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Continued fraction expansions for q-tangent and q-cotangent functions

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For 3 different versions of q-tangent resp. q-cotangent functions, we compute the continued fraction expansion explicitly, by guessing the relative quantities and proving the recursive relation afterwards. It is likely that these are the only instances with a “nice” expansion. Additional formulæ of a similar type are also provided.

Keywords: q-tangent, q-cotangent, continued fraction.

To Philippe Flajolet for 30 years of inspiration

1 Philippe Flajolet and continued fractions

In a paper that was written on the occasion of Philippe Flajolet’s 50th birthday (26) and discussed his various research areas, we wrote about his contributions to continued fractions:

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Continued fractions

The papers (8; 9; 10) deal with the interplay of continued fractions and combinatorics. Let us consider lattice paths, consisting of steps NORTHEAST, EAST, SOUTHEAST, starting at the origin, returning to the x-axis after n steps, and never being negative. The possible steps are denoted by the letters $\{a, b, c\}$, and an index i is additionally used when a step starts at altitude i. Thus, such a lattice path is a *word* in the variables $\{a_0, a_1, \dots, b_0, b_1, \dots, c_1, \dots\}$.

The set of all paths (a *formal language*) is given by the infinite continued fraction

$$\cfrac{1}{1-c_0-\cfrac{a_0|b_1}{1-c_1-\cfrac{a_1|b_2}{1-c_2-\cfrac{a_2|b_3}{\dots}}}},$$

where $(u|v)/w$ denotes $uw^{-1}v$, and w^{-1} is the quasi-inverse of languages (or formal power series).

There are many consequences of this *continued fraction theorem*, e. g. finite versions describe lattice paths of bounded height. Counting leads to a replacement of noncommuting variables by commuting variables. For instance, replacing all the variables by z gives the continued fraction for $\sum M_n z^n$, M_n being a Motzkin number. Many combinatorial objects can be described by such lattice paths, with suitable specializations. Some examples: Set partitions (also with several restrictions), permutations (via tournament trees), involutions, etc. Some later developments can be found in (15; 20).

There are also applications to Computer Science, since several dynamic data structures can be described in this way, the simplest being a stack, but also Dictionaries, Priority queues, Linear lists, Symbol tables, and subspecies of these. Operations like Insertion, Query, Deletion have then an obvious interpretation in the path diagram. Several notions of costs can be discussed with conveniently in terms of continued fractions. These concepts were worked out in collaboration with Chéno, Françon, Puech, and Vuillemin (12; 14; 3; 13; 17; 18).

Numbertheoretic aspects of continued fractions

Gauss studied expansions of complex numbers into continued fractions. Here is an example.

$$\frac{35470}{99661}+\frac{315}{9961}i=\cfrac{1}{3-\cfrac{1}{5+\cfrac{1}{\frac{1}{3}+\frac{7}{5}i}}}.$$

Consecutive digits are obtained by the recursive rule

$$\psi(z)=\lceil\Re(z)\rfloor+\frac{\epsilon(z)}{\psi\Bigl(\dfrac{\epsilon(z)}{z-\lceil\Re(z)\rfloor}\Bigr)},$$

where $\lceil x\rfloor=\lceil x-\frac{1}{2}\rceil$ and $\epsilon(z)=\operatorname{sign}(\Re(z)-\lceil\Re(z)\rfloor)$. The algorithm terminates if the resulting number falls into the domain $\{z\in\mathbb{C}\mid 0\le\Re(z)\le\frac{1}{2}\text{ and }|z|\ge 1\}$.

The average number of steps of this algorithm (in various continuous and discrete models) turns out to be linear, and the constant(s) involve the interesting quantity

$$\sum_{d\ge1}\frac{(-1)^d}{d^2}\sum_{c=1}^{d}\frac{1}{c^2},$$

which is expressible in terms of the remarkable constants $\zeta(3)$ and $\mathrm{Li}_4(\frac{1}{2})$ (a *tetralogarithm*).

This and much more can be found in the papers (28; 5; 6; 11). The work (11) is a survey paper and covers much more general reduction schemes (transformation), e. g. the binary representation. The average-case analysis of these usually involves interesting numerical constants, like Wirsing's, Lévy's, Hensley's, and Vallée's constant. This is a quite challenging domain, with relations to Functional Analysis.

The paper (8) has since 1998, when the previous lines were published, become a *classic*, and it was reprinted by *Discrete Mathematics* in a volume that comprised the most influential papers of the journal since its beginning (22).

Since 1998, Flajolet's research on continued fractions has not stopped; here are the more recent papers on the subject (16; 21; 2).

It is my hope that *Philippe* (as I am allowed to call him) will like my own research on *continued fractions* as well.

We always represent our continued fractions in the form

$$\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{a_3+\cfrac{z}{\ddots}}}}$$

since this is convenient for our computations. It would be easy, however, to transform it, say, into the form:

$$\cfrac{zb_1}{1+\cfrac{zb_2}{1+\cfrac{zb_3}{1+\cfrac{zb_4}{\ddots}}}}$$

Set $a_0 = 1$, then $b_i = \dfrac{1}{a_{i-1}a_i}$ for all $i = 1, 2, \ldots$.

