Solving the kernel perfect problem by (simple) forbidden subdigraphs for digraphs in some families of generalized tournaments and generalized bipartite tournaments

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A digraph such that every proper induced subdigraph has a kernel is said to be *kernel perfect* (KP for short) (*critical kernel imperfect* (CKI for short) resp.) if the digraph has a kernel (does not have a kernel resp.). The unique CKI-tournament is \vec{C}_3 and the unique KP-tournaments are the transitive tournaments, however bipartite tournaments are KP. In this paper we characterize the CKI- and KP-digraphs for the following families of digraphs: locally in-/out-semicomplete, asymmetric arc-locally in-/out-semicomplete, asymmetric 3-quasi-transitive and asymmetric 3-anti-quasi-transitive TT_3 -free and we state that the problem of determining whether a digraph of one of these families is CKI is polynomial, giving a solution to a problem closely related to the following conjecture posted by Bang-Jensen in 1998: the kernel problem is polynomially solvable for locally in-semicomplete digraphs.

Keywords: kernel, perfect graph, kernel perfect digraph, locally in-/out-semicomplete digraph, asymmetric arc-locally in-/out-semicomplete digraph, asymmetric 3-(anti)-quasi-transitive digraph

1 Introduction

A family which is a generalization of tournaments is a family of digraphs that in some way preserves basic structures of the tournaments, an interesting survey of generalizations of tournaments can be found in Bang-Jensen and Gutin (1998). Generalizations of tournaments have been widely studied, more than 300 papers have been published in this topic and this has improved the understanding of topics such as hamiltonicity, domination and pancyclicity, properties that digraphs of some families of generalized tournaments preserve, see Bang-Jensen and Gutin (1998); Bang-Jensen and Huang (1995). The locally semicomplete digraphs introduced in Bang-Jensen (1990) are among the families which have been most studied. Locally in-semicomplete digraphs and locally out-semicomplete digraphs are also generalizations

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of tournaments. A digraph D is *locally in-semicomplete* (resp. *locally out-semicomplete*) if for any vertex $v \in V(D)$, the in-neighborhood (resp. out-neighborhood) induces a semicomplete digraph in D. A digraph D is *locally semicomplete* if it is both locally in-semicomplete and locally out-semicomplete. Observe that locally semicomplete digraphs are locally in-semicomplete and locally out-semicomplete, but the converse is not true (see Figure 1), the class of locally in-semicomplete digraphs (resp. locally)



Fig. 1: A locally-in semicomplete digraph which is not a locally semicomplete digraph.

out-semicomplete) is a quite wider class than the class of locally semicomplete digraphs. The arc-locally semicomplete digraphs was defined in Bang-Jensen (2004), this definition is somehow close related to the definition of locally semicomplete digraphs, although they are generalizations of bipartite tournaments and it is surprising so few results there are on this family, properties such as hamiltonicity and pancyclism are known. A family of digraphs is a *family of generalized bipartite tournaments* if the digraphs preserve in some way basic structures of bipartite tournaments. The arc-locally in-semicomplete, arc-locally outsemicomplete, 3-quasi-transitive and 3-anti-quasi-transitive digraphs are families of generalized bipartite tournaments. A digraph D is *arc-locally in-semicomplete (arc-locally out-semicomplete*, resp.) if, for every arc $uv \in D$ and every pair of vertices x, y such that $x \in N^-(u)$ and $y \in N^-(v)$ ($x \in N^+(u)$ and $y \in N^+(v)$, resp.), x, y are adjacent. A digraph D is *arc-locally semicomplete* if D is arc-locally in-semicomplete and arc-locally out-semicomplete. In Bang-Jensen (2004), a digraph D was defined as 3-quasi-transitive (3-anti-quasi-transitive resp.) if, for every directed (anti-directed resp.) 4-path uvwx, u and x are adjacent.

Kernels are an important topic in the theory of digraphs, they where introduced in Neumann and Morgenstern (1944) and has received lot of attention due to its theoretical interest as well as its many applications in different areas such as game theory, argumentation theory, logic, logic programming and artificial intelligence, see Le Bars (2000); Creignou (1995); Dimopoulos and Torres (1996); Dung (1995). A kernel in a digraph is an independent and absorbent set of vertices (a subset of vertices S of a digraph D is an absorbent set if for every vertex $u \in V(D) \setminus S$ there is a vertex $v \in S$ such that $uv \in A(D)$, for surveys of kernels see Bang-Jensen and Gutin (1998); Haynes et al. (1998). The problem of deciding whether a digraph D has a kernel is \mathcal{NP} -complete, see Chvátal (1973), even for planar digraphs with in-degree and out-degree at least 1 and degree at most three, see Fraenkel (1981). Due to the difficulty of this topic, the study of kernels is centered in sufficient conditions to asure the existence of kernels and the study of fixed classes of digraphs having a kernel or fixed classes of kernel perfect digraphs. The existence of kernels in digraphs with a perfect underlying graph has been studied in Galeana-Sánchez and Rojas-Monroy (2006); the existence of kernels in locally in-/out-semicomplete digraphs has been studied in Galeana-Sánchez (1995, 1997); and the existence of kernels in arc-locally in-/out-semicomplete digraphs has been studied in Galeana-Sánchez (2006). The Strong Perfect Graph Conjecture, stated by C. Berge in 1960 and proved in Chudnovsky et al. (2006) states that a graph G is perfect if and only if G contains neither C_{2n+1} nor the complement of C_{2n+1} , $n \geq 2$, as an induced subgraph. Many authors have contributed to obtain nice properties and interesting characterizations of Perfect Graphs, see Chvátal and Berge (1984); Ramírez-Alfonsín and Reed (2001). The underlying graph G_D of a digraph D is the graph on the vertex set $V(G_D) = V(D)$ whith $\{u, v\} \in E(G_D)$ if and only if $\{uv, vu\} \cap A(D) \neq \emptyset$. The digraphs of some families of generalized tournaments have a perfect graph as underlying graph, for instance *quasi-transitive digraphs* (for every directed path uvw there is an arc between u and w) and *semicomplete multipartite tournaments* (the underlying graph is a complete multipartite graph).

