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Semitotal domination in trees *

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In this paper, we study a parameter that is squeezed between arguably the two important domination parameters, namely the domination number, $\gamma(G)$, and the total domination number, $\gamma_t(G)$. A set S of vertices in G is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number, $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. We observe that $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. In this paper, we give a lower bound for the semitotal domination number of trees and we characterize the extremal trees. In addition, we characterize trees with equal domination and semitotal domination numbers.

Keywords: domination, semitotal domination, tree

1 Introduction

Let $G = (V, E)$ be a graph without isolated vertices with vertex set V of order $n(G) = |V|$ and edge set E of size $m(G) = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d(v) = |N(v)|$. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ between u and v is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of a graph G which is denoted by $diam(G)$. A *leaf* of G is a vertex of degree 1 and a *support vertex* of G is a vertex adjacent to a leaf. Denote the sets of leaves and support vertices of G by $L(T)$ and $S(T)$, respectively. Let $l(T) = |L(T)|$ and $s(T) = |S(T)|$. A *double star* is a tree that contains exactly two vertices that are not leaves.

A dominating set in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A total dominating set of a graph G with no isolated vertex is a set D of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A dominating (total dominating) set of G of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a $\gamma(G)$-set ($\gamma_t(G)$-set).

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The concept of semitotal domination in graphs was introduced and studied by Goddard, Henning and McPillan [3]. A set S of vertices in a graph G with no isolated vertices is a semitotal dominating set of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. A semitotal dominating set of G of cardinality $\gamma_{t2}(G)$ is called a $\gamma_{t2}(G)$-set. Clearly, for every graph G with no isolated vertex, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. If the graph G is clear from the context, we simply write γ-set and γ_{t2}-set rather than $\gamma(G)$-set and $\gamma_{t2}(G)$-set, respectively.

An area of research in domination of graphs that has received considerable attention is the study of classes of graphs with equal domination parameters. For any two graph theoretic parameters λ and μ, G is called a (λ, μ)-graph if $\lambda(G) = \mu(G)$. The class of (γ, γ_t)-trees, that is trees with equal domination and total domination numbers, was characterized in [6]. In [4], the authors provided a constructive characterizations of trees with equal domination and paired domination numbers. More results in this area were investigated in [7, 9, 8, 1] and elsewhere. Motivated by these results, we aim to characterize trees with equal domination and semitotal domination numbers. In addition, we give a lower bound for the semitotal domination number of trees and we characterize the extremal trees.

2 A lower bound for semitotal domination number of trees

In this section we give a lower bound for the semitotal domination number of trees and we characterize the extremal trees. First, we shall need the following two observations.

Observation 2.1. *Let G be a connected graph that is not a star. Then,*

(i) *there is a γ-set of G that contains no leaf, and*

(ii)*[5] there is a γ_{t2}-set of G that contains no leaf.*

Theorem 2.2. *If T is a tree of order $n(T) \geq 2$ with $l(T)$ leaves, then $\gamma_{t2}(T) \geq \frac{2[n(T)-l(T)+2]}{5}$.*

Proof: We use induction on $n(T)$. It is easy to see that the result holds for a tree of order $n \leq 8$. Let T be a tree of order $n > 8$ and assume that $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$ for each tree T' with order at most $n-1$. We consider the case that $diam(T) \geq 4$. Otherwise, T is a star or double-star, then $\gamma_{t2}(T)$ has the desired property in theorem. By Observation 2.1(ii), we can obtain a γ_{t2}-set of T, say D, which contains no leaf.

Claim 1. For any vertex $v \in V(T) \setminus L(T)$, v has only one leaf-neighbor when $|N(v) \setminus L(T)| = 1$, and v is not a support vertex when $|N(v) \setminus L(T)| \geq 2$.

Proof: If v is a vertex that has at least two leaf-neighbors and $|N(v) \setminus L(T)| = 1$. We remove one of those leaves and denote the resulting tree by T'. It is easy to observe that $\gamma_{t2}(T') = \gamma_{t2}(T)$. By induction, $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$. And consequently $\gamma_{t2}(T) \geq \frac{2[n(T)-l(T)+2]}{5}$ as $l(T') = l(T)-1$, $n(T') = n(T)-1$.

If v is a support vertex and $|N(v) \setminus L(T)| \geq 2$, we remove a leaf-neighbor of v and the semitotal domination number of the resulting tree is no greater than that of T. Analogously to the previous case, $\gamma_{t2}(T)$ has the desired property in theorem. □

In other words, each support vertex of T has degree two. Let $P = v_0v_1v_2 \cdots v_t$ be a longest path in T such that

(i) $d(v_3)$ as large as possible, and subject to this condition

(ii) $d(v_2)$ as large as possible.

By Claim 1, $d(v_1)$=2 and v_2 is not a support vertex.

Claim 2. $d(v_2) = 2$.

Proof: If $d(v_2) > 2$, it follows from the choice of P and Claim 1 that all neighbors of v_2 are support vertices of degree two, except possibly the vertex v_3.