2 Introduction

In this paper, we consider the functions

$$F(z) = \sum_{n\ge 0} \frac{(-1)^n z^n}{[2n+1]_q!} q^{dn^2},$$

$$G(z) = \sum_{n\ge 0} \frac{(-1)^n z^n}{[2n]_q!} q^{dn^2}.$$

We use standard q-notation:

$$[n]_q := \frac{1-q^n}{1-q}, \qquad [n]_q! := [1]_q[2]_q \ldots [n]_q.$$

For $d=0,1,2$, we will find the following continued fraction expansions:

$$\frac{zF(z)}{G(z)}=\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{a_3+\cfrac{z}{\ddots}}}}$$

(Replacing z by z^2, we get z times a q-tangent function.)

$$\frac{zG(z)}{F(z)}=\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{a_3+\cfrac{z}{\ddots}}}}$$

(Replacing z by z^2, we get z^3 times a q-cotangent function.)

These q-trigonometric functions are variants of Jackson's, see (24).

The instance $d=0$ of the q-tangent appeared in (25), and the instance $d=1$ in (23) and (27). Computer experiments indicate that, apart from trivial variations, these are the only cases where we get "nice" coefficients a_k.

We treat all 6 instances in a systematic way:

We write

$$\frac{zF(z)}{G(z)}=\frac{z}{N_0}=\cfrac{z}{a_1+\cfrac{z}{N_1}}=\cfrac{z}{a_1+\cfrac{z}{a_2+\cfrac{z}{N_2}}}=\dots,$$

and set

$$N_i=\frac{r_i}{s_i}.$$

This means

$$N_i=a_{i+1}+\frac{z}{N_{i+1}}$$

or

$$\frac{z}{N_{i+1}}=\frac{zs_{i+1}}{r_{i+1}}=N_i-a_{i+1}=\frac{r_i}{s_i}-a_{i+1}=\frac{r_i-a_{i+1}s_i}{s_i}.$$

We can set $r_i=s_{i-1}$ and get the recursion

$$s_{i+1}z=s_{i-1}-a_{i+1}s_i.$$

The initial conditions are

$$s_{-1}=G(z)\quad\text{and}\quad s_0=F(z).$$

Note that the a_i's are the unique numbers that make the s_i's power series expansions.

In all instances, we are able to guess the numbers a_k and the power series s_k, and prove the guessed form by induction. In the cotangent case, F and G switch roles, of course. The proof by induction is a routine computation; the challenging part in this line of research is the guessing. Since the proofs are very similar, we present just one of them.

Not all of the results are new; the instance $d = 0$ is of course the *classical* case, and the instance $d = 1$ (tangent case) was published in (23; 27), but the other formulæ are believed to be new. However, to be systematic, we collected all the results.

3 Tangent expansions

Here, we treat, as promised, the three instances $d = 0, 1, 2$.

3.1 $d = 0$

$$a_k = (-1)^{k-1} \frac{[2k-1]_q}{q^{k-1}},$$

$$s_k = (-1)^{\lfloor \frac{k+1}{2} \rfloor} q^{\binom{k+1}{2}} \sum_{n \ge 0} \frac{z^n (-1)^n}{[2n+2k+1]_q!} \prod_{j=1}^{k} [2n+2j]_q.$$

3.2 $d = 1$

$$a_{2k} = -\frac{[4k-1]_q}{q^{(k+1)(2k-1)}},$$

$$a_{2k+1} = [4k+1]_q q^{k(2k-1)}.$$

$$s_{2k} = (-1)^k q^{k^2} \sum_{n \ge 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q q^{n(n+2k)},$$

$$s_{2k+1} = (-1)^{k-1} q^{(k+1)(3k+2)} \sum_{n \ge 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q q^{n(n+2k+2)}.$$

3.3 $d = 2$

$$a_{2k} = -\frac{[4k-1]_q (1 - q^{2k} - q^{2k+1} + q^{4k-1})^2}{(1-q^2)^2 q^{6k-3}},$$

$$a_{2k+1} = \frac{[4k+1]_q (1-q^2)^2 q^{2k-1}}{(1 - q^{2k+2} - q^{2k+3} + q^{4k+3})(1 - q^{2k} - q^{2k+1} + q^{4k-1})}.$$

$$s_{2k} = (-1)^k q^{2k^2} \sum_{n \ge 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q \Big(1 + \frac{q^{2n+2}(1-q^{2k})(1-q^{2k+1})}{1-q^2}\Big) q^{2n(n+2k)},$$

$$s_{2k+1} = (-1)^{k-1} q^{2k^2+6k+3} \sum_{n \ge 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q \frac{(1-q^2) q^{2n(n+2k+2)}}{1 - q^{2k+2} - q^{2k+3} + q^{4k+3}}.$$

4 Cotangent expansions

Here are the three companion expansions related to the previous tangent expansions.