A digraph such that every proper induced subdigraph has a kernel is said to be a kernel perfect digraph (KP-digraph) (critical kernel imperfect digraph (CKI-digraph) resp.) if the digraph D has a kernel (does not have a kernel resp.). In Berge and Duchet (1990) it was conjectured that a graph G is perfect if and only if any orientation by sinks of G is a kernel perfect digraph, the authors considered orientations of G by directing each edge of G in at least one of the two possible directions. An orientation of G is an *orientation by sinks* (or normal) if every induced semicomplete subgraph H of G has an absorbing vertex in H (a vertex $v \in V(H)$ such that $uv \in A(H)$ for every vertex $u \in V(H) \setminus \{v\}$). This conjecture was proved in Berge and Duchet (1990) and in Boros and Gurvich (1996), and it constructs an important bridge between two topics in graph theory: namely colorings and kernels. It is important to stress that a digraph D without induced CKI-digraph is a KP-digraph and so, it does have a kernel. Although there are digraphs with kernels which are not KP. Hence, another tool to decide whether a digraph has a kernel is characterizing the CKI-digraphs; due to a result of the first author and Neumann-Lara Galeana-Sánchez and Neumann-Lara (1991) CKI-digraphs cannot be characterized by forbidden minors. For structural properties of CKI-digraphs see Balbuena et al. (2014); Galeana-Sánchez and Guevara (2016); Galeana-Sánchez and Neumann-Lara (1991); Galeana-Sánchez and Olsen (2016a) and for characterizations of families of CKI-digraphs see Galeana-Sánchez and Olsen (2016a,b). The unique tournament which is CKI is \overline{C}_3 and the unique KP-tournaments are the transitive tournaments, however any induced subdigraph of a bipartite tournament does have a kernel. Hence, bipartite tournaments are KP. In Galeana-Sánchez and Neumann-Lara (1986) it was proved that the semicomplete CKI-digraphs are \vec{C}_3 and the family $D \cong \vec{C}_n(1, \pm 2, \pm 3, \dots, \pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$, where $\vec{C}_m(J)$ is a *circulant* (or *rotational*) *digraph* defined by $V(\overrightarrow{C}_m(J)) = \mathbb{Z}_m$ and $A(\overrightarrow{C}_m(J)) = \{(i,j) : i, j \in \mathbb{Z}_m, j-i \in J\}$, with \mathbb{Z}_m as the cyclic group of integers modulo m ($m \ge 2$) and J is a nonempty subset of $\mathbb{Z}_m \setminus \{0\}$. In Galeana-Sánchez and Olsen (2016b) the authors characterized the locally semicomplete CKI-digraphs as directed odd cycles, $\overrightarrow{C}_7(1,2)$ and $\overrightarrow{D} \cong \overrightarrow{C}_m(1,\pm 2,\pm 3,\ldots,\pm \lfloor \frac{n}{2} \rfloor)$, for $n \ge 4$.

We characterize the CKI-digraphs having a perfect graph as underlying graph, using the relation between perfect graphs and kernel perfect graphs Berge and Duchet (1990); Boros and Gurvich (1996); Galeana-Sánchez (2012) and we characterize the locally in-semicomplete digraphs and the locally outsemicomplete digraphs which are CKI-/KP-digraphs. It is important to stress that the property of being CKI is not preserved for the converse digraph of a CKI digraph, see Duchet and Meyniel (1981), the converse digraph D^{-1} of a digraph D is obtained by reversing the arcs of D. Hence, characterizing the locally in-semicomplete digraphs and the locally out-semicomplete digraphs which are CKI-/KP-digraphs is not the same problem, although for these two families the families of CKI-/KP-digraphs remains the same. Finally, we characterize the asymmetric CKI-/KP-digraphs which are arc-locally in-semicomplete, arc-locally out-semicomplete or 3-anti-quasi-transitive TT_3 -free digraphs as the directed odd cycles and the asymmetric 3-quasi-transitive CKI-digraphs as \vec{C}_3 . Moreover, we state that the problem of determining whether a digraph is CKI is polynomial for digraphs of the following families of digraphs: locally in-/out-semicomplete, asymmetric arc-locally in-/out-semicomplete, 3-quasi-transitive and asymmetric 3-anti-quasi-transitive TT_3 -free. Hence, we give a solution to a problem closely related to the following conjecture posted in Bang-Jensen and Gutin (1998): the kernel problem is polynomially solvable for locally in-semicomplete digraphs.

2 Definitions and prelimitries

For general concepts and notation we refer the reader to Bang-Jensen and Gutin (2008). The paths and cycles considered in this paper are not necessarily directed paths or cycles. We denote the path P by the sequence of its vertices $P = u_0 u_1 \dots u_n$. We say that P is a *directed path* if $u_i u_{i+1} \in A(D)$ for $0 \le i \le n-1$, P is an *anti-directed path* if it has no directed subpath of length 2. Let D be a digraph and H a proper subdigraph of D. An arc $uv \in A(D) \setminus A(H)$ is a *diagonal* of H whenever $u, v \in V(H)$. For a subset of vertices S of a digraph D, D[S] denotes the digraph induced by the vertex set S.