Let u_1 be a neighbor of v_2 outside P, u_2 be the leaf that adjacent to u_1, and $T' = T - \{v_0, u_2\}$. By induction, $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$. In addition, replacing the vertices u_1 and v_1 in D with v_2(If $v_2 \in D$, take $D \setminus \{v_1\}$ instead), we can obtain a semitotal dominating set of T'. That is, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$. Note that $l(T') = l(T)$, $n(T') = n(T) - 2$. Therefore, $\gamma_{t2}(T) \geq \frac{2[n(T)-l(T)+2]}{5}$. □

We know that $v_1 \in D$ and exactly one of v_2 and v_3 belongs to D. Without loss of generality, $v_3 \in D$ (Otherwise, we replace the vertex v_2 in D with v_3, and the resulting set is also a γ_{t2}-set of T).

Claim 3. $d(v_3) = 2$.

Proof: By Claim 1 and the assumption that $n > 8$, v_3 is not a support vertex. If $d(v_3) > 2$, it follows from the choice of P and Claim 1 that v_3 has a neighbor of degree two outside P, say v_2', which is either a support vertex or adjacent to a support vertex outside P, say v_1'.

In the former case, we have that $\{v_1, v_3, v_2'\} \subseteq D$. And in the latter case, we have that $\{v_1, v_3, v_1'\} \subseteq D$. Let $T' = T - \{v_0, v_1\}$. By induction, $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$. In either case, we have that $l(T') = l(T)$, $n(T') = n(T) - 2$ and it is easy to see that $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$. Therefore, $\gamma_{t2}(T) \geq \frac{2[n(T)-l(T)+2]}{5}$. □

Claim 4. $d(v_4) = 2$.

Proof: By Claim 1 and the assumption that $n > 8$, v_4 is not a support vertex. If $d(v_4) > 2$, from the choice of P and Claim 1, we only need to consider the case as follows: v_4 has a neighbor outside P, say v_3', which is adjacent to t support vertices $u_1, u_2, \cdots, u_t$, where $t \geq 2$. (In other cases, we always have that $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$, $l(T') = l(T)$ and $n(T') = n(T) - 2$, where $T' = T - \{v_0, v_1\}$. And similar to the proof of Claim 3, $\gamma_{t2}(T)$ has the desired property in theorem.) Let u_i' be the leaf-neighbor of u_i, where $i = 1, 2, \cdots, t$. Let $T' = T - \{u_1', u_2', \cdots, u_t'\}$. By induction, $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$. Note that $\{u_1, u_2, \cdots, u_t\} \subseteq D$. Then $(D \setminus \{u_1, u_2, \cdots, u_t\}) \cup \{v_3'\}$ is a semitotal dominating set of T'. That is, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - t + 1$. In addition, $l(T') = l(T)$, $n(T') = n(T) - t$. Hence, $\gamma_{t2}(T) \geq \frac{2[n(T')-l(T')+2]}{5} + t - 1 = \frac{2[n(T)-t-l(T)+2]}{5} + t - 1 > \frac{2[n(T)-l(T)+2]}{5}$. □

Note $v_1, v_3 \in D$. Then, one of the two cases as following holds: (1) Each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from v_3; (2) There is a vertex of $D \setminus \{v_1, v_3\}$ which is within distance 2 of v_3.

In the former case, let $T' = T - \{v_0, v_1, v_2, v_3, v_4\}$. By induction, $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$. In addition, note that $D \setminus \{v_1, v_3\}$ is a semitotal dominating set of T', $n(T) = n(T') + 5$, $l(T) \geq l(T')$. Hence, $\gamma_{t2}(T) \geq \frac{2[n(T)-l(T)+2]}{5}$.

In the latter case, let $T' = T - \{v_0, v_1\}$. By induction, $\gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5}$. Since $D \setminus \{v_1\}$ is a semitotal dominating set of T', $n(T) = n(T') + 2$, $l(T) = l(T')$. Hence, $\gamma_{t2}(T) \geq \frac{2[n(T)-l(T)+2]}{5}$.

The proof is completed. □

Next, we are ready to provide a constructive characterization of the trees achieving equality in the bound of Theorem 2.2. For our purposes we define a *labeling* of a tree T as a partition $S = (S_A, S_B, S_C)$ of $V(T)$ (This idea of labeling the vertices is introduced in [2]). We will refer to the pair (T, S) as a *labeled tree*. The label or *status* of a vertex v, denoted sta(v), is the letter $x \in \{A, B, C\}$ such that $v \in S_x$.

Let $\mathscr{T}$ be the family of labeled trees that: (i) contains (P_5, S') where S' is the labeling that assigns to the two support vertices of the path P_5 status A, to the two leaves status C and to the center vertex status B (see Fig.1(a)); and (ii) is closed under the two operations $\mathscr{O}_1$ and $\mathscr{O}_2$ that are listed below, which extend the tree T' to a tree T by attaching a tree to the vertex $v \in V(T')$.

Operation $\mathscr{O}_1$: Let v be a vertex with sta$(v) = A$. Add a vertex u and the edge uv. Let sta$(u) = C$.

