(T)|$. A double star is a tree that contains exactly two vertices that are not leaves.

A dominating set in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A total dominating set of a graph $G$ with no isolated vertex is a set $D$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $D$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A dominating (total dominating) set of $G$ of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a $\gamma(G)$-set ($\gamma_t(G)$-set).

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The concept of semitotal domination in graphs was introduced and studied by Goddard, Henning and McPillan [3]. A set $S$ of vertices in a graph $G$ with no isolated vertices is a semitotal dominating set of $G$ if every vertex in $S$ is within distance 2 of another vertex of $S$. The semitotal domination number, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of $G$. A semitotal dominating set of $G$ of cardinality $\gamma_{t2}(G)$ is called a $\gamma_{t2}(G)$-set. Clearly, for every graph $G$ with no isolated vertex, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. If the graph $G$ is clear from the context, we simply write $\gamma$-set and $\gamma_{t2}$-set rather than $\gamma(G)$-set and $\gamma_{t2}(G)$-set, respectively.

An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters. For any two graph theoretic parameters $\lambda$ and $\mu$, $G$ is called a $(\lambda, \mu)$-graph if $\lambda(G) = \mu(G)$. The class of $(\gamma, \gamma_t)$-trees, that is trees with equal domination and total domination numbers, was characterized in [6]. In [4], the authors provided a constructive characterizations of trees with equal domination and paired domination numbers. More results in this area were investigated in [7, 5, 8, 1] and elsewhere. Motivated by these results, we aim to characterize trees with equal domination and semitotal domination numbers. In addition, we give a lower bound for the semitotal domination number of trees and we characterize the extremal trees.

2 A lower bound for semitotal domination number of trees

In this section we give a lower bound for the semitotal domination number of trees and we characterize the extremal trees. First, we shall need the following two observations.

**Observation 2.1.** Let $G$ be a connected graph that is not a star. Then,

(i) there is a $\gamma$-set of $G$ that contains no leaf, and

(ii) there is a $\gamma_{t2}$-set of $G$ that contains no leaf.

**Theorem 2.2.** If $T$ is a tree of order $n(T) \geq 2$ with $l(T)$ leaves, then $\gamma_{t2}(T) \geq 2\lceil n(T) - l(T) + 2 \rceil$. 

**Proof:** We use induction on $n(T)$. It is easy to see that the result holds for a tree of order $n \leq 8$. Let $T$ be a tree of order $n > 8$ and assume that $\gamma_{t2}(T') \geq 2\lceil l(T') - (l(T') + 2) \rceil$ for each tree $T'$ with order at most $n - 1$. We consider the case that $\text{diam}(T) \geq 4$. Otherwise, $T$ is a star or double-star, then $\gamma_{t2}(T)$ has the desired property in theorem. By Observation 2.1(ii), we can obtain a $\gamma_{t2}$-set of $T$, say $D$, which contains no leaf.

**Claim 1.** For any vertex $v \in V(T) \setminus L(T)$, $v$ has only one leaf-neighbor when $|N(v) \setminus L(T)| = 1$, and $v$ is not a support vertex when $|N(v) \setminus L(T)| \geq 2$.

**Proof:** If $v$ is a vertex that has at least two leaf-neighbors and $|N(v) \setminus L(T)| = 1$. We remove one of those leaves and denote the resulting tree by $T'$. It is easy to observe that $\gamma_{t2}(T') = \gamma_{t2}(T)$. By induction, $\gamma_{t2}(T') \geq 2\lceil l(T') - (l(T') + 2) \rceil$. And consequently $\gamma_{t2}(T) \geq 2\lceil l(T) - (l(T) + 2) \rceil$ as $l(T') = l(T) - 1$, $n(T') = n(T) - 1$.

If $v$ is a support vertex and $|N(v) \setminus L(T)| \geq 2$, we remove a leaf-neighbor of $v$ and the semitotal domination number of the resulting tree is no greater than that of $T$. Analogously to the previous case, $\gamma_{t2}(T)$ has the desired property in theorem.

In other words, each support vertex of $T$ has degree two. Let $P = v_0v_1v_2 \cdots v_l$ be a longest path in $T$ such that
Let \( v \) be adjacent to a support vertex outside \( \gamma \) induction, of degree two, except possibly the vertex \( v \).

\[ \text{Proof:} \] If \( P \) then the choice of \( P \) and Claim 1 that all neighbors of \( v \) are support vertices of degree two, except possibly the vertex \( v \).

By Claim 1, \( d(v_1) = 2 \) and \( v_2 \) is not a support vertex.

\[ \text{Claim 2.} \quad d(v_2) = 2. \]

\[ \text{Proof:} \] If \( d(v_2) > 2 \), it follows from the choice of \( P \) and Claim 1 that all neighbors of \( v_2 \) are support vertices of degree two, except possibly the vertex \( v_3 \).

Let \( u_1 \) be a neighbor of \( v_2 \) outside \( P \), \( u_2 \) be the leaf that adjacent to \( u_1 \), and \( T' = T - \{v_0, u_2\} \). By induction, \( \gamma_{v_2}(T') \geq \frac{2n(T') - l(T') + 2}{5} \). In addition, replacing the vertices \( u_1 \) and \( v_1 \) in \( D \) with \( v_2 \) (if \( v_2 \in D \), take \( D \setminus \{v_1\} \) instead), we can obtain a semitotal dominating set of \( T' \). That is, \( \gamma_{v_2}(T') \leq \gamma_{v_2}(T) - 1 \). Note that \( l(T') = l(T) \), \( n(T') = n(T) - 2 \). Therefore, \( \gamma_{v_2}(T) \geq \frac{2n(T) - l(T) + 2}{5} \).