Remark 1 Directed odd cycles and $\vec{C}_7(1,2)$ are CKI-digraphs. Directed even cycles are KP-digraphs.

The following remark is a consequence of the definition of KP-digraphs and CKI-digraphs.

Remark 2 If D is a CKI-digraph (or a KP-digraph), then D has no proper induced CKI-subdigraph. In particular, D has no proper induced subdigraph isomorphic to a directed odd cycle.

Circulant digraphs are regular, vertex transitive and isomorphic to its converse digraph (the *converse* digraph D^{-1} of a digraph D is obtained by reversing the arcs of D). Thus, if a circulant digraph D is CKI, then D^{-1} is also CKI. This is not true in general, due to Duchet and Meyniel (1981). A graph is a *perfect graph*, if for every induced subgraph, the clique number equals the chromatic number.

Theorem 1 (Strong Perfect Graph Theorem; Chudnovsky et al. (2006)) A graph G is not perfect if and only if G has as an induced subgraph

- (*i*) an odd cycle on at least 5 vertices or
- *(ii) the complement of an odd cycle on at least 7 vertices.*

Let *H* be a subdigraph of *G*. An *absorbing vertex* in *H* is a vertex $v \in V(H)$ such that $uv \in A(H)$ for every vertex $u \in V(H) \setminus \{v\}$. A digraph D_G is an orientation (or biorientation) of a graph *G* if $V(D_G) = V(G)$ and $\{u, v\} \in E(G)$ if and only if $\{uv, vu\} \cap A(D) \neq \emptyset$, and an orientation D_G of *G* is an *orientation by sinks* if every complete subgraph of *G* has an absorbing vertex in D_G .

Theorem 2 (Berge and Duchet (1990); Boros and Gurvich (1996)) A graph G is perfect if and only if every orientation by sinks of G is a KP-digraph.

For another relation between kernels and perfect graphs see Galeana-Sánchez (2012).

Theorem 3 (Galeana-Sánchez and Neumann-Lara (1986)) A semicomplete digraph D is a CKIdigraph if and only if $D \cong \overrightarrow{C}_3$ or $D \cong \overrightarrow{C}_n(1, \pm 2, \pm 3, \dots, \pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

Theorem 4 (Galeana-Sánchez and Olsen (2016b)) A locally semicomplete digraph D is a CKI-digraph if and only if D is an odd cycle, $D \cong \overrightarrow{C}_7(1,2)$ or $D \cong \overrightarrow{C}_n(1,\pm 2,\pm 3,\ldots,\pm |\frac{n}{2}|)$, for some $n \ge 4$.

3 Generalized tournaments

In this section we characterize the CKI-digraphs with a perfect underlying graph and the locally in- and the locally out-semicomplete CKI-digraphs. As a consequence of Remark 2, Theorems 2 and 3 we have the following characterization of CKI-digraphs having a perfect underlying graph.

Theorem 5 Let D be a digraph such that the underlying graph, G_D , is a perfect graph. Then D is CKI if and only if $D \cong \overrightarrow{C}_3$ or $D \cong \overrightarrow{C}_n(1, \pm 2, \pm 3, \dots, \pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

Proof: Let D be a CKI-digraph such that G_D is a perfect graph. By Theorem 2, D is not oriented by sinks, and therefore D has an induced semicomplete subdigraph H with no sink. Hence, H is not KP and contains an induced subdigraph H' which is semicomplete and CKI. By Remark 2, D has no proper induced CKI-subdigraph, therefore D = H'. Hence, G_D is complete, D is semicomplete and by Theorem 3, $D \cong \vec{C}_3$ or $D \cong \vec{C}_n(1, \pm 2, \pm 3, \dots, \pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

A digraph is a *semicomplete multipartite tournament* if the underlying graph is a complete multipartite graph. A digraph is *quasi-transitive* if for every directed path uvw there is an arc between u and w. We have the following corollary.

Corollary 1 Let D be a semicomplete multipartite digraph or a quasi-transitive digraph. Then D is CKI if and only if $D \cong \overrightarrow{C}_3$ or $D \cong \overrightarrow{C}_m(1, \pm 2, \pm 3, \dots, \pm \lfloor \frac{m}{2} \rfloor)$ for some $m \ge 4$.

Proof: The underlying graph of a semicomplete multipartite digraph is a complete multipartite graph, which is a perfect graph and the underlying graph of a quasi-transitive digraph is a comparability graph, see Duchet (1984), which is a perfect graph. Hence, the result follows. \Box

As a consequence of Theorem 5 and Corollary 1 we have the following result.

Theorem 6 Let D be a digraph such that the underlying graph is a perfect graph. Then D is KP if and only if D has no induced subdigraph H such that $H \cong \vec{C}_3$ or $H \cong \vec{C}_n(1, \pm 2, \pm 3, \ldots, \pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$. In particular, semicomplete multipartite digraphs and quasi-transitive digraphs are KP if and only if they have no induced subdigraph H with $H \cong \vec{C}_3$ or $H \cong \vec{C}_m(1, \pm 2, \pm 3, \ldots, \pm \lfloor \frac{n}{2} \rfloor)$, for some $m \ge 4$.