Operation $\mathscr{O}_2$: Let v be a vertex with sta$(v) = C$ that has degree one. Add a path $u_1u_2u_3u_4u_5$ and the edge u_1v. Let sta(u_1) =sta$(u_5) = C$, sta(u_2) =sta$(u_4) = A$, sta$(u_3) = B$.

The two operations $\mathscr{O}_1$ and $\mathscr{O}_2$ are illustrated in Fig.1(b), (c).

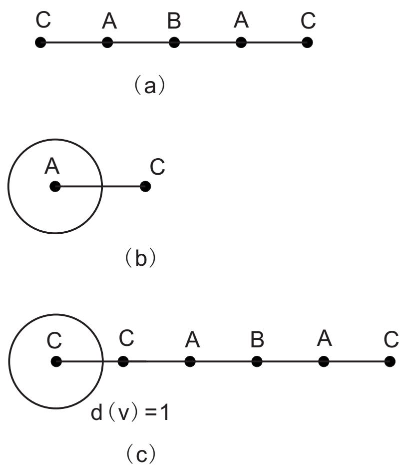

Fig. 1

Let $(T, S) \in \mathscr{T}$ be a labeled tree for some labeling S. Then there is a sequence of labeled trees (T_0, S_0), $(T_1, S_1), \cdots, (T_{k-1}, S_{k-1})$, (T_k, S_k) such that $(T_0, S_0) = (P_5, S')$, $(T_k, S_k) = (T, S)$. The labeled tree (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations $\mathscr{O}_1$ and $\mathscr{O}_2$, where $i \in \{1, 2, \cdots, k\}$. We call the number of terms in such a sequence of labeled trees that is used to construct (T, S), the *length* of the sequence. Clearly, the above sequence has length k. We remark that a sequence of labeled trees used to construct (T, S) is not necessarily unique.

We take an example to make it easier for reader to understand the family $\mathscr{T}$. In Fig.2, $(P_5, S') \in \mathscr{T}$, (H_1, S_1) is obtained from (P_5, S') by operation $\mathscr{O}_2$, (H_2, S_2) is obtained from (H_1, S_1) by repeated applications of operation $\mathscr{O}_1$, and (H_3, S) is obtained from (H_2, S_2) by operation $\mathscr{O}_2$. Thus, (H_1, S_1), (H_2, S_2), $(H_3, S) \in \mathscr{T}$. For $T \in \{P_5, H_1, H_2, H_3\}$, it is easy to see that the set, say D, consisting of the vertices labeled A in T is a γ_{t2}-set of T. In particular, $|D| = \frac{2[n(T)-l(T)+2]}{5}$.

Before presenting our main result, we present a few preliminary results and observations.

Observation 2.3. *Let T be a tree and let S be a labeling of T such that $(T, S) \in \mathscr{T}$. Then, T has the following properties:*

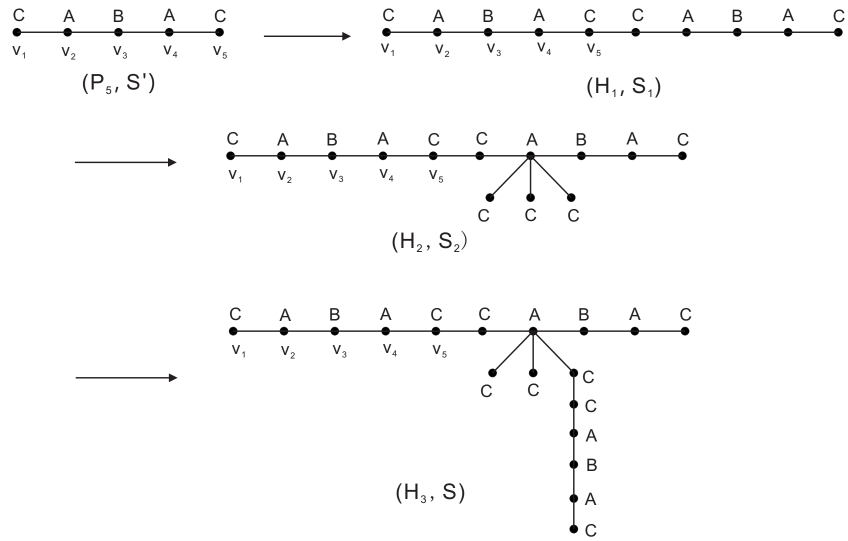

Fig. 2

(a) Every support vertex is labeled A and every leaf is labeled C.

(b) $|S_A| = 2|S_B|$.

(c) The set S_A is a semitotal dominating set of T.

(d) The set S_A and S_B are independent sets.

(e) Every vertex labeled B has degree two and its neighbors labeled A.

Lemma 2.4. *Let T be a tree and let S be a labeling of T such that $(T, S) \in \mathscr{T}$. Then, $\gamma_{t2}(T) = \frac{2[n(T)-l(T)+2]}{5}$.*

Proof: First, we are ready to show that $|S_A| = \frac{2[n(T)-l(T)+2]}{5}$. We proceed by induction on the length k of a sequence required to construct the labeled tree (T, S).