We know that \( v_1 \in D \) and exactly one of \( v_2 \) and \( v_3 \) belongs to \( D \). Without loss of generality, \( v_3 \in D \) (Otherwise, we replace the vertex \( v_2 \) in \( D \) with \( v_3 \), and the resulting set is also a \( \gamma_{v_2} \)-set of \( T \).

\[ \text{Claim 3.} \quad d(v_3) = 2. \]

\[ \text{Proof:} \] By Claim 1 and the assumption that \( n > 8 \), \( v_3 \) is not a support vertex. If \( d(v_3) > 2 \), it follows from the choice of \( P \) and Claim 1 that \( v_3 \) has a neighbor of degree two outside \( P \), say \( v_2 \), which is either a support vertex or adjacent to a support vertex outside \( P \), say \( v_1 \).

Let \( T' = T - \{v_0, v_1\} \). By induction, \( \gamma_{v_2}(T') \geq \frac{2n(T') - l(T') + 2}{5} \). In either case, we have that \( l(T') = l(T) \), \( n(T') = n(T) - 2 \) and it is easy to see that \( \gamma_{v_2}(T') \leq \gamma_{v_2}(T) - 1 \). Therefore, \( \gamma_{v_2}(T) \geq \frac{2n(T) - l(T) + 2}{5} \).

\[ \text{Claim 4.} \quad d(v_4) = 2. \]

\[ \text{Proof:} \] By Claim 1 and the assumption that \( n > 8 \), \( v_4 \) is not a support vertex. If \( d(v_4) > 2 \), from the choice of \( P \) and Claim 1, we only need to consider the case as follows: \( v_4 \) has a neighbor outside \( P \), say \( v_3 \), which is adjacent to \( t \) support vertices \( u_1, u_2, \ldots, u_t \), where \( t \geq 2 \). (In other cases, we always have that \( \gamma_{v_2}(T') \leq \gamma_{v_2}(T) - 1 \), \( l(T') = l(T) \) and \( n(T') = n(T) - 2 \), where \( T' = T - \{v_0, v_1\} \). And similar to the proof of Claim 3, \( \gamma_{v_2}(T) \) has the desired property in theorem.) Let \( u_i' \) be the leaf-neighbor of \( u_i \), where \( i = 1, 2, \ldots, t \). Let \( T' = T - \{u_1', u_2', \ldots, u_t'\} \). By induction, \( \gamma_{v_2}(T') \geq \frac{2n(T') - l(T') + 2}{5} \). Note that \( \{u_1, u_2, \ldots, u_t\} \subseteq D \). Then \( (D \setminus \{u_1, u_2, \ldots, u_t\}) \cup \{v_3\} \) is a semitotal dominating set of \( T' \). That is, \( \gamma_{v_2}(T') \leq \gamma_{v_2}(T) + t - 1 \). In addition, \( l(T') = l(T) \), \( n(T') = n(T) - t \). Hence, \( \gamma_{v_2}(T) \geq \frac{2n(T) - l(T) + 2}{5} + t - 1 = \frac{2n(T) - l(T) + 2}{5} + t - 1 > \frac{2n(T) - l(T) + 2}{5} \).

Note \( v_1, v_3 \in D \). Then, one of the two cases as following holds: (1) Each vertex of \( D \setminus \{v_1, v_3\} \) is at distance at least 3 from \( v_3 \); (2) There is a vertex of \( D \setminus \{v_1, v_3\} \) which is within distance 2 of \( v_3 \).

In the former case, let \( T'' = T - \{v_0, v_1, v_2, v_3\} \). By induction,  \( \gamma_{v_2}(T'') \geq \frac{2n(T'') - l(T'') + 2}{5} \). In addition, note that \( D \setminus \{v_1, v_3\} \) is a semitotal dominating set of \( T'' \), \( n(T) = n(T') + 5 \), \( l(T) \geq l(T') \). Hence, \( \gamma_{v_2}(T) \geq \frac{2n(T) - l(T) + 2}{5} \).

In the latter case, let \( T'' = T - \{v_0, v_1\} \). By induction,  \( \gamma_{v_2}(T'') \geq \frac{2n(T'') - l(T'') + 2}{5} \). Since \( D \setminus \{v_1\} \) is a semitotal dominating set of \( T'' \), \( n(T') = n(T') + 2 \), \( l(T) = l(T') \). Hence, \( \gamma_{v_2}(T) \geq \frac{2n(T) - l(T) + 2}{5} \).

The proof is completed.
Next, we are ready to provide a constructive characterization of the trees achieving equality in the bound of Theorem 2.2. For our purposes we define a labeling of a tree $T$ as a partition $S = (S_A, S_B, S_C)$ of $V(T)$. (This idea of labeling the vertices is introduced in [2]). We will refer to the pair $(T, S)$ as a labeled tree. The label or status of a vertex $v$, denoted $sta(v)$, is the letter $x \in \{A, B, C\}$ such that $v \in S_x$.