The following lemma determines the orientation of some induced subdigraphs in asymmetric locally in-semicomplete digraphs or asymmetric locally out-semicomplete digraphs. An arc $uv \in A(D)$ is asymmetric if $vu \notin A(D)$ (symmetric if $vu \in A(D)$ resp.). A digraph is asymmetric if all its arcs are asymmetric arcs. Observe that if $n \ge 5$, \overline{G}_D is an induced cycle on n vertices if and only if G_D is an antihole on n vertices.

Lemma 1 Let D be a locally in-semicomplete digraph or a locally out-semicomplete digraph. If G_D is an induced cycle on at least 4 vertices, then D is a directed cycle. If \overline{G}_D is an induced cycle on at least 5 vertices, then $D \cong \overrightarrow{C}_{2n+1}(2, -3, 4, -5, ..., (-1)^n n)$.

Proof: Let *D* be a locally in-semicomplete digraph (locally out-semicomplete digraph resp.). Suppose that G_D is an induced cycle $C = u_0u_1 \dots u_{n-1}u_0$, with $n \ge 4$. Since *C* is induced and *D* is locally in-semicomplete, (locally out-semicomplete resp.), there is no vertex $u_i \in V(C)$ such that $u_{i-1}, u_{i+1} \in N^-(u_i)$, $(u_{i-1}, u_{i+1} \in N^+(u_i)$ resp.), with indices taken modulo *n*. Thus, *C* is an induced directed cycle on at least 4 vertices. Hence, if *D* is a locally in-/out-semicomplete digraph and G_D is an induced cycle on at least *n* vertices, $n \ge 4$, then *D* is a induced directed cycle on *n* vertices.

If D is a locally in-semicomplete digraph or a locally out-semicomplete digraph and \overline{G}_D is an induced cycle on 5 vertices, then G_D is an induced cycle on 5 vertices and D is a directed cycle on 5 vertices.

Let D be a locally in-semicomplete digraph. Suppose that \overline{G}_D is an induced cycle $C = u_0 u_1 \dots u_{n-1} u_0$, with $n \ge 6$. Observe that u_i and u_{i+1} are not adjacent in D for $0 \le i \le n-1$ with indices taken modulo n and since D is locally in-semicomplete, $N^+(u_i) \cap N^+(u_{i+1}) = \emptyset$.

Let $u_i u_{i+j} \in A(D)$. Clearly, $2 \leq j \leq n-2$. If $j \geq 3$ and $u_{i+1}u_{i+j} \in A(D)$, then $u_i, u_{i+1} \in N^-(u_{i+j})$, a contradiction because $\{u_i, u_{i+1}\} \notin E(G_D)$. Thus, if $j \geq 3$, it follows that $u_{i+1}u_{i+j} \notin E(G_D)$.



Fig. 2: The dotted arcs are not arcs of D.

A(D), see Figure 2. Analogously, if $j \le n-3$, then $u_{i-1}u_{i+j} \notin A(D)$, see Figure 2. Since C is an induced cycle in \overline{G}_D , it follows that for $u_i u_{i+j} \in A(D)$

(a) if
$$j \ge 3$$
, then $u_{i+j}u_{i+1} \in A(D)$, (b) if $j \le n-3$, then $u_{i+j}u_{i-1} \in A(D)$. (1)

First we prove that D is asymmetric, and then we determine the arcs of D.

For a contradiction, suppose that $\{u_i, u_{i+j}\}$ is a symmetric arc for some $j \ge 4$. Since $u_i u_{i+j} \in A(D)$ it follows by 1(a) that $u_{i+j}u_{i+1} \in A(D)$, see Figure 3; since $u_{i+j}u_i \in A(D)$ and $n - ((i+j)-i) \le n-4$, it follows by 1(b) that $u_i u_{i+j-1} \in A(D)$, see Figure 3. In this case $u_{i+j-1}u_{i+1} \notin A(D)$, because Dis locally in-semicomplete and u_{i+j-1} and u_{i+j} are not adjacent; and $u_{i+1}u_{i+j-1} \notin A(D)$, because u_i and u_{i+1} are not adjacent. Thus, $\{u_{i+j-1}, u_{i+1}\} \notin E(G_D)$, see Figure 3, a contradiction because



Fig. 3: The dotted arcs are not arcs of D.

 $j \ge 4$ and C is an induced cycle in \overline{G}_D . Thus, if $j \ge 4$, then $\{u_i, u_{i+j}\}$ is an asymmetric arc. By a dual argumentation, using first 1(b) and then 1(a), we obtain that if $j \le n - 4$, then $\{u_i, u_{i+j}\}$ is an asymmetric arc.

If $|V(D)| \ge 7$, then every edge $\{u_i, u_{i+j}\} \in E(\overline{G}_D)$ is an asymmetric arc in D because $j = (i+j) - i \ge 4$ or $n-j = n - ((i+j)-i) \ge 4$, that is at least one of the induced paths in $\overline{G}_D \ u_0 u_1 \dots u_j$ or $u_j u_{j+1} \dots u_0$ has 4 arcs. Hence, D is asymmetric if $|V(D)| \ge 7$.

If |V(D)| = 6, it follows that $\{u_i, u_{i+j}\}$ is asymmetric if j = 2, 4. For a contradiction, suppose that $\{u_0, u_3\}$ is an asymmetric arc. By 1(a), $u_0u_2, u_0u_4, u_3u_1, u_3u_5 \in A(D)$. In this case $u_2u_5 \notin A(D)$, because D is locally in-semicomplete, $u_3u_5 \in A(D)$ and u_2 and u_3 are not adjacent in D; and $u_5u_2 \notin A(D)$, because $u_0u_2 \in A(D)$ and u_0 and u_5 are not adjacent in D. Thus, $\{u_2, u_5\} \notin E(G_D)$, a contradiction. Hence, D is asymmetric.