4.1 $d = 0$

$a_1 = 1$, and for $k \geq 1$

$$\begin{aligned} a_{2k} &= \frac{[4k-1]_q[2k-1]_q^2[2k]_q^2}{q^{6k-5}(1+q)^2}, \\ a_{2k+1} &= -\frac{[4k+1]_q(1+q)^2q^{2k-2}}{[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q}. \end{aligned}$$

$$\begin{aligned} s_{2k} &= (-1)^k q^{k(2k-1)} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q \bigg([2n+4k+1]_q + \frac{q^2[2k]_q[2k-1]_q}{1+q} \bigg), \\ s_{2k+1} &= (-1)^k q^{2k^2+5k+1} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q \frac{1+q}{[2k+1]_q[2k+2]_q}. \end{aligned}$$

4.2 $d = 1$

$a_1 = 1$, and for $k \geq 1$

$$\begin{aligned} a_{2k} &= \frac{[4k-1]_q[k(2k-1)]_q^2}{q^{(2k-1)(k+1)}}, \\ a_{2k+1} &= -\frac{[4k+1]_q q^{k(2k-1)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q}. \end{aligned}$$

$$\begin{aligned} s_{2k} &= (-1)^k q^{k^2} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q [2n+2k^2+3k+1]_q q^{n(n+2k)}, \\ s_{2k+1} &= \frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q q^{n(n+2k+2)}. \end{aligned}$$

4.3 $d = 2$

$a_1 = 1$, and for $k \geq 1$

$$\begin{aligned} a_{2k} &= \frac{[4k-1]_q[2k-1]_q^2[2k]_q^2}{q^{6k-3}(1+q)^2}, \\ a_{2k+1} &= -\frac{[4k+1]_q(1+q)^2q^{2k-1}}{[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q}. \end{aligned}$$

$$s_{2k} = (-1)^k q^{2k^2} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q q^{2n(n+2k)} \left([2n+4k+1]_q + \frac{q^{2n+2}[2k]_q[2k-1]_q}{1+q} \right),$$

$$s_{2k+1} = \frac{(-1)^k q^{2k^2+6k+3}(1+q)}{[2k+2]_q[2k+1]_q} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q q^{2n(n+2k+2)}.$$

5 Proof of the cotangent case $d = 1$

We have by inspection that $s_0 = G(z)$, and compute

$$\begin{aligned}
s_1 &= \frac{1}{z}\big(s_{-1} - s_0\big) \\
&= \frac{1}{z}\big(F(z) - G(z)\big) \\
&= \frac{1}{z} \sum_{n\geq 0} \frac{z^n(-1)^n q^{n^2}}{[2n+1]_q!} \frac{1-q-1+q^{2n+1}}{1-q} \\
&= \sum_{n\geq 1} \frac{z^{n-1}(-1)^n q^{n^2}}{[2n+1]_q!} \frac{-q(1-q^{2n})}{1-q} \\
&= \sum_{n\geq 1} \frac{z^{n-1}(-1)^{n-1} q^{n^2+1}}{[2n-1]_q![2n+1]_q} \\
&= q^2 \sum_{n\geq 0} \frac{z^n(-1)^n q^{n(n+2)}}{[2n+1]_q![2n+3]_q},
\end{aligned}$$

which checks, so we have the basis for our induction. And now we must show for all n that

$$\begin{aligned}
[z^n]\big(s_{2k} - a_{2k+2}s_{2k+1}\big) &= [z^{n-1}]s_{2k+2}, \\
[z^n]\big(s_{2k-1} - a_{2k+1}s_{2k}\big) &= [z^{n-1}]s_{2k+1}.
\end{aligned}$$

Let us start with the first one:

$$\begin{aligned}
[z^n]\big(s_{2k} - a_{2k+2}s_{2k+1}\big) = (-1)^k q^{k^2} & \frac{(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q [2n+2k^2+3k+1]_q q^{n(n+2k)} \\
&- \frac{[4k+3]_q[(k+1)(2k+1)]_q^2}{q^{(2k+1)(k+2)}} \times \\
&\times \frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q} \frac{(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1} [2n+2j]_q q^{n(n+2k+2)} \\
&= \frac{(-1)^{n+k} q^{k^2}}{[2n+4k+1]_q!} \prod_{j=1}^{2k} [2n+2j]_q [2n+2k^2+3k+1]_q q^{n(n+2k)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(-1)^{n+k}q^{k^2}[4k+3]_q[(k+1)(2k+1)]_q}{[2n+4k+1]_q![2n+4k+3]_q}\prod_{j=1}^{2k}[2n+2j]_q q^{n(n+2k+2)}\\
&=\frac{(-1)^{n+k}q^{(n+k)^2}}{[2n+4k+1]_q![2n+4k+3]_q}\prod_{j=1}^{2k}[2n+2j]_q\times\\
&\quad\times\Big([2n+4k+3]_q[2n+2k^2+3k+1]_q-q^{2n}[4k+3]_q[(k+1)(2k+1)]_q\Big)\\
&=\frac{(-1)^{n+k}q^{(n+k)^2}}{[2n+4k+1]_q![2n+4k+3]_q}\prod_{j=1}^{2k}[2n+2j]_q[2n]_q[2n+2k^2+7k+4]_q.
\end{aligned}
$$

On the other hand

$$
[z^{n-1}]s_{2k+2}=(-1)^{k-1}q^{(k+1)^2}\frac{(-1)^{n-1}}{[2n+4k+3]_q!}\prod_{j=1}^{2k+2}[2n-2+2j]_q[2n+2k^2+7k+4]_q q^{(n-1)(n+2k+1)},
$$

which is the same, as it should.