When $k = 0$, $(T, S) = (P_5, S')$, and so $|S_A| = 2$. This establishes the base case. Let $k \geq 1$ and assume that if the length of sequence used to construct a labeled tree $(T^*, S^*) \in \mathscr{T}$ is less than k, then $|S_A^*| = \frac{2[n(T^*)-l(T^*)+2]}{5}$. Now, $(T, S) \in \mathscr{T}$ and let (T_0, S_0), $(T_1, S_1), \cdots, (T_{k-1}, S_{k-1})$, (T_k, S_k) be a sequence of length k used to construct (T, S), where $(T_0, S_0) = (P_5, S')$, $(T_k, S_k) = (T, S)$, (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations $\mathscr{O}_1$ and $\mathscr{O}_2$, $i \in \{1, 2, \cdots, k\}$. Let $T^* = T_{k-1}$ and $S^* = S_{k-1}$. Note that $(T_{k-1}, S_{k-1}) \in \mathscr{T}$. By the inductive hypothesis, $|S_A^*| = \frac{2[n(T^*)-l(T^*)+2]}{5}$. (T, S) can be obtained from (T^*, S^*) by operation $\mathscr{O}_1$ or $\mathscr{O}_2$.

In the former case, we have that $n(T) = n(T^*) + 1$, $l(T) = l(T^*) + 1$, and $|S_A| = |S_A^*|$. Thus, $|S_A| = \frac{2[n(T^*)-l(T^*)+2]}{5} = \frac{2[n(T)-1-l(T)+1+2]}{5} = \frac{2[n(T)-l(T)+2]}{5}$.

In the latter case, we have that $n(T) = n(T^*) + 5$, $l(T) = l(T^*)$ and $|S_A| = |S_A^*| + 2$. Thus, $|S_A| = \frac{2[n(T^*)-l(T^*)+2]}{5} + 2 = \frac{2[n(T)-5-l(T)+2]}{5} + 2 = \frac{2[n(T)-l(T)+2]}{5}$.

By Observation 2.3(c), we have that $\gamma_{t2}(T) \leq \frac{2[n(T)-l(T)+2]}{5}$. Combining Theorem 2.2, we conclude that $\gamma_{t2}(T) = \frac{2[n(T)-l(T)+2]}{5}$. Moreover, S_A is a γ_{t2}-set of T. □

Theorem 2.5. *Let T be a nontrivial tree, then $\gamma_{t2}(T) = \frac{2[n(T)-l(T)+2]}{5}$ if and only if $(T, S) \in \mathscr{T}$ for some labeling S.*

Proof: The sufficiency follows immediately from Lemma 2.4. So we prove the necessity only. The proof is by induction on the order of T. The result is immediate for $n \leq 5$. For the inductive hypothesis, let $n \geq 6$ and moreover, $diam(T) \geq 4$ (If $diam(T) \leq 3$, T is a star or a double star, and then $\gamma_{t2}(T) > \frac{2[n(T)-l(T)+2]}{5}$, a contradiction). Assume that for every nontrivial tree T' of order less than n with $\gamma_{t2}(T') = \frac{2[n(T')-l(T')+2]}{5}$, we have that $(T', S^*) \in \mathscr{T}$ for some labeling S^*. Let T be a tree of order n satisfying $\gamma_{t2}(T) = \frac{2[n(T)-l(T)+2]}{5}$. Let $P = v_1v_2\cdots v_t$ be a longest path in T such that

(i) $d(v_4)$ as large as possible, and subject to this condition

(ii) $d(v_3)$ as large as possible.

Let D be a γ_{t2}-set of T which contains no leaf.

Claim 1. Each support vertex has exactly one leaf-neighbor.

Proof: If not, assume that there is a support vertex u which is adjacent to at least two leaves. Deleting one of its leaf-neighbors, say u_1, and denote the resulting tree by T'. D is still a semitotal dominating set of T'. That is, $\gamma_{t2}(T') \leq \gamma_{t2}(T) = \frac{2[n(T)-l(T)+2]}{5} = \frac{2[n(T')+1-l(T')-1+2]}{5} = \frac{2[n(T')-l(T')+2]}{5}$. Combining Theorem 2.2, we have that $\gamma_{t2}(T') = \frac{2[n(T')-l(T')+2]}{5}$. By the inductive hypothesis, $(T', S^*) \in \mathscr{T}$ for some labeling S^*. Since u is still a support vertex in T', by Observation 2.3(a), the vertex u has label A in S^*. Let S be obtained from the labeling S^* by labeling the vertex u_1 with label C. Then, (T, S) can be obtained from (T', S^*) by operation $\mathscr{O}_1$. Thus, $(T, S) \in \mathscr{T}$. $\square$

By Claim 1, we can assume that $d(v_2) = 2$. Now, we consider the vertex v_3. If v_3 is a support vertex, then $v_2, v_3 \in D$. Let T' be the tree which is obtained from T by subdividing the edge v_2v_3. It is easy to see that D is still a semitotal dominating set of T', and it means that $\frac{2[n(T)-l(T)+2]}{5} = \gamma_{t2}(T) \geq \gamma_{t2}(T') \geq \frac{2[n(T')-l(T')+2]}{5} = \frac{2[n(T)+1-l(T)+2]}{5} = \frac{2[n(T)-l(T)+3]}{5}$, a contradiction. So, v_3 is not a support vertex.