In order to prove that $D \cong \overrightarrow{C}_{2m+1}(2, -3, 4, -5, \dots, (-1)^m m)$, we need the following claim.

Claim 1 Let j be an integer, with $2 \le j \le n-2$. If $u_k u_{k+j} \in A(D)$ for some vertex $u_k \in V(D)$, then $u_i u_{i+j} \in A(D)$ for every vertex $u_i \in V(D)$.

Proof of Claim 1. Let k, j be integers such that $u_k u_{k+j} \in A(D)$. If j = 2, 3, then by 1(b), it follows that $u_{k+j}u_{k-1} \in A(D)$, see Figure 4(b). Consider the arc $u_{k+j}u_{k-1}$, by 1(b), it follows that $u_{k-1+j} \in A(D)$. Consider the arc u_{k-1+j} , by 1(b), it follows that $u_{k-1+j}u_{k-2} \in A(D)$.

Consider the arc $u_{k-1+j}u_{k-2}$, by 1(b), it follows that $u_{k-2}u_{k-2+j} \in A(D)$. Continuing these two steps it follows that for j = 2, 3, if $u_k u_{k+j} \in A(D)$ for some vertex $u_k \in V(D)$, then $u_i u_{i+j} \in A(D)$ for every vertex $u_i \in V(D)$. Analogously, using 1(a), if j = n - 2, n - 3 and $u_k u_{k+j} \in A(D)$, then $u_i u_{i+j} \in A(D)$ for every $u_i \in V(D)$.



Fig. 4: The dotted arcs are not arcs of D.

If 3 < j < n-3, then by 1(a), it follows that $u_{k+j}u_{k+1} \in A(D)$, see Figure 4(a). Consider the arc $u_{k+j}u_{k+1}$, by 1(a), it follows that $u_{k+1}u_{k+1+j} \in A(D)$. Consider the arc $u_{k+1}u_{k+1+j}$, by 1(a), it follows that $u_{k+1+j}u_{k+2} \in A(D)$. Consider the arc $u_{k+1+j}u_{k+2}$, by 1(a), it follows that $u_{k+2}u_{k+2+j} \in A(D)$. Continuing these two steps it follows that for $4 \le j \le n-4$, if $u_ku_{k+j} \in A(D)$, then $u_iu_{i+j} \in A(D)$ for every $u_i \in V(D)$.

If $u_0u_2 \in A(D)$, then $u_3u_0 \in A(D)$ and in this case $u_0u_4 \in A(D)$. Continuing this argument we obtain that $u_0u_{2i}, u_{2i+1}u_0 \in A(D)$. If the order of D is equal to n = 2m, then if m is even $u_0u_m \in A(D)$ (if m is odd, $u_mu_0 \in A(D)$ resp.) and by Claim 1 it follows that $u_iu_{i+m} \in A(D)$ ($u_{i+m}u_i \in A(D)$) resp.). In particular for i = m we obtain that $u_mu_0 \in A(D)$ ($u_0u_m \in A(D)$ resp.) implying that $\{u_0, u_m\}$ is a symmetric arc, a contradiction. Hence, the order of D is odd. Let n = 2m + 1. If $u_0u_2 \in A(D)$, then by Claim 1 it follows that $D \cong \overrightarrow{C}_{2m+1}(2, -3, \ldots, (-1)^m(m))$. If $u_2u_0 \in A(D)$, by a dual argument, we obtain $D \cong \overrightarrow{C}_{2m+1}(-2, 3, \ldots, (-1)^{m-1}(m))$. Since $\overrightarrow{C}_{2m+1}(J) \cong \overrightarrow{C}_{2m+1}(-J)$, the result follows for in-semicomplete digraphs.

The case when D is a out-semicomplete digraph and \overline{G}_D is an induced cycle on at least 6 vertices is obtained by a dual argumentation, considering locally out-semicomplete digraphs instead of locally in-semicomplete digraphs and changing the orientation of all arcs in the proof.

We use the Strong Perfect Graph Theorem in Chudnovsky et al. (2006) and Lemma 1 to characterize the locally in-semicomplete CKI-digraphs.

Theorem 7 If D is a locally in-semicomplete CKI-digraph or a locally out-semicomplete CKI-digraph, then D is a directed odd cycle, $D \cong \vec{C}_7(1,2)$ or $D \cong \vec{C}_n(1,\pm 2,\pm 3,\ldots,\pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

Proof: Let D be a locally in-semicomplete CKI-digraph or a locally out-semicomplete CKI-digraph. By Theorem 5, if G_D is a perfect graph, then $D \cong \overrightarrow{C}_3$ or $D \cong \overrightarrow{C}_n(1, \pm 2, \pm 3, \ldots, \pm \lfloor \frac{n}{2} \rfloor)$ for some $n \ge 4$. Assume that G_D is not a perfect graph. By Theorem 1, G_D has an induced odd cycle C on at least 5 vertices or \overrightarrow{G}_D has an induced odd cycle C on at least 7 vertices. If G_D has an induced odd cycle C, then by Lemma 1, D[C] is a directed odd cycle, which is CKI. Let $C = u_0u_1 \ldots u_{2n}u_0$. By Remark 2, D[C]is not a proper subdigraph. Hence, D is a directed odd cycle. If \overrightarrow{G}_D has an induced odd cycle C, then by Lemma 1, $D \cong \overrightarrow{C}_{2n+1}(2, -3, 4, -5, \ldots, (-1)^n n)$.