Let $\mathcal{T}$ be the family of labeled trees that: (i) contains $(P_5, S')$ where $S'$ is the labeling that assigns to the two support vertices of the path $P_5$ status $A$, to the two leaves status $C$ and to the center vertex status $B$ (see Fig.1(a)); and (ii) is closed under the two operations $O_1$ and $O_2$ that are listed below, which extend the tree $T'$ to a tree $T$ by attaching a tree to the vertex $v \in V(T')$.

Operation $O_1$: Let $v$ be a vertex with $sta(v) = A$. Add a vertex $u$ and the edge $uv$. Let $sta(u) = C$.

Operation $O_2$: Let $v$ be a vertex with $sta(v) = C$ that has degree one. Add a path $u_1u_2u_3u_4u_5$ and the edge $u_5v$. Let $sta(u_1) = sta(u_5) = C$, $sta(u_2) = sta(u_4) = A$, $sta(u_3) = B$.

The two operations $O_1$ and $O_2$ are illustrated in Fig.1(b), (c).

![Fig. 1](image)

Let $(T, S) \in \mathcal{T}$ be a labeled tree for some labeling $S$. Then there is a sequence of labeled trees $(T_0, S_0)$, $(T_1, S_1), \ldots, (T_{k-1}, S_{k-1}), (T_k, S_k)$ such that $(T_0, S_0) = (P_5, S')$, $(T_k, S_k) = (T, S)$. The labeled tree $(T_i, S_i)$ can be obtained from $(T_{i-1}, S_{i-1})$ by one of the operations $O_1$ and $O_2$, where $i \in \{1, 2, \ldots, k\}$. We call the number of terms in such a sequence of labeled trees that is used to construct $(T, S)$, the length of the sequence. Clearly, the above sequence has length $k$. We remark that a sequence of labeled trees used to construct $(T, S)$ is not necessarily unique.

We take an example to make it easier for reader to understand the family $\mathcal{T}$. In Fig.2, $(P_5, S') \in \mathcal{T}$, $(H_1, S_1)$ is obtained from $(P_5, S')$ by operation $O_2$, $(H_2, S_2)$ is obtained from $(H_1, S_1)$ by repeated applications of operation $O_1$, and $(H_3, S)$ is obtained from $(H_2, S_2)$ by operation $O_2$. Thus, $(H_1, S_1)$, $(H_2, S_2), (H_3, S) \in \mathcal{T}$. For $T \in \{P_5, H_1, H_2, H_3\}$, it is easy to see that the set, say $D$, consisting of the vertices labeled $A$ in $T$ is a $\gamma_2$-set of $T$. In particular, $|D| = \frac{2(n(T) - (T) + 2)}{5}$.

Before presenting our main result, we present a few preliminary results and observations.

**Observation 2.3.** Let $T$ be a tree and let $S$ be a labeling of $T$ such that $(T, S) \in \mathcal{T}$. Then, $T$ has the following properties:
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![Diagram](image)

Fig. 2

(a) Every support vertex is labeled A and every leaf is labeled C.
(b) \( |S_A| = 2|S_B| \).
(c) The set \( S_A \) is a semitotal dominating set of \( T \).
(d) The set \( S_A \) and \( S_B \) are independent sets.
(e) Every vertex labeled B has degree two and its neighbors labeled A.

**Lemma 2.4.** Let \( T \) be a tree and let \( S \) be a labeling of \( T \) such that \((T, S) \in \mathcal{F}\). Then, \( \gamma_{t2}(T) = \frac{2[n(T) - l(T) + 2]}{5} \).

**Proof:** First, we are ready to show that \( |S_A| = \frac{2[n(T) - l(T) + 2]}{5} \). We proceed by induction on the length \( k \) of a sequence required to construct the labeled tree \((T, S)\).

When \( k = 0 \), \((T, S) = (P_5, S')\), and so \( |S_A| = 2 \). This establishes the base case. Let \( k \geq 1 \) and assume that if the length of sequence used to construct a labeled tree \((T^*, S^*) \in \mathcal{F}\) is less than \( k \), then \( |S_A^*| = \frac{2[n(T^*) - l(T^*) + 2]}{5} \). Now, \((T, S) \in \mathcal{F}\) and let \((T_0, S_0), (T_1, S_1), \ldots, (T_{k-1}, S_{k-1}), (T_k, S_k)\) be a sequence of length \( k \) used to construct \((T, S)\), where \((T_0, S_0) = (P_5, S')\), \((T_k, S_k) = (T, S)\), \((T_i, S_i)\) can be obtained from \((T_{i-1}, S_{i-1})\) by one of the operations \( \theta_1 \) or \( \theta_2 \), \( i \in \{1, 2, \ldots, k\} \). Let \( T^* = T_{k-1} \) and \( S^* = S_{k-1} \). Note that \((T_{k-1}, S_{k-1}) \in \mathcal{F}\). By the inductive hypothesis, \( |S_A^*| = \frac{2[n(T^*) - l(T^*) + 2]}{5} \). \((T, S)\) can be obtained from \((T^*, S^*)\) by operation \( \theta_1 \) or \( \theta_2 \).

In the former case, we have that \( n(T) = n(T^*) + 1 \), \( l(T) = l(T^*) + 1 \), and \( |S_A| = |S_A^*| \). Thus, \( |S_A| = \frac{2[n(T^*) - l(T^*) + 2]}{5} + 1 = \frac{2[n(T) - l(T) + 2]}{5} \).