Assume that $d(v_3) \geq 3$. Then, it follows from the choice of P that v_3 is adjacent to a support vertex, say u, which does not belong to P. Clearly, $u, v_2 \in D$. Moreover, $v_3 \notin D$. Otherwise, we subdivide the edges v_2v_3 and uv_3, and yield a similar contradiction as above.

If u is within distance two from a vertex in $D \setminus \{u, v_2\}$, we have that $\frac{2[n(T)-2-(l(T)-1)+2]}{5} \leq \gamma_{t2}(T') \leq \gamma_{t2}(T) - 1 = \frac{2[n(T)-l(T)+2]}{5} - 1$, where $T' = T - \{v_1, v_2\}$. It is impossible. It follows that $v_4 \notin D$, but in this case, let T'' be the component of $T - v_3v_4$ containing the vertex v_4, and $\frac{2[n(T)-l(T)+2]}{5} = \gamma_{t2}(T) \geq \gamma_{t2}(T'') + \gamma_{t2}(T - T'') \geq \frac{2[n(T'')-l(T'')+2]}{5} + \frac{2[n(T-T'')-l(T-T'')+2]}{5} \geq \frac{2[n(T)-(l(T)+1)+4]}{5} = \frac{2[n(T)-l(T)+3]}{5}$, a contradiction. Therefore, $d(v_3) = 2$.

From the choice of D, $v_2 \in D$, and without loss of generality, $v_4 \in D$ (If $v_4 \notin D$, then $v_3 \in D$, replacing v_3 in D with v_4, and we obtain a new γ_{t2}-set of T).

Assume that $d(v_4) \geq 3$. We have that the following conclusion.

Claim 2. $N(v_4) \setminus \{v_3, v_5\} \subset L(T)$.

Proof: Assume that there exists a vertex $v_3' \in N(v_4) \setminus \{v_3, v_5\}$ which is not a leaf, it follows from the choice of P and Claim 1 that v_3' is either a support vertex or adjacent to a support vertex outside P, say v_2'. In particular, $d(v_3') = 2$ (From the choice of P). In either case, let $T' = T - \{v_1, v_2\}$. Observe that $n(T) = n(T') + 2$, $l(T) = l(T')$, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$. Then, we have that $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1 = \frac{2[n(T)-l(T)+2]}{5} - 1 = \frac{2[n(T')+2-l(T')+2]}{5} - 1 = \frac{2[n(T')-l(T')+2]}{5} - \frac{1}{5} < \frac{2[n(T')-l(T')+2]}{5}$, contradicting Theorem 2.2. It concludes that $\emptyset \neq N(v_4) \setminus \{v_3, v_5\} \subset L(T)$. $\square$

So $d(v_4) = 2$ or $N(v_4) \setminus \{v_3, v_5\} \subset L(T)$. Moreover, all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from v_4 (If not, let $T' = T - \{v_1, v_2\}$. Observe that $n(T) = n(T') + 2$, $l(T) = l(T')$, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$. We can obtain a contradiction by an argument similar to the proof of Claim 2).

If $d(v_5) = 1$, by Claim 1 and the choice of P, $T = P_5$, contradicting the assumption that $n \geq 6$. So assume that $d(v_5) \geq 3$, since all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from v_4, each neighbor of v_5 is neither a leaf nor a support vertex. From the choice of P and Claim 1, we only need to consider the case as follows: v_5 has a neighbor outside P, say v_4', which is adjacent to t support vertices $u_1, u_2, \cdots, u_t$, where $t \geq 2$. (In other cases, let $T' = T - \{v_1, v_2\}$. Observe that $n(T) = n(T') + 2$, $l(T) = l(T')$, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$. We can always obtain contradictions by an argument similar to the proof of Claim 2). Let u_i' be the leaf-neighbor of u_i, where $i = 1, 2, \cdots, t$, and $T' = T - \{u_1', u_2', \cdots, u_t'\}$. Note that $\{u_1, u_2, \cdots, u_t\} \subseteq D$. Then $(D \setminus \{u_1, u_2, \cdots, u_t\}) \cup \{v_4'\}$ is a semitotal dominating set of T'. That is, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - t + 1$. In addition, $l(T') = l(T)$, $n(T') = n(T) - t$. Hence, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - t + 1 = \frac{2[n(T)-l(T)+2]}{5} - t + 1 = \frac{2[n(T')+t-l(T')+2]}{5} - t + 1 = \frac{2[n(T')-l(T')+2]}{5} + \frac{2t}{5} - t + 1 < \frac{2[n(T')-l(T')+2]}{5}$, contradicting Theorem 2.2. Therefore, $d(v_5) = 2$.