If n > 3, then $2n + 1 \ge 9$. Let $i, j, k \ge 3$ be odd integers such that i + j + k = 2n + 1. Since $u_{i+j+k} = u_0$, we have the proper asymmetric $\vec{C}_3 = u_0 u_i u_{i+j} u_{i+j+k}$, which contradicts Remark 2,

because \overrightarrow{C}_3 and D are CKI-digraphs. Since C has at least 7 vertices, n = 3 and $D[V(C)] = \overrightarrow{C}_7(2, -3)$, which is isomorphic to $\overrightarrow{C}_7(1, 2)$. By Remark 1, $\overrightarrow{C}_7(1, 2)$ is CKI, then by Remark 2, D[V(C)] is not a proper subdigraph and $D \cong \overrightarrow{C}_7(1, 2)$. Hence, if \overline{G}_D has an induced odd cycle, then D is $\overrightarrow{C}_7(1, 2)$, which is CKI.

Thus, if *D* is a locally in-/out-semicomplete CKI-digraph, then *D* is a directed odd cycle, $D \cong \overrightarrow{C}_7(1,2)$ or $D \cong \overrightarrow{C}_n(1,\pm 2,\pm 3,\ldots,\pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

Characterizing the locally in-semicomplete CKI-digraphs and the locally out-semicomplete CKI-digraphs is not the same problem, because there are digraphs such that D is CKI but D^{-1} is not CKI, see Duchet and Meyniel (1981). Although, in Theorem 7, it turned out that the digraphs that characterize the locally insemicomplete CKI-digraphs and the locally out-semicomplete CKI-digraphs remains the same. Using the fact that locally semicomplete digraphs are both locally in-semicomplete and locally out-semicomplete, we have the following result as a corollary of Theorem 7.

Corollary 2 (Galeana-Sánchez and Olsen (2016b)) If D is a locally semicomplete CKI-digraph, then D is a directed odd cycle, $D \cong \overrightarrow{C}_7(1,2)$ or $D \cong \overrightarrow{C}_n(1,\pm 2,\pm 3,\ldots,\pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

As a consequence of Theorem 7 and Corollary 2, it turns out that the characterizations of the CKIdigraphs of locally semicomplete digraphs and locally in-/out-semicomplete families are the same. Hence, we have the following result.

Theorem 8 Locally in-semicomplete digraphs, locally out-semicomplete digraphs and locally semicomplete digraphs are KP if and only if they have no induced subdigraph H such that H is a directed odd cycle, $H \cong \vec{C}_7(1,2)$ or $H \cong \vec{C}_n(1,\pm 2,\pm 3,\ldots,\pm \lfloor \frac{n}{2} \rfloor)$, for some $n \ge 4$.

4 Generalized bipartite tournaments

In this section we characterize the arc-locally in-semicomplete, arc-locally out-semicomplete, 3-quasitransitive and 3-anti-quasi-transitive CKI-digraphs. The following result is a reformulation of the original result.

Theorem 9 (Theorem 4.3 Galeana-Sánchez and Neumann-Lara (1984)) Let D be a CKI-digraph, which is not a directed odd cycle. For every vertex $u_0 \in V(D)$ there is a directed cycle $C = u_0u_1 \dots u_{2n}u_0$ such that C has no diagonal u_iu_i with $j \in \{0\} \cup \{1, 3, \dots, 2n - 1\}$, $i \in \{0, 1, \dots, 2n\}$.

Theorem 10 Let D be a CKI-digraph, which is not a directed odd cycle. For every vertex $u_0 \in V(D)$ there exist a directed cycle $C = u_0u_1 \dots u_{2n}u_0$ with $n \ge 2$ such that

- *i*) C has a diagonal.
- *ii*) C has no diagonal $u_i u_j$ with $j \in \{0\} \cup \{1, 3, ..., 2n 1\}$, $i \in \{0, 1, ..., 2n\}$.
- *iii*) *C* has no diagonal $u_{2i-1}u_{2j}$ with $0 < i < j \le n$.

Proof: *i*) follows by Remark 2. By Theorem 9, for each vertex $u_0 \in V(D)$ there is a directed cycle $C = u_0u_1 \dots u_{2n}u_0$ such that C has no diagonal u_iu_j with $j \in \{0\} \cup \{1, 3, \dots, 2n-1\}$ and $i \in \{0, 1, \dots, 2n\}$. Hence, *ii*) follows. Consider a directed cycle C satisfying *ii*) of minimum length. If C has a diagonal $u_{2i+1}u_{2j}$ with $0 < i < j \le n$, then $C' = u_0u_1 \dots u_{2i+1}u_{2j} \dots u_{2n}u_0$ is a cycle satisfying *ii*) and C' is shorter than C, contradicting the choice of C and *iii*) is proved. \Box A not necessarily directed path P = uvwx is an H_1 -path if $u \to v \leftarrow w \leftarrow x$; an H_2 -path if $u \leftarrow v \leftarrow w \to x$; an H_3 -path if P is a directed path and an H_4 -path if P is an anti-directed path. For i = 1, 2, 3, 4, Bang-Jensen defined a digraph to be H_i -free, if every H_i -path uvwx has an arc between u and x. In Bang-Jensen (2004) the arc-locally in-/out-semicomplete and the 3-quasi-(anti-)transitive digraphs where defined in terms of H_i -free digraphs, i = 1, 2, 3, 4. The family of H_1 -free digraphs (H_2 -free digraphs, resp.) is the family of arc-locally in-semicomplete digraphs (arc-locally out-semicomplete digraphs, resp.). The H_3 -free digraphs are the 3-quasi-transitive digraphs and the H_4 -free digraphs are the 3-quasi-transitive (acyclic) tournament on 3 vertices.

