Discrete Mathematics and Theoretical Computer Science DMTCS vol. **12:5**, 2010, 125–138

Some properties of semiregular cages

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received 28th November 2008, revised 5th March 2010, accepted 5th November 2010.

A graph with degree set $\{r, r+1\}$ is said to be semiregular. *A semiregular cage* is a semiregular graph with given girth g and the least possible order. First, an upper bound on the diameter of semiregular graphs with girth g and order close enough to the minimum possible value is given in this work. As a consequence, these graphs are proved to be maximally connected when the girth $g \geq 7$ is odd. Moreover an upper bound for the order of semiregular cages is given and, as an application, every semiregular cage with degree set $\{r, r+1\}$ is proved to be maximally connected for $g \in \{6, 8\}$, and when $g = 12$ for $r \geq 7$ and $r \neq 20$. Finally it is also shown that every $(\{r, r+1\}; g)$-cage is 3-connected.

Keywords: cage, degree set, girth, connectivity

1 Introduction

Throughout this work only undirected simple graphs without loops or multiple edges are considered. Unless stated otherwise, we follow [13] for terminology and definitions not explicitly given here.

Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. Given a proper subset X of $V(G)$, let $[X, V(G) \setminus X]$ denote the set of edges with one end in X and the other in $V(G) \setminus X$. We denote by $N(v)$ the *neighborhood* of a vertex v, the *degree* of a vertex v is $|N(v)| = d(v)$, and the *minimum degree* of G is denoted by $\delta(G)$. The *edge degree* of $uv \in E(G)$ is equal to $d(u) + d(v) - 2$, and $\xi(G)$ stands for the *minimum edge degree of* G. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest path between them. The *eccentricity* of a vertex u is the largest distance between u and any other vertex of the graph. The *diameter* of G is denoted by $diam(G)$ and is the maximum of the eccentricities of the vertices of the graph. The *connectivity* and *edge connectivity* of G, denoted respectively by $\kappa(G)$, $\lambda(G)$, are linked by the Whitney inequality $\kappa(G) \leq \lambda(G) \leq \delta(G)$. A graph G is *maximally connected* if $\kappa(G) = \delta(G)$, and *maximally edge connected* if $\lambda(G) = \delta(G)$. A *restricted edge cut* of a graph G is a set of edges whose deletion yields a nonconnected graph without isolated vertices. A graph G is said to be *optimally restricted edge connected* if the minimum cardinality of a restricted edge cut is equal to $\xi(G)$, the minimum edge degree of G.

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The *degree set* D of a graph G is the set of distinct degrees of the vertices of G. A $(D;g)$-*graph* is a graph with degree set D and girth g. The *frequency* of each degree in D is the number of vertices of the graph having this degree. A $(D;g)$-*cage* is a $(D;g)$-graph with the least possible order. A *biregular cage* is a $(D;g)$-cage with degree set $D=\{r,m\}$, $m>r$. A *semiregular cage* is a biregular cage with degree set $D=\{r,r+1\}$.

The concept of $(D;g)$-cages was proposed by Chartrand, Gould, and Kapoor [12]. If $D=\{r\}$, $(D;g)$-cages coincide with $(r;g)$-cages, which have been intensely studied since their introduction by Tutte [28]. See the survey by Wong [29] or the book by Holton and Sheehan [21] or the recent survey by Exoo and Jajcay [17]. The existence of $(r;g)$-cages was proved by Erdős and Sachs in the early 1960s [16], and using this result Chartrand, Gould and Kapoor [12] proved the existence of $(D;g)$-cages.

Construction of $(D;g)$-cages is a challenging topic as well as a very difficult task. This goal has only been achieved for a few pairs $(D;g)$, and most of them correspond to the case $D=\{r\}$. Even the problem of determining the value of the order of a $(D;g)$-cage, denoted by $n(D;g)$, is a difficult one and is open for most of degree sets D and girths g. A natural approach is to try to estimate upper and lower bounds as close as possible for $n(D;g)$ (written $n(r;g)$ instead of $n(\{r\};g)$ when $D=\{r\}$).

For the case of $(r;g)$-cages, Lazebnik, Ustimenko and Woldar [23], improving a previous result of Sauer [26], obtained the following upper bound for $n(r;g)$:

Theorem 1.1 *[23] Let $r\geq 2$ and $g\geq 5$ be integers, and let q denote the smallest odd prime power verifying $r\leq q$. Then*

$$n(r;g)\leq 2rq^{\frac{3g}{4}-a}, \tag{1}$$

where $a=4,\frac{11}{4},\frac{7}{2},\frac{13}{4}$ for $g\equiv 0,1,2,3 \pmod 4$ respectively.

Recently, the above result has been improved for $g=6,8,12$ in several references listed in the following theorem.

Theorem 1.2 *Let $q\geq 2$ be a prime power and $g=6,8,12$.*

(i) [1, 6] $n(q;6)\leq 2(q^2-1)$;

(ii) [6] $n(q-1;6)\leq 2(q^2-q-2)$;

(iii) [2] $n(k;6)\leq 2(qk-2)$ for all $k\leq q-1$;

(iv) [7] $n(k;8)\leq 2q(qk-1)$ for all $k\leq q$;

(v) [18] $n(q;8)\leq 2q(q^2-2)$ if q is a square;

(vi) [3] $n(k;12)\leq 2kq^2(q^2-1)$ for all $k\leq q$.

As far as the lower bounding of $n(D;g)$ is concerned, Downs, Gould, Mitchem and Saba [14] obtained the following bound, by counting the vertices emerging from a vertex with maximum degree.