Let T' be the component of $T - v_5v_6$ containing v_6. If v_6 is not a leaf in T', then $n(T) = n(T') + 5 + s$, $l(T) = l(T') + 1 + s$, where s is the number of the leaf-neighbors of v_4. Since all vertices in $D \setminus \{v_2, v_4\}$ are distance at least three from v_4, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 2$. It follows that $\gamma_{t2}(T') \leq \frac{2[n(T)-l(T)+2]}{5} - 2 = \frac{2[n(T')+5+s-l(T')-1-s+2]}{5} - 2 = \frac{2[n(T')-l(T')+2]}{5} - \frac{2}{5} < \frac{2[n(T')-l(T')+2]}{5}$, contradicting Theorem 2.2. It means that v_6 is a leaf in T', and $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 2 = \frac{2[n(T)-l(T)+2]}{5} - 2 = \frac{2[n(T')+5+s-l(T')-s+2]}{5} - 2 = \frac{2[n(T')-l(T')+2]}{5}$. Combining Theorem 2.2, we have that $\gamma_{t2}(T') = \frac{2[n(T')-l(T')+2]}{5}$. By the inductive hypothesis, $(T', S^*) \in \mathscr{T}$ for some labeling S^*. Since v_6 is a leaf in T', by Observation 2.3(a), the vertex v_6 has label C in S^*.

If $d(v_4) = 2$, let S be obtained from the labeling S^* by labeling the vertices v_1 and v_5 with label C, the vertices v_2 and v_4 with label A, the vertex v_3 with label B. Then, (T, S) can be obtained from (T', S^*) by operation $\mathscr{O}_2$. Thus, $(T, S) \in \mathscr{T}$.

If $\emptyset \neq N(v_4) \setminus \{v_3, v_5\} \subset L(T)$, by Claim 1, v_4 has exactly one leaf-neighbor. Let S_1^* be obtained from the labeling S^* by labeling the vertices v_1 and v_5 with label C, the vertices v_2 and v_4 with label A, the vertex v_3 with label B. S be obtained from the labeling S_1^* by labeling the leaf-neighbor of v_4 with label C. Then, (T'', S_1^*) can be obtained from (T', S^*) by operation $\mathscr{O}_2$, and (T, S) can be obtained from (T'', S_1^*) by operation $\mathscr{O}_1$, where T'' is obtained from T by deleting the leaf-neighbor of v_4. Thus, $(T, S) \in \mathscr{T}$. $\square$

3 A characterization of (γ, γ_{t2})-trees

Before presenting a characterization of (γ, γ_{t2})-trees, we shall need some additional notation.

Take a star with the center vertex x. A subdivided star, denoted by X, is obtained from the star by subdividing all edges once. And the tree obtained from the star by subdividing exactly one of the edges once is denoted by Y.

An *almost dominating set* (ADS) of G relative to a vertex v is a set of vertices of G that dominates all vertices of G, except possibly for v. The *almost domination number* of G relative to v, denoted $\gamma(G; v)$, is the minimum cardinality of an ADS of G relative to v. An ADS of G relative to v of cardinality $\gamma(G; v)$ we call a $\gamma(G; v)$-set.

In order to state the characterization of trees with equal domination and semitotal domination numbers, we introduce the four types of operations as follows.

Operation $\mathscr{O}_1$: Add a path P_1 and join it to a vertex of T, which is in some γ_{t2}-set of T.

Operation $\mathscr{O}_2$: Add a path P_2 or P_5 and join one of its leaves to a vertex v of T, where $\gamma(T;v) = \gamma(T)$.

Operation $\mathscr{O}_3$: Add a subdivided star X with at least two leaves and join the center vertex x to a vertex of T.

Operation $\mathscr{O}_4$: Add Y with three leaves and join a leaf-neighbor of the center vertex x to a vertex of T.

We define the family $\mathscr{O}$ as:

$\mathscr{O} = \{T|T$ is obtained from P_4 by a finite sequence of operations $\mathscr{O}_i$, $i = 1, 2, 3, 4\}$. We show first that every tree in the family $\mathscr{O}$ has equal domination and semitotal domination numbers.

Lemma 3.1. *If $T \in \mathscr{O}$, then T is a (γ, γ_{t2})-tree.*

Proof: The proof is by induction on the number $h(T)$ of operations required to construct the tree T. Observe that $T = P_4$ when $h(T) = 0$, and clearly $\gamma(T) = \gamma_{t2}(T)$. This establishes the base case. Assume that $k \geq 1$ and each tree $T' \in \mathscr{O}$ with $h(T') < k$ is a (γ, γ_{t2})-tree. Let $T \in \mathscr{O}$ be a tree with $h(T) = k$. Then T can be obtained from a tree $T' \in \mathscr{O}$ with $h(T') < k$ by one of the operations $\mathscr{O}_i$, $i = 1, 2, 3, 4$. By induction, T' is a (γ, γ_{t2})-tree. By Observation 2.1(i), we can obtain a γ-set of T, say S, which contains no leaf. Now we can distinguish four cases as follows:

Case 1. T is obtained from T' by operation $\mathscr{O}_1$.

In this case, T is obtained from T' by adding a path P_1 and joining it to a vertex of T', which is in some γ_{t2}-set of T', say D'. Note that D' is also a semitotal dominating set of T. That is, $\gamma_{t2}(T') \geq \gamma_{t2}(T)$. On the other hand, we have that $\gamma(T') = \gamma_{t2}(T')$. Moreover, since the set S contains no leaf of T, we have that S is a dominating set of T', and then $\gamma(T') \leq \gamma(T)$. Hence, $\gamma(T) \leq \gamma_{t2}(T) \leq \gamma_{t2}(T') = \gamma(T') \leq \gamma(T)$. Consequently we must have equality throughout this inequality chain. In particular, $\gamma(T) = \gamma_{t2}(T)$.