Theorem 11 The unique asymmetric 3-quasi-transitive CKI-digraph is \vec{C}_3 . The directed odd cycles are the only asymmetric CKI-digraphs which are arc-locally in-semicomplete digraphs, arc-locally outsemicomplete digraphs or 3-anti-quasi-transitive TT_3 -free digraphs.

Proof: The asymmetric 3-quasi-transitive digraph \vec{C}_3 is CKI. Suppose, for a contradiction, that D is an asymmetric 3-quasi-transitive CKI-digraph, which is not \vec{C}_3 . By hypothesis D is H_3 -free and since directed odd cycles of order at least 5 have induced directed paths of order 4, D is not an odd cycle. Let $u_0 \in V(D)$. By Theorem 10, D has a directed odd cycle of minimum length $C = u_0 u_1 \dots u_{2n} u_0$, $n \ge 2$, such that C has no diagonal $u_i u_j$ with $i \in \{0, 1, \dots, 2n\}$ and $j \in \{0\} \cup \{1, 3, 5, \dots, 2n - 1\}$. Since D is CKI, C is not induced and C has at least five vertices. By definition, the H_3 -path $u_{2n-1}u_{2n}u_0u_1$ must have a diagonal between the vertices u_{2n-1} and u_1 , that is a diagonal between two vertices with odd subindices, contradicting the choice of C in Theorem 10 ii). Hence, the unique asymmetric 3-quasi-transitive CKI-digraphs is \vec{C}_3 .

Directed odd cycles are asymmetric arc-locally in-semicomplete (arc-locally out-semicomplete resp.) [3-anti-quasi-transitive TT_3 -free resp.] CKI-digraphs. Suppose, for a contradiction, that D is an asymmetric arc-locally in-semicomplete (arc-locally out-semicomplete resp.) [3-anti-quasi-transitive TT_3 -free resp.] CKI-digraph, which is not a directed odd cycle. Let $u_0 \in V(D)$. By Theorem 10, D has a directed odd cycle of minimum length $C = u_0u_1 \dots u_{2n}u_0$, $n \ge 2$, such that C has no diagonal u_iu_j with $i \in \{0, 1, \dots, 2n\}$ and $j \in \{0\} \cup \{1, 3, 5, \dots, 2n - 1\}$. Observe that C has at least five vertices. Let vw be a diagonal, with $v, w \in \{u_0, u_1, \dots, u_{2n}\}$ and the index are taken modulo 2n + 1.

We have two cases: (i) The cycle C and its diagonal vw induce a directed cycle containing u_0u_1 or (ii) the cycle C and its diagonal vw induce a directed cycle avoiding u_0u_1 .

(i) The directed cycle C and its diagonal vw induce a directed cycle containing the arc u_0u_1 .

By Theorem 10 *ii*) and *iii*), the diagonal is $vw=u_{2i}u_{2j}$ with $0 < i < j \le n$ ($0 < i < j \le n$ resp.) [$1 < i + 1 < j \le n$ resp.]. By definition, the H_1 -path $u_{2i-1}u_{2i}u_{2j}u_{2j-1}$ must have a diagonal between the vertices u_{2i-1} and u_{2j-1} (the H_2 -path $u_{2i+1}u_{2i}u_{2j}u_{2j+1}$ must have a diagonal between the vertices u_{2i+1} and u_{2j+1} resp.) [the H_4 -path $u_{2j-1}u_{2j}u_{2i}u_{2i+1}$ must have a diagonal between the vertices u_{2i+1} and u_{2j-1} resp., (2i + 1 < 2j - 1 because D is TT_3 -free)], in each case vw is a diagonal between two vertices with subindices which are odd or equal to zero (if j = n, then $2j + 1 \equiv 0 \pmod{2n + 1}$), contradicting the choice of C in Theorem 10 *ii*). Hence, C has no diagonal inducing a directed cycle containing the arc u_0u_1 .

(*ii*) The directed cycle C and its diagonal vw induce a directed cycle avoiding the arc u_0u_1 . By the choice of C, we may assume that $vw = u_iu_{2j}$ with i = 0 and $1 \le j < n$ ($1 \le j < n$ resp.) [since D is TT_3 -free, 1 < j < n resp.] or $i \ne 0$ and $2j + 2 < i \le 2n$ (by Remark 2, D has no \overrightarrow{C}_3).

If i = 0, since vw is a diagonal, it follows that $1 \le j < n$ ($1 \le j < n$ resp.) [since D is TT_3 -free,



1 < j < n resp.]. By definition, the H_1 -path $u_{2n}u_0u_{2j}u_{2j-1}$ must have a diagonal between the vertices



 u_{2n} and u_{2j-1} , since 2j-1 is odd, by Theorem 10 ii), $u_{2j-1}u_{2n}$ is a diagonal of C, contradicting the minimality of C in Theorem 10 *iii*), see Figure 5(a) (the H_2 -path $u_1u_0u_{2i}u_{2i+1}$ must have a diagonal between the vertices u_{2i+1} and u_1 , a diagonal between two vertices with odd subindices, a contradiction to the choice of C in Theorem 10 ii), see Figure 5(b) resp.) [the anti-directed path $u_{2i-1}u_{2i}u_0u_1$ must have a diagonal between the vertices u_{2j-1} and u_1 ($u_{2j-1} \neq u_1$ because D is TT₃-free), a diagonal between two vertices with odd subindices, contradicting the choice of C in Theorem 10 ii), see Figure 5(c) resp.].