Theorem 1.3 *[14] Let $D=\{a_1,a_2,\ldots,a_k\}$ be a set of positive integers with $2\leq a_1<a_2<\cdots<a_k$, and $g\geq 3$. Then*

$$n(D;g)\geq n_0(D;g)=\begin{cases}1+\sum\limits_{i=1}^{t}a_k(a_1-1)^{i-1} & \text{if } g=2t+1,\\ 1+\sum\limits_{i=1}^{t-1}a_k(a_1-1)^{i-1}+(a_1-1)^{t-1} & \text{if } g=2t.\end{cases} \tag{2}$$

Expression (2) is easily seen to hold when $D = \{r\}$ by replacing both a_k and a_1 with r.

A $(D; g)$-cage satisfying $n(D; g) = n_0(D; g)$ is called a ***minimal*** *$(D; g)$-cage*. Kapoor, Polimeni and Wall [22] proved that $(D; 3)$-cages are minimal, i.e., $n(D; 3) = n_0(D; 3) = 1 + a_k$.

Chartrand, Gould and Kapoor [12] proved that $(\{2, m\}; g)$-cages are minimal. They also proved that $n(\{r, m\}; 4) = n_0(\{r, m\}; 4) = r + m$ for any $r \geq 2$. Moreover, Downs, Gould, Mitchem and Saba [14] proved that $n(\{3, m\}; 5) = n_0(\{3, m\}; 5) = 1 + 3m$ and $n(\{3, m\}; 7) = n_0(\{3, m\}; 7) = 1 + 7m$, for $m \geq 4$; and $n(\{3, m\}; 9) = n_0(\{3, m\}; 9) = 1 + 15m$, for $m \geq 6$. Araujo, Balbuena, and Valenzuela [5] provided some constructions of $(\{r, m\}; g)$-cages in the case $g \in \{5, 7, 11\}$, when $r - 1$ is a prime power, k an even integer and $m = k(r - 1)$.

Yuansheng and Liang [30] proved that $n(\{r, m\}; 6) \geq 2(rm - m + 1)$, and they conjectured that $n(\{r, m\}; 6) = 2(rm - m + 1)$. Moreover, they proved that the conjecture is true when $m - 1$ is a prime power and also for any m and $r = 3, 4, 5$. Constructions of minimal $(\{r, m\}; 6)$-cages when $r - 1$ is a prime power and $m = k(r - 1) + 1$ for $k \geq 2$ are also provided [4]. For the case where the girth is even and greater than 6, no new results have been achieved.

In the Table 1 we present some of the known exact values of $n(\{r, m\}; g)$.

$n(\{r,m\};g)$	$g=5$	$g=6$	$g=7$	$g=8$	g=9	$g=11$
$r=3$	$3m+1$ $m \geq 4$ [14]	$4m+2$ $m \geq 4$ [30]	$7m+1$ $m \geq 4$ [14]	$8m+\frac{m}{3}+5$ $m=3k$ [5]	$1+15m$ $m \geq 6$ [14]	
$r=4$	$4m+1$ $m \geq 5$ [19]	$6m+2$ $m \geq 5$ [19]				
$5 \leq r < m$		$2(rm-m+1)$ $m-1=p^\alpha$ [4, 30]				
$3 \leq r < m$ $r-1=p^\alpha$	$1+rm$ $m=k(r-1)$ $k \geq 2$ even [5]	$2(rm-m+1)$ $m=k(r-1)+1$ $k \geq 2$ [4], or $m=kr, k \geq 2$ [5]	$1+m(r^2-r+1)$ $m=k(r-1)$ $k \geq 2$ even [5]			$1+m\frac{(r-1)^5-1}{r-2}$ $m=k(r-1)$ $k \geq 2$ even [5]

Tab. 1: Exact values of $n(\{r, m\}; g)$ (p^α denotes *a prime power*).

Concerning other structural properties of interest of $(D; g)$-cages, some results for $(r; g)$-cages have been extended by Balbuena and Marcote [10, 11].

Theorem 1.4 *[11] Let $D = \{a_1, a_2, \ldots, a_k\}$ with $2 \leq a_1 < a_2 < \cdots < a_k$, and let g_1, g_2 be two integers such that $3 \leq g_1 < g_2$. Then $n(D; g_1) < n(D; g_2)$ provided that any of the following conditions hold:*

(i) $g_1 = 3$*;*

(ii) some $a_i \in D$ is even and has frequency at least two;

(iii) every $a_i \in D$ is even;

(iv) some $a_i \in D$ is odd and $g_1 \geq k - 1$;

(v) $k \leq 5$*;*

(vi) some $a_i \in D$ is odd for $i \geq \lfloor (k + 8)/3 \rfloor$ and has frequency at least 3.

Theorem 1.5 *[10] Let $D = \{r, m\}$ with $2 \leq r < m$, and let G be a $(D; g)$-cage. Then the diameter of G is at most g if one of the following assertions hold:*

(i) r *is even and the frequency of* r *is at least two;*

(ii) $m = r + 1$.

Apart from the order and the diameter, also the connectivity of $(D; g)$-cages is a basic goal to approach. In the following theorem we list some useful known sufficient conditions on the diameter of a graph in terms of the girth to guarantee optimal results for some parameters accounting for its connectivity.