Case 2. T is obtained from T' by operation $\mathscr{O}_2$.

First, suppose that T is obtained from T' by adding a path P_2 and joining one of its vertices, say u, to a vertex v of T', where $\gamma(T';v) = \gamma(T')$. Let D' be a γ_{t2}-set of T'. Clearly, $D' \cup \{u\}$ is a semitotal dominating set of T. That is, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1$. On the other hand, because $u \in S$, the set $S \setminus \{u\}$ can dominate all vertices of T', except possibly the vertex v. It follows from the condition $\gamma(T';v) = \gamma(T')$ that $\gamma(T) - 1 \geq \gamma(T')$. Therefore, $\gamma(T) \leq \gamma_{t2}(T) \leq \gamma_{t2}(T') + 1 = \gamma(T') + 1 \leq \gamma(T)$. It means that $\gamma(T) = \gamma_{t2}(T)$.

Next, suppose that T is obtained from T' by adding a path P_5 and joining one of its leaves to a vertex v of T, where $\gamma(T;v) = \gamma(T)$. Analogously to the previous arguments, we can deduce that $\gamma(T) = \gamma_{t2}(T)$.

Case 3. T is obtained from T' by operation $\mathscr{O}_3$.

In this case, T is obtained from T' by adding a subdivided star X with at least two leaves and joining the center vertex x to a vertex of T'. The set D_1 consists of a γ_{t2}-set of T' together with all support vertices of X. Clearly, D_1 is a semitotal dominating set of T. Assume that X contains t leaves ($t \geq 2$). Then, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + t$. Moreover, it is easy to see that $\gamma(T) - t \geq \gamma(T')$. So, $\gamma(T) \leq \gamma_{t2}(T) \leq \gamma_{t2}(T') + t = \gamma(T') + t \leq \gamma(T)$. Consequently we must have equality throughout this inequality chain. In particular, $\gamma(T) = \gamma_{t2}(T)$.

Case 4. T is obtained from T' by operation $\mathscr{O}_4$.

In this case, we can prove $\gamma(T) = \gamma_{t2}(T)$ similar to the proof of Case 3. □

Lemma 3.2. *If T is a (γ, γ_{t2})-tree, then $T \in \mathscr{O}$.*

Proof: We only need to consider the case that $n(T) \geq 6$ and $diam(T) \geq 4$. Otherwise, $T = P_4$ or T can be obtained from P_4 by repeated applications of operation $\mathscr{O}_1$. We proceed by induction on the order $n(T)$ of a (γ, γ_{t2})-tree T. Assume that the result is true for all (γ, γ_{t2})-tree T' of order $n(T') < n(T)$. By Observation 2.1(ii), we can obtain a γ_{t2}-set of T, say D, which contains no leaf. Let $P = v_0v_1v_2 \cdots v_s$ be a longest path of T such that

(i) $d(v_3)$ as large as possible, and subject to this condition

(ii) $d(v_2)$ as large as possible.

Let z be a support vertex of T which has at least two leaf-neighbors. We remove one of these leaves and denote the resulting tree by T'. Note that D is still a semitotal dominating set of T'. That is, $\gamma_{t2}(T') \leq \gamma_{t2}(T)$. By Observation 2.1(i), there is a γ-set of T', say S', which contains no leaf. Clearly, $z \in S'$ and then S' is also a dominating set of T. Therefore, $\gamma(T') \leq \gamma_{t2}(T') \leq \gamma_{t2}(T) = \gamma(T) \leq \gamma(T')$. Consequently we must have equality throughout this inequality chain. In particular, $\gamma(T') = \gamma_{t2}(T')$ and z is in a $\gamma_{t2}(T')$-set. By induction, $T' \in \mathscr{O}$. And then, T is obtained from T' by operation $\mathscr{O}_1$. So, we assume that each support vertex of T is adjacent to exactly one leaf, for otherwise, we are done. For this reason, $d(v_1) = 2$.

We can distinguish two cases as follows.

Case 1. v_2 is a support vertex of T.

In this case, $v_1, v_2 \in D$. Because of $diam(T) \geq 4$, $|D| \geq 3$. And then, one of the two cases as following holds: (1) Each vertex of $D \setminus \{v_1, v_2\}$ is at distance at least 3 from v_2; (2) There is a vertex of $D \setminus \{v_1, v_2\}$ which is within distance 2 of v_2.