And now to the second one:

$$
\begin{aligned}
[z^n]\big(s_{2k-1}-a_{2k+1}s_{2k}\big)&=\frac{(-1)^{k-1}q^{k(3k-1)}}{[k(2k-1)]_q}\frac{(-1)^n}{[2n+4k-1]_q!}\prod_{j=1}^{2k-1}[2n+2j]_q q^{n(n+2k)}\\
&+\frac{[4k+1]_q q^{k(2k-1)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q}\times\\
&\times(-1)^k q^{k^2}\frac{(-1)^n}{[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q[2n+2k^2+3k+1]_q q^{n(n+2k)}\\
&=\frac{(-1)^{n+k-1}q^{k(3k-1)+n(n+2k)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q\times\\
&\times\Big([2n+4k+1]_q[(k+1)(2k+1)]_q-[4k+1]_q[2n+2k^2+3k+1]_q\Big)\\
&=\frac{(-1)^{n+k-1}q^{k(3k-1)+n(n+2k)}}{[k(2k-1)]_q[(k+1)(2k+1)]_q[2n+4k+1]_q!}\prod_{j=1}^{2k}[2n+2j]_q[2n]_q[k(2k-1)]_q q^{4k+1}\\
&=\frac{(-1)^{n+k-1}q^{3k(k+1)+1+n(n+2k)}}{[(k+1)(2k+1)]_q[2n+4k+1]_q!}\prod_{j=0}^{2k}[2n+2j]_q.
\end{aligned}
$$

On the other hand,

$$
[z^{n-1}]s_{2k+1}=\frac{(-1)^k q^{(k+1)(3k+2)}}{[(k+1)(2k+1)]_q}\frac{(-1)^{n-1}}{[2n+4k+1]_q!}\prod_{j=1}^{2k+1}[2n-2+2j]_q q^{(n-1)(n+2k+1)},
$$

which is the same, so that our proof is finished.

6 Another tangent expansion

$$F(z) = \sum_{n\geq 0} \frac{(-1)^n z^n}{[2n+1]_q!},$$

$$G(z) = \sum_{n\geq 0} \frac{(-1)^n z^n}{[2n]_q!} q^{2n}.$$

$$a_{2k} = -\frac{[4k-1]_q q^{2k-3}(1-q^2)^2}{(1-q^{2k-3}-q^{2k-2}+q^{4k-3})(1-q^{2k-1}-q^{2k}+q^{4k+1})},$$

$$a_{2k+1} = \frac{[4k+1]_q(1-q^{2k-1}-q^{2k}+q^{4k+1})^2}{q^{6k-2}(1-q^2)^2}.$$

$$s_{2k} = \frac{(-1)^{k-1}q^{2k^2+3k-1}(1-q^2)}{1-q^{2k-1}-q^{2k}+q^{4k+1}} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q![2n]_q} \prod_{j=1}^{2k}[2n+2j]_q,$$

$$s_{2k+1} = (-1)^{k-1}q^{2k^2+k} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q![2n]_q} \prod_{j=1}^{2k+1}[2n+2j]_q\Big(q^{4k+2n+3} - \frac{(1-q^{2k+1})(1-q^{2k+2})}{1-q^2}\Big).$$

7 The cotangent expansion as a companion

$$F(z) = \sum_{n\geq 0} \frac{(-1)^n z^n}{[2n+1]_q!} q^{2n},$$

$$G(z) = \sum_{n\geq 0} \frac{(-1)^n z^n}{[2n]_q!}.$$

For $k \geq 0$,

$$a_{2k} = \frac{[4k-1]_q[2k-1]_q^2[2k]_q^2}{q^{6k-6}(1+q)^2},$$

$$a_{2k+1} = -\frac{[4k+1]_q(1+q)^2q^{2k-3}}{[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q},$$

and $a_1 = 1$.

$$s_{2k} = \frac{(-1)^k q^{2k^2-k}}{1-q} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+1]_q!} \prod_{j=1}^{2k}[2n+2j]_q\Big(\frac{1-q^{2k+1}-q^{2k+2}+q^{4k+1}}{1-q^2} - q^{2n+4k+1}\Big),$$

$$s_{2k+1} = \frac{(-1)^k q^{2k^2+5k}(1+q)}{[2k+2]_q[2k+1]_q} \sum_{n\geq 0} \frac{z^n(-1)^n}{[2n+4k+3]_q!} \prod_{j=1}^{2k+1}[2n+2j]_q.$$

8 A generalization

Our computer calculations suggested to go for a generalization of the previous results. Note that $h=0$ is the instance studied before.

The first 3 expansions are for the tangent, the following 3 are for the cotangent.