Theorem 1.6 *Let G be a graph with minimum degree $\delta \geq 2$, diameter $diam(G)$ and girth g. Then,*

(i) [27] if $diam(G) \leq 2\lfloor (g-1)/2 \rfloor - 1$, then $\kappa(G) = \delta$;

(ii) [27] if $diam(G) \leq 2\lfloor (g-1)/2 \rfloor$, then $\lambda(G) = \delta$;

(iii) [8] if $diam(G) \leq g - 2$, then G is optimally restricted edge connected.

Going back to the framework of $(D; g)$-cages, the connectivity of semiregular cages was studied by Balbuena et al. [9]. They proved the following result.

Theorem 1.7 *[9] Every $(\{r, r+1\}; g)$-cage is maximally edge connected. And every $(\{3, 4\}; g)$-cage is maximally connected.*

This paper is devoted to semiregular cages, and is organized as follows. In Section 2 we prove that the diameter of an $(\{r, r+1\}; g)$-graph whose order is close enough to the (minimal) bound $n_0(\{r, r+1\}; g)$ has diameter at most $g - 2$. As an application every $(\{r, r+1\}; g)$-graph with odd girth and order close enough to $n_0(\{r, r+1\}; g)$ is shown to be maximally connected. In Section 3 we present an upper bound on the order of a semiregular cage. Finally, Section 4 deals with the connectivity of semiregular cages. With the help of the aforementioned new upper bound on $n(\{r, r+1\}; g)$, semiregular cages are proved to be maximally connected when $g = 6, 8$, and when $g = 12$ for $r \geq 7$ and $r \neq 20$. Furthermore, it is also shown that every $(\{r, r+1\}; g)$-cage with $r \geq 4$ and $g \geq 6$ is 3-connected, extending a previous result obtained by the authors in [9] for $r = 3$.

2 Diameter of $(\{r, r+1\}; g)$-graphs with small order

In what follows an upper bound on the diameter of semiregular graphs with small order is given.

Theorem 2.1 *Let G be an $(\{r, r+1\}; g)$-graph with $r \geq 2$ and $g \geq 6$ on at most $n_0(\{r, r+1\}; g) + r - 1$ vertices. Then the diameter is at most $diam(G) \leq g - 2$.*

Proof: Let us first prove the following claim.

Claim: No two vertices of degree $r + 1$ are adjacent.

Suppose that there exists an edge $xy \in E(G)$ such that $d(x) = d(y) = r + 1$, then if g is odd it follows

$$\begin{aligned} |V(G)| &\geq 1 + (r+1) \sum_{i=0}^{(g-3)/2} (r-1)^i + \sum_{i=0}^{(g-5)/2} (r-1)^i \\ &= n_0(\{r, r+1\}; g) + \sum_{i=0}^{(g-5)/2} (r-1)^i \end{aligned}$$

which is a contradiction to the hypothesis. And if g is even then

$$\begin{aligned}|V(G)| &\geq 2+2r\sum_{i=0}^{g/2-2}(r-1)^i\\ &= 1+r\sum_{i=0}^{g/2-2}(r-1)^i+(r-1)^{g/2-1}+2\sum_{i=0}^{g/2-2}(r-1)^i\\ &= n_0(\{r,r+1\};g)+\sum_{i=0}^{g/2-2}(r-1)^i\end{aligned}$$

which is again a contradiction to the hypothesis. Then any two vertices of degree $r+1$ are at distance at least two. $\square$

To continue the proof, first suppose that G contains two vertices u and v of degree r such that $d(u,v)\geq g-1$. Let us consider the graph $G'=G\cup\{uv\}$. If G' contains a vertex of degree r, then G' is an $(\{r,r+1\};g)$-graph having two vertices of degree $r+1$ at distance one, contradicting the *Claim*. If there are no vertices of degree r in G', then G' is an $(r+1;g)$-graph and

$$|V(G')|=|V(G)|\geq n(r+1;g)>n_0(\{r,r+1\};g)+r-1.$$

Therefore $d(u,v)\leq g-2$ for all two vertices u,v of degree r. Next let us see that any vertex u of degree $r+1$ has eccentricity at most $\lceil (g+1)/2\rceil$.

Let us consider the subgraph H induced by the sets of vertices

$$\bigcup_{i=0}^{(g-1)/2}N_i(u),\quad \text{if } g \text{ is odd},\qquad \bigcup_{i=0}^{g/2-1}N_i(e),\quad \text{if } g \text{ is even},$$

e being an edge incident with u. Then

$$|V(H)|\geq\begin{cases}1+(r+1)\displaystyle\sum_{i=1}^{(g-1)/2}(r-1)^{i-1}=n_0(\{r,r+1\};g), & \text{if } g \text{ is odd},\\ 2\displaystyle\sum_{i=0}^{g/2-1}(r-1)^i+\sum_{i=0}^{g/2-2}(r-1)^i=n_0(\{r,r+1\};g), & \text{if } g \text{ is even}.\end{cases}$$

Clearly if $|V(G)|=|V(H)|$, then $G=H$ and so for all $x\in V(G)$, $d(u,x)\leq (g-1)/2$ when g is odd, and $d(u,x)\leq g/2$ for g even; hence the theorem is valid. Thus assume that $|V(G)|>|V(H)|$ and let $R=V(G)\setminus V(H)$. If some vertex $s\in R$ is not joined to some vertex in H, then $|V(R)|\geq d(s)+1\geq r+1$, which means that $|V(G)|\geq |V(H)|+|R|\geq n_0(\{r,r+1\};g)+r+1$, a contradiction to the hypothesis. Therefore, any vertex of R is joined to some vertex of H yielding for all $x\in V(G)$:

$$d(u,x)\leq\begin{cases}(g+1)/2, & \text{for } g \text{ odd},\\ g/2+1, & \text{for } g \text{ even},\end{cases}$$

hence the claimed eccentricity of u holds.