In the former case, if $d(v_3) \geq 3$, let v_2' be a neighbor of v_3 outside P. From the choice of P and D, it is not difficult to verify that the component of $T - v_2'v_3$ containing the vertex v_2' is a subdivided star with at least two leaves, say X. Suppose that X contains t leaves. The set obtained by deleting all support vertices of X from D is denoted by D', is still a semitotal dominating set of $T - X$. So, $\gamma_{t2}(T - X) \leq \gamma_{t2}(T) - t$. On the other hand, the set consists of a γ-set of $T - X$ together with all support vertices of X is a dominating set of T. For this reason, $\gamma(T) \leq \gamma(T - X) + t$. Therefore, $\gamma(T-X) \leq \gamma_{t2}(T-X) \leq \gamma_{t2}(T)-t = \gamma(T)-t \leq \gamma(T-X)$. It concludes that $\gamma(T-X) = \gamma_{t2}(T-X)$. By induction, $T - X \in \mathscr{O}$. Then, T is obtained from $T - X$ by operation $\mathscr{O}_3$. If $d(v_3) = 2$, then the component of $T - v_3v_4$ containing v_3 is a tree Y with three leaves. With a similar discussion as above, one can prove that T is obtained from $T - Y$ by operation $\mathscr{O}_4$.

In the latter case, let $T' = T - \{v_0, v_1\}$. Clearly, the inequality chain $\gamma(T') \leq \gamma_{t2}(T') \leq \gamma_{t2}(T) - 1 = \gamma(T) - 1 \leq \gamma(T')$ holds. And then, $\gamma(T') = \gamma_{t2}(T')$. By induction, $T' \in \mathscr{O}$. Further, we have that $\gamma(T) = \gamma(T') + 1 \geq \gamma(T'; v_2) + 1 \geq \gamma(T)$. That is, $\gamma(T') = \gamma(T'; v_2)$. Hence, T is obtained from T' by operation $\mathscr{O}_2$.

Case 2. v_2 is not a support vertex of T.

In this case, if $d(v_2) \geq 3$, then all neighbors of v_2 outside P are support vertices, each of which has exactly one leaf-neighbor. Clearly, the component of $T - v_2v_3$ containing the vertex v_2 is a subdivided star with at least two leaves. Let D' be the set which is obtained from D by deleting all support vertices of the subdivided star. Next, one of the two cases as following holds: (1) Each vertex of D' is at distance at least 3 from v_1; (2) There is a vertex of D' which is within distance 2 of v_1. In both cases, the same arguments as Case 1 shows that $T \in \mathscr{O}$.

We may assume that $d(v_2) = 2$ by means of the above discussion. Without loss of generality, $v_2 \notin D$ (Otherwise, replacing v_2 in D with v_3, and the resulting set is also a γ_{t2}-set of T), and then $v_3 \in D$. We may assume that $|D| \geq 3$, for otherwise, we are done.

If there exists a vertex of $D \setminus \{v_1, v_3\}$ is within distance 2 of v_3. Analogously to Case 1, T is obtained from T' by operation $\mathscr{O}_2$, where $T' = T - \{v_0, v_1\}$, and $T \in \mathscr{O}$.

Thus, each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from v_3. From the choice of P and D, v_3 has only one neighbor outside P which is a leaf or $d(v_3) = 2$.

In the former case, we consider $T' = T - v_0$ and it is easy to show that $T \in \mathscr{O}$. In the latter case, suppose that $d(v_4) \geq 3$ and let v_3' be a neighbor of v_4 outside P. From the choice of v_2 , v_3 and D, the component of $T - v_3'v_4$ containing v_3' is either a subdivided star or a P_4. We only need to consider the second case. Let v_2' be the neighbor of v_3' on the P_4, and v_1' be the remaining neighbor of v_2' on the P_4. Clearly, $v_1' \in D$. Since each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from v_3, $v_2' \in D$. Replacing v_2' in D with v_3', and the resulting set is also a γ_{t2}-set of T. Take $T' = T - \{v_0, v_1\}$, and it can be deduced that $T' \in \mathscr{O}$ and T is obtained from T' by operation $\mathscr{O}_2$.

Hence, we may assume that $d(v_4) = 2$. We know that $v_1, v_3 \in D$. Because each vertex of $D \setminus \{v_1, v_3\}$ is at distance at least 3 from v_3. Then, let $T' = T - \{v_0, v_1, v_2, v_3, v_4\}$. Observe that $D \setminus \{v_1, v_3\}$ is a semitotal dominating set of T'. Moreover, we have that $\gamma(T') \leq \gamma_{t2}(T') \leq \gamma_{t2}(T) - 2 = \gamma(T) - 2 \leq \gamma(T')$. Thus, $\gamma(T') = \gamma_{t2}(T')$. By induction, $T' \in \mathscr{O}$. In addition, let D' be a $\gamma(T'; v_5)$-set of T' and $D'' = D' \cup \{v_1, v_4\}$. We can see that D'' dominates all vertices of T. That is, $\gamma(T'; v_5) + 2 \geq \gamma(T)$. It follows from $\gamma(T) = \gamma(T') + 2 \geq \gamma(T'; v_5) + 2 \geq \gamma(T)$ that $\gamma(T'; v_5) = \gamma(T')$. Hence, T is obtained from T' by operation $\mathscr{O}_2$.

The proof is completed. □

As an immediate consequence of Lemmas 3.1 and 3.2 we have the following characterization of (γ, γ_{t2})-trees.

Theorem 3.3. *A tree T is a (γ, γ_{t2})-tree if and only if $T \in \mathscr{O}$.*

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