In the latter case, we have that \( n(T) = n(T^*) + 5 \), \( l(T) = l(T^*) \) and \( |S_A| = |S_A^*| + 2 \). Thus, \( |S_A| = \frac{2[n(T^*) - l(T^*) + 2]}{5} + 2 = \frac{2[n(T) - l(T) + 2]}{5} + 2 \).

By Observation 2.3(c), we have that \( \gamma_{t2}(T) \leq \frac{2[n(T) - l(T) + 2]}{5} \). Combining Theorem 2.2, we conclude that \( \gamma_{t2}(T) = \frac{2[n(T) - l(T) + 2]}{5} \). Moreover, \( S_A \) is a \( \gamma_{t2} \)-set of \( T \).

**Theorem 2.5.** Let \( T \) be a nontrivial tree, then \( \gamma_{t2}(T) = \frac{2[n(T) - l(T) + 2]}{5} \) if and only if \((T, S) \in \mathcal{F}\) for some labeling \( S \).
**Proof:** The sufficiency follows immediately from Lemma 2.4. So we prove the necessity only. The proof is by induction on the order of $T$. The result is immediate for $n \leq 5$. For the inductive hypothesis, let $n \geq 6$ and moreover, $\text{diam}(T) \geq 4$ (If $\text{diam}(T) \leq 3$, $T$ is a star or a double star, and then $\gamma_2(T) > \frac{2n(\text{diam}(T))-l(T)+2}{5}$, a contradiction). Assume that for every nontrivial tree $T'$ of order less than $n$ with $\gamma_2(T') = \frac{2[n(T')]-l(T')+2}{5}$, we have that $(T', S^*) \in \mathcal{F}$ for some labeling $S^*$. Let $T$ be a tree of order $n$ satisfying $\gamma_2(T) = \frac{2n(T)-l(T)+2}{5}$. Let $P = v_1v_2\ldots v_6$ be a longest path in $T$ such that

(i) $d(v_4)$ as large as possible, and subject to this condition
(ii) $d(v_5)$ as large as possible.

Let $D$ be a $\gamma_2$-set of $T$ which contains no leaf.

**Claim 1.** Each support vertex has exactly one leaf-neighbor.

**Proof:** If not, assume that there is a support vertex $u$ which is adjacent to at least two leaves. Deleting one of its leaf-neighbors, say $u_1$, and denote the resulting tree by $T'$. $D$ is still a semitotal dominating set of $T'$. That is, $\gamma_2(T') \leq \gamma_2(T) = \frac{2n(T)-l(T)+2}{5} = \frac{2n(T')+1-l(T')-1+2}{5} = \frac{2n(T')-l(T')+2}{5}$. Combining Theorem 2.2, we have that $\gamma_2(T') = \frac{2n(T')-l(T')+2}{5}$. By the inductive hypothesis, $(T', S^*) \in \mathcal{F}$ for some labeling $S^*$. Since $u$ is still a support vertex in $T'$, by Observation 2.3(a), the vertex $u$ has label $A$ in $S^*$. Let $S$ be obtained from the labeling $S^*$ by labeling the vertex $u_1$ with label $C$. Then, $(T, S)$ can be obtained from $(T', S^*)$ by operation $\mathcal{O}_1$. Thus, $(T, S) \in \mathcal{F}$. \qed

By Claim 1, we can assume that $d(v_2) = 2$. Now, we consider the vertex $v_3$. If $v_3$ is a support vertex, then $v_2, v_3 \in D$. Let $T'$ be the tree which is obtained from $T$ by subdividing the edge $v_2v_3$. It is easy to see that $D$ is still a semitotal dominating set of $T'$, and it means that $\frac{2n(T')-l(T')-1}{5} = \gamma_2(T') \geq \gamma_2(T') \geq \frac{2n(T')-l(T')-1}{5} = \frac{2n(T)-l(T)+2}{5}$, a contradiction. So, $v_3$ is not a support vertex.

Assume that $d(v_3) \geq 3$. Then, it follows from the choice of $P$ that $v_3$ is adjacent to a support vertex, say $u$, which does not belong to $P$. Clearly, $u, v_2 \in D$. Moreover, $v_3 \not\in D$. Otherwise, we subdivide the edges $v_2v_3$ and $uv_3$, and yield a similar contradiction as above.

If $u$ is within distance two from a vertex in $D \setminus \{u, v_2\}$, we have that $\frac{2n(T)-2-(l(T)-1)+2}{5} \leq \gamma_2(T') \leq \gamma_2(T) - 1 = \frac{2n(T)-l(T)+2}{5} - 1$, where $T' = T - \{v_1, v_$. It is impossible. It follows that $v_3 \not\in D$, but in this case, let $T''$ be the component of $T - v_3v_4$ containing the vertex $v_4$, and $\frac{2n(T')-l(T')}{5} = \frac{2n(T) - l(T) + 2}{5} = \gamma_2(T) \geq \gamma_2(T''') + \gamma_2(T - T'') \geq \frac{2[n(T''')-l(T'')+2]}{5} + \frac{2[n(T-T'')]-l(T-T'')}{5} + 2 \frac{[n(T-T'')]-l(T-T'')}{5} \geq \frac{2n(T)-l(T)+1}{5} = \frac{2n(T)-l(T)+3}{5}$, a contradiction. Therefore, $d(v_3) = 2$.