Therefore the diameter of G is at most $g-2$. $\square$

The next result follows combining Theorem 1.6 and Theorem 2.1.

Theorem 2.2 *Let G be an $(\{r, r+1\}; g)$-graph with $r \geq 2$. Then G is maximally connected if $g \geq 7$ is odd and the order is at most $n_0(\{r, r+1\}; g) + r - 1$.*

In [10] was proved that all minimal $(\{r, m\}; g)$-cages are 2-connected. Hence Theorem 2.2 is an improvement of this result when $m = r + 1$ and the order is close enough to the minimal bound.

3 Upper bounds on the order of $(\{r, r+1\}; g)$-cages

Note that $n(\{r, r+1\}; g) < n(r+1; g)$ for every $g \geq 4$. Indeed, if G is an $(r+1; g)$-cage with $g \geq 4$, then for every $v \in V(G)$ the graph $G - v$ is an $(\{r, r+1\}; g')$-graph with $g' \geq g$, hence $n(\{r, r+1\}; g) \leq n(\{r, r+1\}; g') \leq |V(G)| - 1 < n(r+1; g)$, the first inequality due to Theorem 1.4. Next an upper bound for $n(\{r, r+1\}; g)$ is given.

Theorem 3.1 *Let $r \geq 2$ and $g \geq 6$ be two integers. Then*

$$n(\{r, r+1\}; g) \leq n(r+1; g) - \begin{cases} \frac{(r+1)r^{(g-3)/2} - 2}{r-1} & \text{if } g \text{ is odd,} \\ \frac{2r^{(g-2)/2} - 2}{r-1} & \text{if } g \text{ is even and } n(r+1; g) > n_0(r+1; g). \end{cases}$$

Moreover if g is even and $n(r+1; g) = n_0(r+1; g)$, then $n(\{r, r+1\}; 6) = 2r^2$ and $n(\{r, r+1\}; g) \leq 1 + 2r^{(g-6)/2}(r^2 - 1)$ for $g = 8, 12$.

Proof: Let G be an $(r+1; g)$-cage. Let us distinguish two cases.

Case 1. $g \geq 7$ **odd.** In this case note that G is not a minimal cage, since minimal cages for g odd only exist when $r + 1 = 2$ (cycles), $g = 3$ (complete graphs), and $g = 5$ and $r + 1 = 3, 7$ (and possibly) 57; see [20]. This means that the diameter is $diam(G) \geq (g+1)/2$.

Let $u \in V(G)$ be a vertex of maximum eccentricity, and let us consider the following sets of neighbors:

$$N_i(u) = \{z \in V(G) : d(u, z) = i\}, \quad i = 0, 1, \ldots, diam(G).$$

Note that $N_0(u) = \{u\}$ and $N_1(u) = N(u)$. Now consider the induced subgraph T spanned by the vertices within distance $\mu \leq (g-3)/2$ from u. Since G has girth g, it is clear that T is a tree.

Observe that $d_T(z) = r + 1$ for all $z \in N_i(u)$ with $i \leq (g-5)/2$, $d_T(z) = 1$ for all $z \in N_{(g-3)/2}(u)$, and the order of T is

$$|V(T)| = \sum_{i=0}^{\frac{g-3}{2}} |N_i(u)| = 1 + (r+1) \sum_{i=0}^{\frac{g-5}{2}} r^i = \frac{(r+1)r^{(g-3)/2} - 2}{r-1}.$$

Let $G^* = G - V(T)$. Notice that both $N_{(g-1)/2}(u)$ and $N_{(g+1)/2}(u)$ are proper subsets of $V(G^*)$. Furthermore $d_{G^*}(z) = r$ if $z \in N_i(u)$ with $i = (g-1)/2$ and $d_{G^*}(z) = r + 1$ if $z \in N_i(u)$ with $i > (g-1)/2$. Then G^* is an $(\{r, r+1\}; g^*)$-graph with $g^* \geq g$ and from Theorem 1.4 it follows

$$\begin{aligned} n(\{r, r+1\}; g) &\leq n(\{r, r+1\}; g^*) \\ &\leq |V(G^*)| \\ &= |V(G)| - |V(T)| \\ &= n(r+1; g) - \frac{(r+1)r^{(g-3)/2} - 2}{r-1}. \end{aligned}$$

Case 2. $g \geq 6$ **even.** Let $e = uv \in E(G)$ and

$$N_i(e) = \{z \in V(G) : d(\{u, v\}, z) = i\},\ i = 0, 1, \ldots, l,\ \text{with } l \geq \frac{g}{2} - 1.$$

Consider the subgraph T induced by the vertices within distance $\mu \leq (g-4)/2$ from e. Since G has girth g, then T is a tree. Notice that $d_T(z) = r+1$ for all $z \in N_i(e)$ with $i \leq (g-6)/2$ and $d_T(z) = 1$ for all $z \in N_{(g-4)/2}(e)$. The order of T is

$$|V(T)| = 2(1 + r + r^2 + \ldots + r^{(g-4)/2}) = 2\sum_{i=0}^{\frac{g-4}{2}} r^i = \frac{2r^{(g-2)/2} - 2}{r-1}.$$

We need to distinguish two subcases.

Subcase 2.1: Suppose that G is not a minimal cage of even girth, hence $n(r+1; g) > n_0(r+1; g)$.