We only give the quantities a_k's, but finding the expressions for the s_k's would be a good project for a student.

$$F_h(z)=\sum_{n\ge0}\frac{(-1)^nz^n}{[2n+1]_q!}\prod_{j=1}^{h}[2n+2j]_q q^{dn^2},$$
$$G_h(z)=\sum_{n\ge0}\frac{(-1)^nz^n}{[2n]_q!}\prod_{j=1}^{h}[2n+2j]_q q^{dn^2}.$$

8.1 $d=0$

$$a_{2k}=-[4k-1+2h]_q q^{-2k+1-2h},$$
$$a_{2k+1}=[4k+1+2h]_q q^{-2k}.$$

8.2 $d=1$

$$a_{2k}=-[4k-1+2h]_q q^{-2k^2-k(2h+1)+1},$$
$$a_{2k+1}=[4k+1+2h]_q q^{2k^2+k(2h-1)}.$$

8.3 $d=2$

$$a_{2k}=-\frac{[4k-1+2h]_q([k]_{q^2}-q^{2k+1+2h}[k-1]_{q^2})^2}{q^{6k-3+2h}},$$
$$a_{2k+1}=\frac{[4k+1+2h]_q q^{2k-1+2h}}{([k]_{q^2}-q^{2k+1+2h}[k-1]_{q^2})([k+1]_{q^2}-q^{2k+3+2h}[k]_{q^2})}.$$

8.4 $d=0$

$a_1=1$, and for $k\ge1$

$$a_{2k}=\frac{[4k-1+2h]_q[2k-1+2h]_q^2[2k]_q^2}{q^{6k-5+2h}(1+q)^2},$$
$$a_{2k+1}=-\frac{[4k+1+2h]_q(1+q)^2q^{2k-2}}{[2k-1+2h]_q[2k]_q[2k+1+2h]_q[2k+2]_q}.$$

8.5 $d = 1$

$a_1 = 1$, and for $k \geq 1$

$$a_{2k} = \frac{[4k-1+2h]_q[k(2k-1+2h)]_q^2}{q^{(2k-1)(k+1)+2kh}},$$
$$a_{2k+1} = -\frac{[4k+1+2h]_q q^{k(2k-1)+2kh}}{[k(2k-1+2h)]_q[(k+1)(2k+1+2h)]_q}.$$

8.6 $d = 2$

$a_1 = 1$, and for $k \geq 1$

$$a_{2k} = \frac{[4k-1+2h]_q[2k-1+2h]_q^2[2k]_q^2}{q^{6k-3+2h}(1+q)^2},$$
$$a_{2k+1} = -\frac{[4k+1+2h]_q(1+q)^2 q^{2k-1+2h}}{[2k-1+2h]_q[2k]_q[2k+1+2h]_q[2k+2]_q}.$$

9 More continued fraction expansions

The following expansions are not new, but they fit the same pattern, and they are very beautiful, so I decided to include them here to please *Philippe*.

We employ the classic notation $(x;q)_n := (1-x)(1-xq)\dots(1-xq^{n-1})$.

Let
$$G(z) = \sum_{n\geq 0} \frac{(y;q)_n z^n}{(x;q)_n}.$$

Then we have the continued fraction expansion

$$\frac{z}{G(z)} = \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cfrac{z}{a_4 + \cfrac{z}{\ddots}}}}}$$

with $a_1 = 1$, and for $k \geq 1$

$$a_{2k} = \frac{(x;q)_{k-1}(1-xq^{2k-2})(yq;q)_{k-1}}{(1-y)(yq)^{k-1}(\frac{x}{y};q)_{k-1}(q;q)_{k-1}},$$
$$a_{2k+1} = -\frac{(1-y)y^{k-1}(\frac{x}{y};q)_{k-1}(1-xq^{2k-1})(q;q)_{k-1}}{(x;q)_k(yq;q)_k}.$$

For the proof, we notice that a_1 is an exceptional value, and we only start the recursion with

$$s_0 = 1, \qquad s_1 = \sum_{n\ge 0} \frac{(y;q)_{n+1} z^n}{(x;q)_{n+1}}.$$

Note that the numbers a_i are uniquely determined by annihilating the constant term in $s_{i-1} - a_{i+1}s_i$, making s_{i+1} a power series expansion. Our claim follows now by the following *explicit* formulæ (for $k \ge 0$)

$$\begin{aligned} s_{2k} &= \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{k-1}} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_{k-1} (yq;q)_{n+k}}{(xq^k;q)_{n+k}}, \\ s_{2k+1} &= (1-y) y^k (\tfrac{x}{y};q)_k (-1)^k q^{\binom{k+1}{2}} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_k (yq^{k+1};q)_n}{(x;q)_{n+2k+1}}, \end{aligned}$$

provided we are able to establish these formulæ by induction via the recursion. The initial values follow by inspection, and the induction step must be split into two computations, according to the parity of the indices.