From the choice of $D$, $v_2 \in D$, and without loss of generality, $v_4 \in D$ (If $v_4 \not\in D$, then $v_3 \in D$, replacing $v_3$ in $D$ with $v_4$, and we obtain a new $\gamma_2$-set of $T$).

Assume that $d(v_4) \geq 3$. We have that the following conclusion.

**Claim 2.** $N(v_4) \setminus \{v_3, v_5\} \subset L(T)$.

**Proof:** Assume that there exists a vertex $v'_3 \in N(v_4) \setminus \{v_3, v_5\}$ which is not a leaf, it follows from the choice of $P$ and Claim 1 that $v'_3$ is either a support vertex or adjacent to a support vertex outside $P$, say $v'_2$. In particular, $d(v'_2) = 2$ (From the choice of $P$). In either case, let $T' = T - \{v_1, v_2\}$. Observe that $n(T) = n(T') + 2, l(T) = l(T'), \gamma_2(T') \leq \gamma_2(T) - 1$. Then, we have that $\gamma_2(T') \leq \gamma_2(T) = \frac{2n(T)-l(T)+2}{5} - 1 = \frac{2[n(T)+2]-l(T')}{5} - 1 = \frac{2[n(T)-l(T)+2]}{5} - \frac{1}{5} < \frac{2n(T')-l(T')+2}{5}$, contradicting Theorem 2.2. It concludes that $\emptyset \neq N(v_4) \setminus \{v_3, v_5\} \subset L(T)$. \qed
So $d(v_4) = 2$ or $N(v_4) \setminus \{v_3, v_5\} \subseteq L(T)$. Moreover, all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from $v_4$ (if not, let $T' = T - \{v_1, v_2\}$). Observe that $n(T) = n(T') + 2$, $l(T) = l(T')$, $\gamma_{2}(T') \leq \gamma_{2}(T) - 1$. We can obtain a contradiction by an argument similar to the proof of Claim 2).

If $d(v_3) = 1$, by Claim 1 and the choice of $P$, $T = P_3$, contradicting the assumption that $n \geq 6$. So assume that $d(v_3) \geq 3$, since all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from $v_4$, each neighbor of $v_3$ is neither a leaf nor a support vertex. From the choice of $P$ and Claim 1, we only need to consider the case as follows: $v_3$ has a neighbor outside $P$, say $v_4'$, which is adjacent to $t$ support vertices $u_1, u_2, \ldots, u_t$, where $t \geq 2$. (In other cases, let $T' = T - \{v_1, v_2\}$). Observe that $n(T) = n(T') + 2$, $l(T) = l(T')$, $\gamma_{2}(T') \leq \gamma_{2}(T) - 1$. We can always obtain contradictions by an argument similar to the proof of Claim 2).

Let $v_1'$ be the leaf-neighbor of $u_i$, where $i = 1, 2, \ldots, t$, and $T' = T - \{v_2, v_4\}$. Note that \{u_1, u_2, \ldots, u_t\} $\subseteq D$. Then $(D \setminus \{u_1, u_2, \ldots, u_t\} \cup \{v_3\}$ is a semitotal dominating set of $T'$. That is, $\gamma_{2}(T') \leq \gamma_{2}(T) - 1$. In addition, $l(T') = l(T), n(T') = n(T) - t$. Hence, $\gamma_{2}(T') \leq \gamma_{2}(T) - t + 1 = \frac{2n(T) - l(T) + 2}{5} - t + 1 = \frac{2[n(T') - l(T') + 2]}{5} + \frac{2t}{5} - t + 1 < \frac{2[n(T') - l(T') + 2]}{5}$, contradicting Theorem 2.2. Therefore, $d(v_3) = 2$.

Let $T''$ be the component of $T - v_{5}v_{6}$ containing $v_{6}$. If $v_{6}$ is not a leaf in $T''$, then $n(T) = n(T'') + 5 + s$, $l(T') = l(T'') + 1 + s$, where $s$ is the number of the leaf-neighbors of $v_4$. Since all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from $v_4$, $\gamma_{2}(T'') \leq \gamma_{2}(T) - 2$. It follows that $\gamma_{2}(T'') \leq \frac{2[n(T') - l(T'') + 2]}{5} - 2 = \frac{2[n(T') - l(T'') + 2]}{5} - \frac{2}{5} < \frac{2[n(T') - l(T'') + 2]}{5}$, contradicting Theorem 2.2. It means that $v_6$ is a leaf in $T''$, and $\gamma_{2}(T'') \leq \gamma_{2}(T) - 2 = \frac{2[n(T') - l(T'') + 2]}{5} - 2 = \frac{2[n(T'') + 5 + s - l(T'') - s + 2]}{5} - 2 = \frac{2[n(T'') - l(T'') + 2]}{5}$. Combining Theorem 2.2, we have that $\gamma_{2}(T'') = \frac{2[n(T') - l(T'') + 2]}{5}$. By the inductive hypothesis, $(T'', S^*) \in \mathcal{S}$ for some labeling $S^*$. Since $v_6$ is a leaf in $T''$, by Observation 2.3(a), the vertex $v_6$ has label $C$ in $S*$.