Let $G^* = G - V(T)$. Note that in this case $N_{(g-2)/2}(e)$ and $N_{g/2}(e)$ are proper subsets of G^*. Furthermore, $d_{G^*}(z) = r$ if $z \in N_{(g-2)/2}(e)$ and $d_{G^*}(z) = r+1$ if $z \in N_i(e)$ for $i > (g-2)/2$. Hence G^* is an $(\{r, r+1\}, g^*)$-graph with girth $g^* \geq g$ and from Theorem 1.4 it follows

$$\begin{aligned} n(\{r, r+1\}; g) &\leq n(\{r, r+1\}; g^*) \\ &\leq |V(G^*)| \\ &= |V(G)| - |V(T)| \\ &= n(r+1; g) - \frac{2r^{(g-2)/2} - 2}{r-1}. \end{aligned}$$

Subcase 2.2: Suppose that G is a minimal cage of even girth.

Note that in this case $diam(G) = g/2$ with $g = 6, 8, 12$ and $n(r+1; g) = n_0(r+1; g)$.

If $g = 6$, G is the incidence graph of a projective plane $\prod = (\mathcal{P}, \mathcal{L})$ of order r, then all the lines $L \in \mathcal{L}$ have $r+1$ points and every point $p \in \mathcal{P}$ is incident with $r+1$ lines. Let $\mathcal{L}_p$ be the set of lines incident with the point p, and let A be a line such that $p \notin A$. Let $B \in \mathcal{L}_p$ and denote by q the point $\{q\} = A \cap B$.

Let us remove from the projective plane $\prod$ the point p and all the lines of the set $\mathcal{L}_p - B$, and also remove the line A and all its points except the point q. Denote by $\prod^*$ the obtained incidence structure, i.e., $\prod^* = (\mathcal{P} \setminus (\{p\} \cup (A - q)), \mathcal{L} \setminus ((\mathcal{L}_p - B) \cup \{A\}))$. Hence $\prod^*$ has $r^2 + r + 1 - (r+1) = r^2$ points and $r^2 + r + 1 - (r+1) = r^2$ lines.

Observe that $\prod^*$ has exactly $r-1$ lines of cardinality $r+1$, which are all the lines through point q except line B, which has cardinality r because B has lost the point p. The rest of the lines have cardinality r since they have lost the point that shared with A (different from q). Moreover, $\prod^*$ has exactly $r-1$ points incident with $r+1$ lines, which are all the points of B different from q. Each of the remaining points p' is incident with r lines because it has lost the line through point p and p'.

Therefore the incidence graph corresponding to $\prod^*$ is an $(\{r, r+1\}; 6)$-graph with $2r^2$ vertices. Taking into account the lower bound $n(\{r, r+1\}; 6) \geq 2r^2$ proved by Yuansheng and Liang [30] we get that $n(\{r, r+1\}; 6) = 2r^2$.

For $g = 8, 12$, we proceed as in the paper [3]. Let T_{uv} be the induced subgraph spanned by the vertices within distance $\mu \leq (g-6)/2$ from the edge $e = uv$. Let $u_1 \in N(u) - v$ and $v_1 \in N(v) - u$. Let $H_{u_1 v_1}$

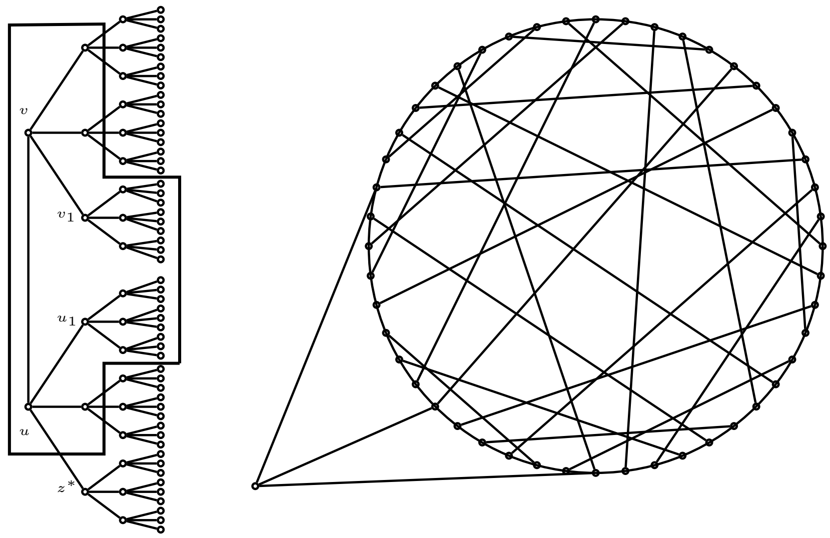

Fig. 1: The deleted vertices in the $(4,8)$-cage are shown in the box. The resulting graph is a $(\{3,4\};8)$-graph on 49 vertices.

be the following subset of vertices

$$H_{u_1v_1} = \bigcup_{j=\frac{g-4}{2}}^{\frac{g-2}{2}} N_j(e) \cap (N_{j-1}(u_1) \cup N_{j-1}(v_1)).$$

Let $z^* \in N_{(g-6)/2}(e) - (N_{(g-8)/2}(u_1) \cup N_{(g-8)/2}(v_1))$, which can be chosen because $r+1 \geq 3$. Let $G^* = G - (V(T_{uv}) - z^*) - V(H_{u_1v_1})$, see Figure 1.