Hence, $i \neq 0$ and since, $j \geq 1$, it follows that $4 \leq 2j + 2 < i \leq 2n$. If i is odd, by definition, the H_1 -path $u_{i-1}u_iu_{2j}u_{2j-1}$ must have a diagonal between the vertices u_{i-1} and u_{2j-1} (the H_2 -path $u_{i+1}u_iu_{2j}u_{2j+1}$ must have a diagonal between the vertices u_{2j+1} and u_{i+1} resp.) [the anti-directed path $u_{2j-1}u_{2j}u_iu_{i+1}$ must have a diagonal between the vertices u_{2j-1} and u_{i+1} ($u_{2j-1} \neq u_1$ because D is TT_3 -free) resp.]. By Theorem 10 ii), the diagonal must be $u_{2j-1}u_{i-1}$ $(u_{2j+1}u_{i+1})$ $[u_{2j-1}u_{i+1}$ resp.] contradicting the minimality of the cycle C in Theorem 10 *iii*). Hence, *i* is even. By definition, the H_1 -path $u_{i-1}u_iu_{2j}u_{2j-1}$ must have a diagonal between the vertices u_{i-1} and u_{2j-1} contradicting the choice of C in Theorem 10 ii) because i - 1 is odd (the H_2 -path $u_{i+1}u_iu_{2j}u_{2j+1}$ must have a diagonal between the vertices u_{2j+1} and u_{i+1} contradicting the choice of C in Theorem 10 ii) because i + 1 is odd or i + 1 = 0 resp.) [the anti-directed path $u_{2i-1}u_{2i}u_{i}u_{i+1}$ must have a diagonal between the vertices u_{2i-1} and u_{i+1} . Since i+1 is odd or i+1=0, it follows that a diagonal between the vertices u_{2i-1} and u_{i+1} contradicts the choice of C in Theorem 10 ii), unless the diagonal is the arc u_0u_1 (i = 2n, i + 1 = 0 and 2j - 1 = 1), because in this case the diagonal of the anti-directed path $u_{2j-1}u_{2j}u_{i}u_{i+1}$ is not a diagonal of the directed cycle C. By Remark 2, the odd cycle $C' = u_2 u_3 \dots u_{2i} u_2$ is not an induced cycle in D and since any diagonal of C' is a diagonal of C, by the choice of C, C' has a diagonal $v'w' = u_{i'}u_{2j'}$ such that $4 < 2j' + 2 < i' \le 2n$ or $4 \le 2j' + 2 < i' < 2n$. Since, v'w' is a diagonal of C, the anti-directed path $u_{2i'-1}u_{2i'}u_{i'+1}$ must have a diagonal between the vertices $u_{2i'-1}$ and $u_{i'+1}$ contradicting the choice of C in Theorem 10 ii) because $3 \le 2j' - 1 < 2n$, and i' + 1 = 0 or i' + 1 is odd resp.].

Both cases lead to a contradiction, thus D is an odd cycle.

As a consequence of Theorem 11 we have the following result.

Theorem 12 An asymmetric 3-quasi-transitive digraph is KP if and only if it has no induced \vec{C}_3 . Asymmetric arc-locally in-/out-semicomplete digraphs and 3-anti-quasi-transitive TT_3 -free digraphs are KP if and only if they have no induced directed odd cycle.

5 Conclusions

For tournaments, there is a unique CKI-digraph namely the \vec{C}_3 , and for each integer *n* the transitive tournament of order *n* is the unique KP-digraph. The families of generalized tournaments considered in this paper have a nice characterization of their CKI-digraphs due to Corollary 1 and Theorem 7, and a nice characterization of their KP-digraphs due to Theorem 6 and Theorem 8. Hence, these families of generalized tournaments somehow preserve the property of the tournaments, that the characterizations of the CKI- and the KP-digraphs are nice.

All bipartite tournaments are KP-digraphs, so there are no bipartite tournaments which are CKI-digraphs. The families of generalized bipartite tournaments considered in this paper have a nice characterization of their CKI-digraphs due to Theorem 11 and a nice characterization of their KP-digraphs due to Theorem 12. Hence, these families of generalized bipartite tournaments somehow preserve the property of the bipartite tournaments, that there is only one class of CKI-digraphs, although not every digraph is KP.

The asymmetric part of a digraph D is the spanning subdigraph of D induced by the asymmetric arcs of D. The asymmetric part of the digraphs considered in this paper is either isomorpic to $\overrightarrow{C}_7(1,2)$ or isomorphic to a cycle. As pointed out in Galeana-Sánchez and Olsen (2016b), deciding whether a digraph D is isomorphic to a directed cycle or isomorphic to $\overrightarrow{C}_7(1,2)$ is polynomial, and for the case when the asymmetric part of the digraph D is a cycle, deciding whether the digraph D is an odd cycle or the digraph D is isomorphic to $\overrightarrow{C}_n(1, \pm 2, \pm 3, \ldots, \pm \lfloor \frac{n}{2} \rfloor)$ for some $n \ge 4$ is also polynomial. Therefore, it is polynomial to determine whether a digraph is CKI if the underlying graph is a perfect graph or if the digraph is semicomplete, semicomplete multipartite, quasi-transitive, locally in-/out-semicomplete, locally semicomplete, asymmetric 3-quasi-transitive, asymmetric arc-locally in-/out-semicomplete and 3-anti-quasi-transitive TT_3 -free.

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