If $d(v_4) = 2$, let $S$ be obtained from the labeling $S^*$ by labeling the vertices $v_1$ and $v_5$ with label $C$, the vertices $v_2$ and $v_4$ with label $A$, the vertex $v_3$ with label $B$. Then, $(T, S)$ can be obtained from $(T'', S^*)$ by operation $\mathcal{O}_2$. Thus, $(T, S) \in \mathcal{S}$.

If $\emptyset \neq N(v_4) \setminus \{v_3, v_5\} \subseteq L(T)$, by Claim 1, $v_4$ has exactly one leaf-neighbor. Let $S^*_1$ be obtained from the labeling $S^*$ by labeling the vertices $v_1$ and $v_5$ with label $C$, the vertices $v_2$ and $v_4$ with label $A$, the vertex $v_3$ with label $B$. $S$ be obtained from the labeling $S^*_1$ by labeling the leaf-neighbor of $v_4$ with label $C$. Then, $(T'', S^*_1)$ can be obtained from $(T'', S^*)$ by operation $\mathcal{O}_2$, and $(T, S)$ can be obtained from $(T'', S^*_1)$ by operation $\mathcal{O}_1$, where $T''$ is obtained from $T$ by deleting the leaf-neighbor of $v_4$. Thus, $(T, S) \in \mathcal{S}$. □

3 A characterization of $(\gamma, \gamma_{2})$-trees

Before presenting a characterization of $(\gamma, \gamma_{2})$-trees, we shall need some additional notation.

Take a star with the center vertex $x$. A subdivided star, denoted by $X$, is obtained from the star by subdividing all edges once. And the tree obtained from the star by subdividing exactly one of the edges once is denoted by $Y$.

An almost dominating set (ADS) of $G$ relative to a vertex $v$ is a set of vertices of $G$ that dominates all vertices of $G$, except possibly for $v$. The almost domination number of $G$ relative to $v$, denoted $\gamma(G; v)$, is the minimum cardinality of an ADS of $G$ relative to $v$. An ADS of $G$ relative to $v$ of cardinality $\gamma(G; v)$ we call a $\gamma(G; v)$-set.
In order to state the characterization of trees with equal domination and semitotal domination numbers, we introduce the four types of operations as follows.

**Operation** \(O_1\): Add a path \(P_1\) and join it to a vertex of \(T\), which is in some \(\gamma_{t2}\)-set of \(T\).

**Operation** \(O_2\): Add a path \(P_2\) or \(P_3\) and join one of its leaves to a vertex \(v\) of \(T\), where \(\gamma(T; v) = \gamma(T)\).

**Operation** \(O_3\): Add a subdivided star \(X\) with at least two leaves and join the center vertex \(x\) to a vertex of \(T\).

**Operation** \(O_4\): Add \(Y\) with three leaves and join a leaf-neighbor of the center vertex \(x\) to a vertex of \(T\).

We define the family \(O\) as:

\[ O = \{ T | T \text{ is obtained from } P_1 \text{ by a finite sequence of operations } O_i, i = 1, 2, 3, 4 \}. \]

We show first that every tree in the family \(O\) has equal domination and semitotal domination numbers.

**Lemma 3.1.** If \(T \in O\), then \(T\) is a \((\gamma, \gamma_{t2})\)-tree.

**Proof:** The proof is by induction on the number \(h(T)\) of operations required to construct the tree \(T\). Observe that \(T = P_1\) when \(h(T) = 0\), and clearly \(\gamma(T) = \gamma_{t2}(T)\). This establishes the base case. Assume that \(k \geq 1\) and each tree \(T' \in O\) with \(h(T') < k\) is a \((\gamma, \gamma_{t2})\)-tree. Let \(T \in O\) be a tree with \(h(T) = k\). Then \(T\) can be obtained from a tree \(T'' \in O\) with \(h(T'') < k\) by one of the operations \(O_i, i = 1, 2, 3, 4\). By induction, \(T''\) is a \((\gamma, \gamma_{t2})\)-tree. By Observation 2.1(i), we can obtain a \(\gamma\)-set of \(T\), say \(S\), which contains no leaf. Now we can distinguish four cases as follows:

**Case 1.** \(T\) is obtained from \(T''\) by operation \(O_1\).

In this case, \(T\) is obtained from \(T''\) by adding a path \(P_1\) and joining it to a vertex of \(T''\), which is in some \(\gamma_{t2}\)-set of \(T''\), say \(D''\). Note that \(D''\) is also a semitotal dominating set of \(T\). That is, \(\gamma_{t2}(T'') \leq \gamma_{t2}(T)\). On the other hand, we have that \(\gamma(T'') = \gamma_{t2}(T'')\). Moreover, since the set \(S\) contains no leaf of \(T\), we have that \(S\) is a dominating set of \(T''\), and then \(\gamma(T'') \leq \gamma(T)\). Hence, \(\gamma(T) \leq \gamma_{t2}(T) \leq \gamma_{t2}(T'') = \gamma(T'') \leq \gamma(T)\). Consequently we must have equality throughout this inequality chain. In particular, \(\gamma(T) = \gamma_{t2}(T)\).