Note that $d_{G^*}(z^*) = r$ and $d_{G^*}(z_i) = r+1$ for every $z_i \in N_{G^*}(z^*)$. Every other vertex of G^* has degree r, since each remaining vertex $v \in N_{(g-4)/2}(e)$ has lost exactly one neighbor in T_{uv}, and each vertex $w \in N_{(g-2)/2}(e)$ has lost exactly one neighbor belonging to either $N_{(g-4)/2}(u_1)$ or $N_{(g-4)/2}(v_1)$. Hence G^* is an $(\{r, r+1\}; g^*)$-graph with girth $g^* \geq g$. Moreover, the order of G^* is:

$$\begin{aligned} |V(G^*)| &= n_0(r+1;g) - \left(2\frac{r^{(g-4)/2}-1}{r-1} - 1\right) - 2(r^{(g-6)/2} + r^{(g-4)/2}) \\ &= 2r^{(g-2)/2} - 2r^{(g-6)/2} + 1 = 2r^{(g-6)/2}(r^2-1) + 1. \end{aligned}$$

In either case the theorem is valid. □

In [4, 30] was obtained that $n(\{r,m\};6) = 2(rm - m + 1)$ for all $m > r \geq 3$ with $m-1$ a prime power. When $m = r+1$ we have $n(\{r, r+1\}; 6) = 2r^2$. In the above Theorem 3.1, this result has been extended for every r for which there exists a projective plane.

4 Vertex connectivity

Every $(\{r, r+1\}; g)$-cage with $r \geq 2$ has been shown to be 2-connected [9]. The main objective of this section is to contribute to answer this question by further approaching the connectedness of semiregular cages. To do that we will use the following known result.

Theorem 4.1 *[25] Let G be a connected graph with minimum degree $\delta \geq 3$, girth g, and order n. Let k be an integer such that $2 \leq k \leq \delta$. Then, G is k-connected if*

$$n \leq \mathcal{N}(\delta, g, k) = \begin{cases} 2\frac{\delta(\delta-1)^{(g-1)/2}-2}{\delta-2} - k & \text{if } g \text{ is odd;} \\ 4\frac{(\delta-1)^{g/2}-1}{\delta-2} - k & \text{if } g \text{ is even.} \end{cases}$$

We also use the following result due to Dusart [15] on the existence of prime powers in short intervals of integers.

Theorem 4.2 *[15] Let r be a positive integer.*

(i) If $r \geq 3275$ then the interval $[r, r(1 + \frac{1}{2ln^2(r)})]$ contains a prime number.

(ii) If $6 \leq r \leq 3276$ then the interval $[r, \frac{7r}{6}]$ contains a prime power.

Using Theorem 4.1 and Theorem 4.2 we obtain the next theorem.

Theorem 4.3 *Let G be an $(\{r, r+1\}; g)$-cage with $r \geq 3$ and $g \geq 6$. Then G is maximally connected provided that any of the following conditions hold.*

(i) $g = 6, 8$.

(ii) $g = 12$ and $r \geq 7$, $r \neq 20$.

Proof: By Theorem 4.1, it is enough to show that $|V(G)| \leq \mathcal{N}(r, g, r)$. Let us distinguish the following cases.

Case (a): Suppose that $n(r+1; g) = n_0(r+1; g)$.

Then $g = 6, 8, 12$. For every $r \geq 3$ if $g = 6, 8$, and for every $r \geq 7$ if $g = 12$ it follows from Theorem 3.1 that:

$$n(\{r, r+1\}; g) \leq \left\{ \begin{array}{ll} 2r^2 & \text{if } g = 6 \\ 1 + 2r^{(g-6)/2}(r^2-1) & \text{if } g = 8, 12 \end{array} \right\} \leq \mathcal{N}(r, g, r) = 4\frac{(r-1)^{g/2}-1}{r-2} - r.$$

Then the result follows from Theorem 4.1. Note that case (a) holds if r is a prime power.

Case (b): Suppose $n(r+1; g) > n_0(r+1; g)$ with $r \in \{q-1, q-2\}$, where q denotes a prime power, and $g = 6, 8, 12$.

Observe that $r \geq 6$. First suppose that $r+1=q$ where $q \geq 7$ is a prime power. Both Theorem 3.1 and Theorem 1.2 yield

$$\begin{aligned} n(\{q-1,q\};g) &\leq n(q;g) - 2\frac{(q-1)^{(g-2)/2}-1}{q-2} \\ &\leq 2q^{(g-6)/2}(q^2-1) - 2\frac{(q-1)^{(g-2)/2}-1}{q-2} \\ &= \begin{cases} 2q^2-2q-2 & \text{if } g=6 \\ 2q^3-2q^2-2 & \text{if } g=8 \\ 2q^5-2q^4+4q^3-8q^2+4q-2 & \text{if } g=12. \end{cases} \end{aligned}$$

Now, it is easy to verify that $n(\{q-1,q\};g) \leq \mathcal{N}(q-1,g,q-1)$ if $q \geq 7$ for $g=6,8$, and if $q \geq 13$ when $g=12$.

Second, suppose that $r+1=q-1$ where q is a prime power; hence $r \geq 14$ and $q \geq 16$ can be assumed. Again from Theorem 3.1 and Theorem 1.2 it follows

$$n(\{q-2,q-1\};g) \leq \begin{cases} 2q^2-4q-2 & \text{if } g=6; \\ 2q^3-4q^2+4q-6 & \text{if } g=8; \\ 2q^5-4q^4+12q^3-36q^2+46q-22 & \text{if } g=12. \end{cases}$$

Now, it is easy to verify that $n(\{q-2,q-1\};g) \leq \mathcal{N}(q-2,g,q-2)$ if $q \geq 16$ for $g=6,8$, and if $q \geq 23$ for $g=12$.