$$\begin{aligned} s_{2k} - a_{2k+2}s_{2k+1} &= \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{k-1}} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_{k-1} (yq;q)_{n+k}}{(xq^k;q)_{n+k}} \\ &\quad - \frac{(x;q)_k (qy;q)_k (1-xq^{2k})}{(qy)^k (\frac{x}{y};q)_k (1-y)(q;q)_k} (1-y) y^k (\tfrac{x}{y};q)_k (-1)^k q^{\binom{k+1}{2}} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_k (yq^{k+1};q)_n}{(x;q)_{n+2k+1}} \\ &= \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{k-1}} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_{k-1} (yq;q)_{n+k}}{(xq^k;q)_{n+k}} - \frac{(1-xq^{2k})}{(q;q)_k} (-1)^k q^{\binom{k}{2}} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_k (yq;q)_{n+k}}{(xq^k;q)_{n+k+1}} \\ &= \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \Bigg[(1-q^k) \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_{k-1} (yq;q)_{n+k}}{(xq^k;q)_{n+k}} - (1-xq^{2k}) \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_k (yq;q)_{n+k}}{(xq^k;q)_{n+k+1}} \Bigg] \\ &= \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_{k-1} (yq;q)_{n+k}}{(xq^k;q)_{n+k+1}} \Big[(1-q^k)(1-xq^{n+2k}) - (1-xq^{2k})(1-q^{n+k}) \Big] \\ &= \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \sum_{n\ge 0} \frac{z^n (q^{n+1};q)_{k-1} (yq;q)_{n+k}}{(xq^k;q)_{n+k+1}} (-1) q^k (1-xq^k)(1-q^n) \\ &= \frac{(-1)^{k+1} q^{\binom{k+1}{2}}}{(q;q)_k} \sum_{n\ge 0} \frac{z^n (q^n;q)_k (yq;q)_{n+k}}{(xq^{k+1};q)_{n+k}} \\ &= z s_{2k+2}; \end{aligned}$$

$$\begin{aligned}
s_{2k-1} - a_{2k+1}s_{2k} &= (1-y)y^{k-1}(\tfrac{x}{y};q)_{k-1}(-1)^{k-1}q^{\binom{k}{2}}\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^k;q)_n}{(x;q)_{n+2k-1}}\\
&\quad+\frac{(1-y)y^{k-1}(\frac{x}{y};q)_{k-1}(q;q)_{k-1}(1-xq^{2k-1})}{(qy;q)_k(x;q)_k}\frac{(-1)^kq^{\binom{k}{2}}}{(q;q)_{k-1}}\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq;q)_{n+k}}{(xq^k;q)_{n+k}}\\
&= (1-y)y^{k-1}(\tfrac{x}{y};q)_{k-1}(-1)^{k-1}q^{\binom{k}{2}}\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^k;q)_n}{(x;q)_{n+2k-1}}\\
&\quad+(1-y)y^{k-1}(\tfrac{x}{y};q)_{k-1}(1-xq^{2k-1})(-1)^kq^{\binom{k}{2}}\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^{k+1};q)_n}{(x;q)_{n+2k}}\\
&= (1-y)y^{k-1}(\tfrac{x}{y};q)_{k-1}(-1)^{k-1}q^{\binom{k}{2}}\\
&\quad\times\Bigg[\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^k;q)_n}{(x;q)_{n+2k-1}}-(1-xq^{2k-1})\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^{k+1};q)_n}{(x;q)_{n+2k}}\Bigg]\\
&= (1-y)y^{k-1}(\tfrac{x}{y};q)_{k-1}(-1)^{k-1}q^{\binom{k}{2}}\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^{k+1};q)_{n-1}}{(x;q)_{n+2k}}\\
&\quad\times\Big[(1-xq^{n+2k-1})(1-yq^k)-(1-xq^{2k-1})(1-yq^{k+n})\Big]\\
&= (1-y)y^{k-1}(\tfrac{x}{y};q)_{k-1}(-1)^{k-1}q^{\binom{k}{2}}\sum_{n\ge0}\frac{z^n(q^{n+1};q)_{k-1}(yq^{k+1};q)_{n-1}}{(x;q)_{n+2k}}\\
&\quad\times(-1)q^{k-1}(1-q^n)yq(1-\tfrac{x}{y}q^{k-1})\\
&= (1-y)y^k(\tfrac{x}{y};q)_k(-1)^kq^{\binom{k+1}{2}}\sum_{n\ge0}\frac{z^n(q^n;q)_k(yq^{k+1};q)_{n-1}}{(x;q)_{n+2k}}\\
&= zs_{2k+1}.
\end{aligned}$$

Remark. Computer experiments indicate that we cannot add additional factors $(u;q)_n$ etc. in either numerator or denominator, as then the expressions for a_i become very messy and don't factor nicely.

We note two special cases explicitly. Set $y=0$, then

$$\begin{aligned}
a_{2k} &= \frac{(x;q)_{k-1}(1-xq^{2k-2})}{x^{k-1}q^{\binom{k}{2}}(q;q)_{k-1}},\\
a_{2k+1} &= -\frac{x^{k-1}q^{\binom{k-1}{2}}(1-xq^{2k-1})(q;q)_{k-1}}{(x;q)_k}.
\end{aligned}$$

Set $x=0$, then

$$a_{2k} = \frac{(yq;q)_{k-1}}{(1-y)(yq)^{k-1}(q;q)_{k-1}},$$

$$a_{2k+1} = -\frac{(1-y)y^{k-1}(q;q)_{k-1}}{(yq;q)_k}.$$

Michael Joseph Schlosser has kindly informed me that the formulæ could be deduced from results in (7).