**Case 2.** \(T\) is obtained from \(T''\) by operation \(O_2\).

First, suppose that \(T\) is obtained from \(T''\) by adding a path \(P_2\) and joining one of its vertices, say \(u\), to a vertex \(v\) of \(T''\), where \(\gamma(T''; v) = \gamma(T'')\). Let \(D''\) be a \(\gamma_{t2}\)-set of \(T''\). Clearly, \(D'' \cup \{u\}\) is a semitotal dominating set of \(T\). That is, \(\gamma_{t2}(T) \leq \gamma_{t2}(T'') + 1\). On the other hand, because \(u \in S\), the set \(S \setminus \{u\}\) can dominate all vertices of \(T''\), except possibly the vertex \(v\). It follows from the condition \(\gamma(T''; v) = \gamma(T'')\) that \(\gamma(T) - 1 \geq \gamma(T'')\). Therefore, \(\gamma(T) \leq \gamma_{t2}(T) \leq \gamma_{t2}(T'') + 1 \leq \gamma(T'') + 1 \leq \gamma(T)\). It means that \(\gamma(T) = \gamma_{t2}(T)\).

Next, suppose that \(T\) is obtained from \(T''\) by adding a path \(P_3\) and joining one of its leaves to a vertex \(v\) of \(T\), where \(\gamma(T; v) = \gamma(T)\). Analogously to the previous arguments, we can deduce that \(\gamma(T) = \gamma_{t2}(T)\).

**Case 3.** \(T\) is obtained from \(T''\) by operation \(O_3\).

In this case, \(T\) is obtained from \(T''\) by adding a subdivided star \(X\) with at least two leaves and joining the center vertex \(x\) to a vertex of \(T''\). The set \(D_1\) consists of a \(\gamma_{t2}\)-set of \(T''\) together with all support vertices of \(X\). Clearly, \(D_1\) is a semitotal dominating set of \(T\). Assume that \(X\) contains \(t\) leaves \((t \geq 2)\). Then, \(\gamma_{t2}(T) \leq \gamma_{t2}(T'') + t\). Moreover, it is easy to see that \(\gamma(T) - t \geq \gamma(T'')\). So, \(\gamma(T) \leq \gamma_{t2}(T) \leq \gamma_{t2}(T'') + t = \gamma(T'') + t \leq \gamma(T)\). Consequently we must have equality throughout this inequality chain. In particular, \(\gamma(T) = \gamma_{t2}(T)\).
Case 4. $T$ is obtained from $T'$ by operation $O_4$.

In this case, we can prove $\gamma(T) = \gamma_{t2}(T)$ similar to the proof of Case 3. \qed

Lemma 3.2. If $T$ is a $(\gamma, \gamma_{t2})$-tree, then $T \in \mathcal{O}$.

Proof: We only need to consider the case that $n(T) \geq 6$ and $\text{diam}(T) \geq 4$. Otherwise, $T = P_4$ or $T$ can be obtained from $P_4$ by repeated applications of operation $O_1$. We proceed by induction on the number $n(T)$ of a $(\gamma, \gamma_{t2})$-tree $T$. Assume that the result is true for all $(\gamma, \gamma_{t2})$-tree $T'$ of order $n(T') < n(T)$. By Observation 2.1(ii), we can obtain a $\gamma_{t2}$-set of $T$, say $D$, which contains no leaf. Let $P = v_0v_1v_2 \cdots v_s$ be a longest path of $T$ such that

(i) $d(v_3)$ as large as possible, and subject to this condition

(ii) $d(v_2)$ as large as possible.

Let $z$ be a support vertex of $T$ which has at least two leaves-neighbors. We remove one of these leaves and denote the resulting tree by $T'$.$^3$ Note that $D$ is still a semitotal dominating set of $T'$. That is, $\gamma_{t2}(T') \leq \gamma_{t2}(T)$. By Observation 2.1(i), there is a $\gamma$-set of $T'$, say $S'$, which contains no leaf. Clearly, $z \in S'$ and then $S'$ is also a dominating set of $T$. Therefore, $\gamma(T') \leq \gamma_{t2}(T') \leq \gamma_{t2}(T) = \gamma(T) \leq \gamma(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T') = \gamma_{t2}(T')$ and $z$ is in a $\gamma_{t2}(T')$-set.

By induction, $T' \in \mathcal{O}$. And then, $T$ is obtained from $T'$ by operation $O_1$. So, we assume that each support vertex of $T$ is adjacent to exactly one leaf, for otherwise, we are done. For this reason, $d(v_1) = 2$.

We can distinguish two cases as follows.

Case 1. $v_2$ is a support vertex of $T$.

In this case, $v_1, v_2 \in D$. Because of $\text{diam}(T) \geq 4$, $|D| \geq 3$. And then, one of the two cases as following holds: (1) Each vertex of $D \setminus \{v_1, v_2\}$ is at distance at least 3 from $v_2$; (2) There is a vertex of $D \setminus \{v_1, v_2\}$ which is within distance 2 of $v_2$.

In the former case, if $d(v_3) \geq 3$, let $v'_2$ be a neighbor of $v_3$ outside $P$. From the choice of $P$ and $D$, it is not difficult to verify that the component of $T - v'_2v_3$ containing the vertex $v'_2$ is a subdivided star with at least two leaves, say $X$. Suppose that $X$ contains $t$ leaves. The set obtained by deleting all support vertices of $X$ from $D$ is denoted by $D'$, is still a semitotal dominating set of $T - X$. So, $\gamma_{t2}(T-X) \leq \gamma_{t2}(T) - t$. On the other hand, the set consists of a $\gamma$-set of $T - X$ together with all support vertices of $X$ which is a dominating set of $T$. For this reason, $\gamma(T) \leq \gamma(T-X) + t$. Therefore, $\gamma(T-X) \leq \gamma_{t2}(T-X) \leq \gamma_{t2}(T) - t = \gamma(T) - t \leq \gamma(T-X)$. It concludes that $\gamma(T-X) = \gamma_{t2}(T-X)$. By induction, $T - X \in \mathcal{O}$. Then, $T$ is obtained from $T - X$ by operation $O_3$. If $d(v_3) = 2$, then the component of $T - v_3v_4$ containing $v_3$ is a tree $Y$ with three leaves. With a similar discussion as above, one can prove that $T$ is obtained from $T - Y$ by operation $O_4$.