Case (c): Suppose $n(r+1;g) > n_0(r+1;g)$, with $r+1 \leq q-2$, where q denotes a prime power, and $g=6,8,12$.

Note that we may assume that $r+1 \notin \{q'-1,q',q'+1\}$ for every prime power q', (otherwise Cases *(a) or (b)* hold). Then $r+1 \geq 21$ and hence $1+\frac{1}{2ln^2(r+1)} \leq \frac{7}{6}$, thus by Theorem 4.2 we may assume that $q \leq \lfloor\frac{7}{6}(r+1)\rfloor$. Again from Theorem 3.1 and Theorem 1.2 it follows that

$$n(\{r,r+1\};g) \leq \begin{cases} 2\lfloor\frac{7}{6}(r+1)\rfloor(r+1)-4-2(r+1) & \text{if } g=6; \\ 2\lfloor\frac{7}{6}(r+1)\rfloor(\lfloor\frac{7}{6}(r+1)\rfloor(r+1)-1)-2(r^2+r+1) & \text{if } g=8. \\ 2\lfloor\frac{7}{6}(r+1)\rfloor^2(\lfloor\frac{7}{6}(r+1)\rfloor^2-1)(r+1)-2(r^4+r^3+r^2+r+1) & \text{if } g=12. \end{cases}$$

As in the above cases, it is very easy to verify that $n(\{r,r+1\};g) \leq \mathcal{N}(r,g,r)$ for $g=6,8$; and for $g=12$ if $r \neq 20$.

As a consequence of the above three cases, the result is valid. □

4.1 Semiregular cages are 3-connected

In this section we prove that most of $(\{r,r+1\};g)$-cages are 3-connected.

Theorem 4.4 *Every $(\{r,r+1\};g)$-cage with $r \geq 3$ and $g \geq 6$ is 3-connected.*

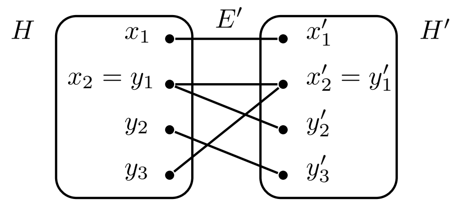

Fig. 2: Construction of G^* .

Proof: The results holds for $r = 3$ by [9], hence asssume $r \geq 4$ for the rest of the proof. Let G be an $(\{r, r+1\}; g)$-cage satisfying the hypothesis. We can suppose that G is 2-connected following [9]. Let $S = \{x, y\}$ be a cutset of G that minimizes the order of the smallest component of the graph obtained by deleting from G a 2-cutset. Let H be a smallest component of $G - S$, notice that $|N(x) \cap V(H)| \geq 2$ and $|N(y) \cap V(H)| \geq 2$, otherwise if $N(x) \cap V(H) = \{z\}$ the set $\{z, y\}$ would be a cut set that leaves a component with fewer vertices than H contradicting the choice of S, and the same occurs if $|N(y) \cap V(H)| = 1$. Furthermore, since $g \geq 6$ it follows that $|N(x) \cap N(y)| \leq 1$. Let us denote $N(x) \cap V(H) = \{x_1, x_2, \ldots, x_\alpha\}$ and $N(y) \cap V(H) = \{y_1, y_2, \ldots, y_\beta\}$, and suppose $\alpha \leq \beta$ without loss of generality. Let H' be a copy of H with $V(H) \cap V(H') = \emptyset$, and for every $v \in V(H)$ let v' denote its copy in H'. Let G^* be a new graph obtained from the union of H and H' and by adding the following edges:

$$E' = \{x_1x'_1, x_2x'_2, \ldots, x_\alpha x'_\alpha, y_1y'_2, y_2y'_3, \ldots, y_\beta y'_1\}.$$

Observe that if $N(x) \cap N(y) \cap V(H) = \{z\}$, then the vertices z and z' are both incident with two new added edges (see Figure 2). Hence, each vertex in G^* has the same degree it had in G and $|V(G^*)| = 2|V(H)| \leq |V(G)| - 2$.

Now, let us show that $g(G^*) = g^* \geq g$. Let C be a cycle of G^*. Since E' is an edge cut of G^* then $|E(C) \cap E'|$ is an even number.

(a) If $|E(C) \cap E'| = 0$, then C corresponds with a cycle of G, therefore C has length at least g.

(b) Suppose $|E(C) \cap E'| = 2$. If $x_ix'_i$ and $x_jx'_j$ are in $E(C)$ for some distinct i, j, then the length $l(C)$ of C is:

$$l(C) \geq d_H(x_i, x_j) + d_{H'}(x'_i, x'_j) + 2 = 2d_H(x_i, x_j) + 2 \geq 2(g-2) + 2 = 2g - 2 > g.$$

And the same occurs if $y_iy'_{i+1}$ and $y_jy'_{j+1}$ are in $E(C)$ for some different i, j. Finally, if $x_ix'_i$ and $y_jy'_{j+1}$ are in $E(C)$, then

$$\begin{aligned} l(C) &\geq d_H(x_i, y_j) + d_{H'}(x'_i, y'_{j+1}) + 2 \\ &= d_H(x_i, y_j) + d_H(x_i, y_{j+1}) + 2 \\ &\geq d_H(y_j, y_{j+1}) + 2 \\ &\geq (g-2) + 2 \\ &= g. \end{aligned}$$