A continued fraction of Ramanujan

This method of proof also applies to a continued fraction of Ramanujan, see (1). In slightly changed notation, we have

$$G(z) = \sum_{n\geq 0} \frac{q^{\binom{n}{2}}(y;q)_n z^n}{(x;q)_n (q;q)_n},$$

and $H(z) = G(z)/G(qz)$. Then

$$\frac{z}{H(z)} = \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{a_3 + \cfrac{z}{a_4 + \cfrac{z}{\ddots}}}}}$$

with

$$a_{2k} = \frac{(1-xq^{2k-2})y^{k-1}(\frac{x}{y};q)_{k-1}(-1)^{k-1}q^{\binom{k-1}{2}}}{(y;q)_k},$$
$$a_{2k+1} = \frac{(1-xq^{2k-1})(y;q)_k(-1)^k}{y^k(\frac{x}{y};q)_k q^{\binom{k+1}{2}}}.$$

Here, the formulæ follow from

$$s_{2k} = y^k(\tfrac{x}{y};q)_k(-1)^k q^{\binom{k}{2}} \sum_{n\geq 0} \frac{z^n (yq^k;q)_n q^{\binom{n+k+1}{2}}}{(x;q)_{n+2k}(q;q)_n},$$
$$s_{2k+1} = \sum_{n\geq 0} \frac{z^n (y;q)_{n+k+1} q^{\binom{n+k+1}{2}}}{(x;q)_{n+2k+1}(q;q)_n}.$$

The celebrated Rogers-Ramanujan continued fraction expansion and companions

Set

$$G(z)=\sum_{n\geq 0}\frac{q^{n^2}z^n}{(q;q)_n}$$

and $H(z)=G(z)/G(qz)$. Then $a_k=q^{-\lfloor\frac{k}{2}\rfloor}$ and

$$\begin{aligned}s_{2k}&=q^{k(k+1)}\sum_{n\geq 0}\frac{z^nq^{n^2+n(2k+1)}}{(q;q)_n},\\ s_{2k+1}&=q^{(k+1)^2}\sum_{n\geq 0}\frac{z^nq^{n^2+n(2k+2)}}{(q;q)_n}.\end{aligned}$$

Companion: Set $H(z)=G(qz)/G(z)$. Then

$$\begin{aligned}a_{2k}&=-\frac{(1-q^k)^2}{(1-q)^2q^{3k-2}},\\ a_{2k+1}&=-\frac{(1-q)^2q^{k-1}}{(1-q^k)(1-q^{k+1})},\end{aligned}$$

and

$$\begin{aligned}s_{2k}&=\sum_{n\geq 0}\frac{z^n(1+q^{n+1}\frac{1-q^k}{1-q})q^{(n+k)^2}}{(q;q)_n},\\ s_{2k+1}&=-\frac{q^k(1-q)}{1-q^{k+1}}\sum_{n\geq 0}\frac{z^nq^{(n+k+1)^2}}{(q;q)_n}.\end{aligned}$$

Another companion: Set $H(z)=G(z)/G(q^2z)$. Then

$$\begin{aligned}a_{2k}&=\frac{(1-q)^2q^{k-2}}{(1-q^k)(1-q^{k+1})},\\ a_{2k+1}&=\frac{(1-q^{k+1})^2}{(1-q)^2q^{3k}}.\end{aligned}$$

Another example

Set

$$G(z)=\sum_{n\ge0}\frac{(z;q^2)_n z^n}{(zq;q^2)_n},$$

then we expand again $z/G(z)$ and get: $a_1=1$, and

$$\begin{aligned}a_{2k}&=\frac{(-1)^{k-1}q^{\binom{k-1}{2}}}{(q;q)_{k-1}},\\ a_{2k+1}&=\frac{(-1)^k(q;q)_{k-1}}{q^{\binom{k+1}{2}}}\end{aligned}$$

and

$$\begin{aligned}s_{2k}&=\frac{(-1)^k q^{k^2}}{(q;q)_{k-1}}\sum_{n\ge0}\frac{z^n q^{\frac{n^2}{2}+\frac{n(2k+1)}{2}}(q;q)_{n+k-1}}{(q;q)_n},\\ s_{2k}&=q^{\binom{k+1}{2}}\sum_{n\ge0}\frac{z^n q^{\frac{n^2}{2}+\frac{n(2k+1)}{2}}(q;q)_{n+k}}{(q;q)_n}.\end{aligned}$$

References

[1] S. Bhargava and C. Adiga, *On some continued fraction identities of Srinivasa Ramanujan*, Proc. A. M. S., **92**, (1984), 13–18.

[2] R. Bacher and P. Flajolet, Pseudo-factorials, elliptic functions, and continued fractions, arXiv:0901.1379 (January 2009), 22 pages, Ramanujan Journal, to appear.