In the latter case, let $T' = T - \{v_0, v_1\}$. Clearly, the inequality chain $\gamma(T') \leq \gamma_{t2}(T') \leq \gamma_{t2}(T) - 1 = \gamma(T) - 1 \leq \gamma(T')$ holds. And then, $\gamma(T') = \gamma_{t2}(T')$. By induction, $T' \in \mathcal{O}$. Further, we have that $\gamma(T) = \gamma(T') + 1 \geq \gamma(T'; v_2) + 1 \geq \gamma(T)$. That is, $\gamma(T') = \gamma(T'; v_2)$. Hence, $T$ is obtained from $T'$ by operation $O_2$.

Case 2. $v_2$ is not a support vertex of $T$.

In this case, if $d(v_2) \geq 3$, then all neighbors of $v_2$ outside $P$ are support vertices, each of which has exactly one leaf-neighbor. Clearly, the component of $T - v_2v_3$ containing the vertex $v_2$ is a subdivided star
with at least two leaves. Let $D'$ be the set which is obtained from $D$ by deleting all support vertices of the subdivided star. Next, one of the two cases as following holds: (1) Each vertex of $D'$ is at distance at least 3 from $v_1$; (2) There is a vertex of $D'$ which is within distance 2 of $v_1$. In both cases, the same arguments as Case 1 shows that $T \in \mathcal{O}$.

We may assume that $d(v_2) = 2$ by means of the above discussion. Without loss of generality, $v_2 \notin D$ (Otherwise, replacing $v_2$ in $D$ with $v_3$, and the resulting set is also a $\gamma_{t_2}$-set of $T$), and then $v_3 \notin D$. We may assume that $|D| \geq 3$, for otherwise, we are done.

If there exists a vertex of $D \setminus \{v_1, v_3\}$ is within distance 2 of $v_3$. Analogously to Case 1, $T$ is obtained from $T'$ by operation $\mathcal{O}_2$, where $T' = T - \{v_0, v_1\}$, and $T \in \mathcal{O}$.

Thus, each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from $v_3$. From the choice of $P$ and $D$, $v_3$ has only one neighbor outside $P$ which is a leaf or $d(v_3) = 2$.

In the former case, we consider $T' = T - v_0$ and it is easy to show that $T \in \mathcal{O}$. In the latter case, suppose that $d(v_3) \geq 3$ and let $v_3'$ be a neighbor of $v_3$ outside $P$. From the choice of $v_2$, $v_3$ and $D$, the component of $T - v_3'v_4$ containing $v_3'$ is either a subdivided star or a $P_4$. We only need to consider the second case. Let $v_2'$ be the neighbor of $v_3'$ on the $P_4$, and $v_1'$ be the remaining neighbor of $v_2'$ on the $P_4$. Clearly, $v_4' \notin D$. Since each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from $v_3$, $v_2' \notin D$. Replacing $v_2'$ in $D$ with $v_3'$, and the resulting set is also a $\gamma_{t_2}$-set of $T$. Take $T' = T - \{v_0, v_1\}$, and it can be deduced that $T' \in \mathcal{O}$ and $T$ is obtained from $T'$ by operation $\mathcal{O}_2$.

Hence, we may assume that $d(v_3) = 2$. We know that $v_1, v_3 \in D$. Because each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from $v_3$. Then, let $T' = T - \{v_0, v_1, v_2, v_3, v_4\}$. Observe that $D \setminus \{v_1, v_3\}$ is a semitotal dominating set of $T'$. Moreover, we have that $\gamma(T') \leq \gamma_{t_2}(T') \leq \gamma_{t_2}(T) = \gamma(T) - 2 \leq \gamma(T)$. Thus, $\gamma(T') = \gamma_{t_2}(T')$. By induction, $T' \in \mathcal{O}$. In addition, let $D'$ be a $\gamma(T'; v_5)$-set of $T'$ and $D'' = D' \cup \{v_1, v_4\}$. We can see that $D''$ dominates all vertices of $T$. That is, $\gamma(T'; v_5) + 2 \geq \gamma(T)$. It follows from $\gamma(T) = \gamma(T') + 2 \geq \gamma(T'; v_5) + 2 \geq \gamma(T)$ that $\gamma(T'; v_5) = \gamma(T')$. Hence, $T$ is obtained from $T'$ by operation $\mathcal{O}_2$.

The proof is completed. \hfill \Box

As an immediate consequence of Lemmas 3.1 and 3.2 we have the following characterization of $(\gamma, \gamma_{t_2})$-trees.

**Theorem 3.3.** A tree $T$ is a $(\gamma, \gamma_{t_2})$-tree if and only if $T \in \mathcal{O}$.

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**References**


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