(c) Suppose $|E(C) \cap E'| \geq 4$. If C contains an (x_i, x_j)-path or an (y_s, y_t)-path, then analogously to (b),

$$l(C) \geq (g-2) + 4 > g.$$

The unique remaining case to consider is when C contains neither an (x_i, x_j)-path nor an (y_s, y_t)-path. Let us consider an (x_i, y_j)-path in H and an (x'_s, y'_t)-path in H'. Observe that we can assume that at most one of the conditions $i = s$, $j = t$ holds; otherwise we consider other (x'_h, y'_p)-path in H' instead of the path (x'_s, y'_t), as $s \neq h$ and $t \neq p$. When $i = s$ $(j \neq t)$ we have

$$\begin{aligned} l(C) &> d_H(x_i, y_j) + d_{H'}(x'_i, y'_t) + 4 \\ &= d_H(x_i, y_j) + d_H(x_i, y_t) + 4 \\ &\geq d_H(y_j, y_t) + 4 \\ &\geq (g-2) + 4 \\ &> g. \end{aligned}$$

Analogously in case $j = t$ $(i \neq s)$. When $i \neq s$ and $j \neq t$, since the union of an (x_i, y_j)-path, an (x_s, y_t)-path and the edges $x_i x, x x_s, y_j y, y y_t$ contains a cycle that corresponds with a cycle in G, it follows that

$$l(C) \geq (g-4) + 4 = g.$$

We have already seen that $g(G^*) = g^* \geq g$ stands for the girth of G^*. Next we consider the degree set of G^*.

If G^* contains some vertex with degree r and some vertex with degree $r+1$, then G^* is an $(\{r, r+1\}; g^*)$-graph with fewer vertices than G, contradicting Theorem 1.4. Hence the graph G^* may be assumed to be a regular graph. If G^* is an $(r+1; g^*)$-graph, then the graph $G^* - v$ where $v \in V(G^*)$ is an $(\{r, r+1\}; g^*)$-graph with fewer vertices than G, yielding a contradiction to Theorem 1.4. Hence G^* is an $(r; g^*)$-graph. Let us show that G^* has $diam(G^*) \geq g - 1$. We distinguish three cases.

Case 1. $N(x) \cap N(y) \cap V(H) \neq \emptyset$, $\alpha \geq 2$ and $\beta \geq 3$.

Since $g > 4$ then $N(x) \cap N(y) \cap V(H) = \{x_\alpha\} = \{y_\beta\}$ may be assumed. Note that

$$d_H(x_i, y_j) \geq g - 4, \text{ for every } x_i \neq y_j$$

since, when $i \neq \alpha$ and $j \neq \beta$, an (x_i, y_j)-path in H joint with the edges $x_i x, x x_\alpha, x_\alpha y, y y_j$ of G contains a cycle in G of length at least g; and when $i = \alpha$ or $j = \beta$, $d_H(x_i, x_j) \geq g - 2$.

Then (for some $y_k \neq y_1$):

$$\begin{aligned} d_{G^*}(y_1, y'_1) &= \min\{d_H(y_1, x_j) + 1 + d_{H'}(x'_j, y'_1), d_H(y_1, y_k) + 1 + d_{H'}(y'_{k+1}, y'_1), 1 + d_{H'}(y'_2, y'_1)\} \\ &\geq \min\{2(g-4) + 1, 2(g-2) + 1, 1 + g - 2\} \\ &= g - 1, \end{aligned}$$

since $g \geq 6$, $2(g-4) + 1 \geq g - 1$.

Case 2. $N(x) \cap N(y) \cap V(H) = \emptyset$, $\alpha \geq 2$ and $\beta \geq 3$.

Then (for some $y_k \notin \{y_1, y_3\}$):

$$\begin{aligned} d_{G^*}(y_3, y_2') &= \min\{d_H(y_3, y_1) + 1, d_H(y_3, x_i) + 1 + d_{H'}(x_i', y_2'), d_H(y_3, y_k) + 1 + d_{H'}(y_{k+1}', y_2')\} \\ &\geq \min\{g - 2 + 1, d_H(y_3, y_2) + 1, 2(g-2) + 1\} \\ &= g - 1. \end{aligned}$$

Case 3. $\alpha = \beta = 2$.

Hence $\lambda(G^*) \leq |E'| = 4$. If $r \geq 5$ then $\lambda(G^*) < r$ and by Theorem 1.6(ii),

$$diam(G^*) \geq \begin{cases} g-1 & \text{if } g \text{ is even,} \\ g & \text{if } g \text{ is odd.} \end{cases}$$

The same occurs if $\lambda(G^*) < r = 4$. If $r = 4 = \lambda(G^*)$, since the set E' is a restricted edge cut because $g > 4$, then Theorem 1.6(iii) implies $diam(G^*) \geq g - 1$ as the minimum edge degree of G^* is equal to $2(r-1) = 6$.

Therefore G^* contains two vertices z_1 and z_2 such that $d_{G^*}(z_1, z_2) \geq g - 1$. By joining the edge $z_1 z_2$ to G^*, the graph $G^* \cup \{z_1 z_2\}$ is an $(\{r, r+1\}; g^*)$-graph with fewer vertices than G and $g^* \geq g$, yielding a contradiction. □

Acknowledgments

This research was supported by the Ministerio de Educación y Ciencia, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-06620-C03-02/MTM; and by Catalonian Government 1298 SGR2009. We would like to thank the anonymous referees for their suggestions and comments.

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