[3] L. Cheno, P. Flajolet, J. Françon, C. Puech and J. Vuillemin, Dynamic data structures: Finite files, limiting profiles and variance analysis, In *18th annual conference on Communciation, Control, and Computing*, 223–232, 1980.

[4] E. Conrad and P. Flajolet, The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion, Séminaire Lotharingien de Combinatoire, **54**, paper B54g, 44 pages, 2006.

[5] H. Dauvé, P. Flajolet and B. Vallée, An analysis of the Gaussian algorithm for lattice reduction, Lecture Notes in Computer Science, **877**, (1994), 144–158.

[6] H. Dauvé, P. Flajolet and B. Vallée, An average-case analysis of the Gaussian algorithm for lattice reduction, Combinatorics, Probability and Computing, **6**, (1997), 397–433.

[7] R. Y. Denis, *On (a) generalization of (a) continued fraction of Gauss*, Int. J. Math. Math. Sci., **4**, (1990), 741–746.

[8] P. FLAJOLET, Combinatorial aspects of continued fractions, Annals of Discrete Mathematics, **8**, (1980), 217–222. Proceedings of "Colloque Franco-Canadien de Combinatoire", Montreal 1979.

[9] P. FLAJOLET, Combinatorial aspects of continued fractions, Discrete Mathematics, **32**, (1980), 125–161.

[10] P. FLAJOLET, On congruences and continued fractions for some classical combinatorial quantities, Discrete Mathematics, **41**, (1982), 145–153.

[11] P. FLAJOLET AND B. VALLÉE, Continued fraction algorithms, functional operators, and structure constants, Theoretical Computer Science, **194**, (1998), 1–34.

[12] P. FLAJOLET, J. FRANÇON AND J. VUILLEMIN, Towards analysing sequences of operations for dynamic data structures, In *Proceedings of the 20th annual symposium on Foundations of Computer Science*, 183–195, 1979.

[13] P. FLAJOLET, J. FRANÇON AND J. VUILLEMIN, Sequences of operations analysis for dynamic data structures, Journal of Algorithms **1**, 111–141, 1980.

[14] P. FLAJOLET, J. FRANÇON AND J. VUILLEMIN, Computing integrated costs of data structures with applications to dictionaries, *Proceedings of the 11th annual ACM symposium on Theory of Computing*, 49–61, 1979.

[15] P. FLAJOLET AND J. FRANÇON, Elliptic functions, continued fractions and doubled permutations, European Journal of Combinatorics, **10**, (1989), 235–241.

[16] P. FLAJOLET AND F. GUILLEMIN, Elliptic functions, continued fractions and doubled permutations, Advances in Applied Probability, **32**, (1989), 750–778.

[17] P. FLAJOLET AND C. PUECH, Analyse de structures de données dynamiques et histoires des fichiers, Questiió **5**, 31–48, 1981.

[18] P. FLAJOLET, C. PUECH AND J. VUILLEMIN, The analysis of simple list structures, Information Sciences **38**, 121–146, 1986.

[19] P. FLAJOLET, J.-C. RAOULT AND J. VUILLEMIN, The number of registers required for evaluating arithmetic expressions, Theoretical Computer Science, **9**, (1979), 99–125.

[20] P. FLAJOLET AND R. SCHOTT, Non-overlapping partition, continued fractions, Bessel functions and a divergent series, European Journal of Combinatorics, **11**, (1990), 421–432.

[21] P. FLAJOLET AND B. VALLÉE, Continued Fractions, Comparison Algorithms, and Fine Structure Constants, In "Constructive, Experimental, and Nonlinear Analysis", M. Théra Ed., (CMS Conf. Proc., 27, Amer. Math. Soc., Providence), pp. 53–82. (Volume in the honour of Jonathan Borwein.)

[22] P. FLAJOLET, Combinatorial aspects of continued fractions, Discrete Mathematics, **306**, (2006), 992–1021.

[23] M. FULMEK, A continued fraction expansion for a q-tangent function, Sém. Loth. Comb., **45**, (2000), [B45b] (5 pages).

[24] F.H. JACKSON, A basic-sine and cosine with symbolic solutions of certain differential equations, Proc. Edinburgh Math. Soc., **22**, (1904), 28–39.

[25] H. PRODINGER, *Combinatorics of geometrically distributed random variables: New q-tangent and q-secant numbers*, Int. J. Math. Math. Sci., **24**, (2000), 825–838.

[26] H. PRODINGER AND W. SZPANKOWSKI, *Philippe Flajolet's research in Combinatorics and Analysis of Algorithms*, Algorithmica, **22**, (1998), 366–387.

[27] H. PRODINGER, *A continued fraction expansion for a q-tangent function: An elementary proof*, Sém. Loth. Comb., **60**, (2008), [B60b] (3 pages).

[28] B. VALLÉE AND P. FLAJOLET, Gauss' reduction algorithm: An average case-analysis, In *Proceedings of the 31th Symposium on Foundations of Computer Science*, (1990), 830–839.