

Sigma Partitioning: Complexity and Random Graphs

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A *sigma partitioning* of a graph G is a partition of the vertices into sets P_1, \dots, P_k such that for every two adjacent vertices u and v there is an index i such that u and v have different numbers of neighbors in P_i . The *sigma number* of a graph G , denoted by $\sigma(G)$, is the minimum number k such that G has a sigma partitioning P_1, \dots, P_k . Also, a *lucky labeling* of a graph G is a function $\ell : V(G) \rightarrow \mathbb{N}$, such that for every two adjacent vertices v and u of G , $\sum_{w \sim v} \ell(w) \neq \sum_{w \sim u} \ell(w)$ ($x \sim y$ means that x and y are adjacent). The *lucky number* of G , denoted by $\eta(G)$, is the minimum number k such that G has a lucky labeling $\ell : V(G) \rightarrow \mathbb{N}_k$. It was conjectured in [Inform. Process. Lett., 112(4):109–112, 2012] that it is **NP**-complete to decide whether $\eta(G) = 2$ for a given 3-regular graph G . In this work, we prove this conjecture. Among other results, we give an upper bound of five for the sigma number of a uniformly random graph.

Keywords: Sigma partitioning; Lucky labeling; Additive coloring; Sigma chromatic number; Computational Complexity; Planar Not-All-Equal 3-SAT; Planar Not-All-Equal 3-SAT Type 2.

1 Introduction

Throughout the paper we denote $\{1, 2, \dots, k\}$ by \mathbb{N}_k . In 2004, Karoński et al. introduced a new coloring of a graph which is generated via edge labeling [19]. Let $f : E(G) \rightarrow \mathbb{N}$ be a labeling of the edges of a graph G by positive integers such that for every two adjacent vertices v and u , $\text{Sum}(v) \neq \text{Sum}(u)$, where $\text{Sum}(v)$ denotes the sum of labels of all edges incident with v . It was conjectured in [19] that three integer labels \mathbb{N}_3 are sufficient for every connected graph, except K_2 . Currently the best bound is five [18]. Regarding the computational complexity of this concept, Dudek and Wajc [13] proved that determining whether a given graph has a labeling of the edges from \mathbb{N}_2 that induces a proper vertex coloring is **NP**-complete. Recently, it was shown that for a given 3-regular graph G deciding whether G has a labeling for the edges from $\{a, b\}$, ($a \neq b$) that induces a proper vertex coloring is **NP**-complete [9].

Lucky labeling and sigma partitioning are two vertex versions of this problem, which were introduced recently by Czerwiński et al. [7] and Chartrand et al. [6]. The *lucky labeling* of a graph G is a function

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$\ell : V(G) \rightarrow \mathbb{N}$, such that for every two adjacent vertices v and u of G , $\sum_{w \sim v} \ell(w) \neq \sum_{w \sim u} \ell(w)$ ($x \sim y$ means that x and y are adjacent). The *lucky number* of G , denoted by $\eta(G)$, is the minimum number k such that G has a lucky labeling $\ell : V(G) \rightarrow \mathbb{N}_k$. Also, the *sigma partitioning* of a graph G is a partition of the vertices into sets P_1, \dots, P_k such that for every two adjacent vertices u and v , there is an index i such that u and v have different numbers of neighbors in P_i . The *sigma number* of a graph G , denoted by $\sigma(G)$, is the minimum number k such that G has a sigma partitioning P_1, \dots, P_k .

There is an alternative definition for the sigma partitioning which is similar to the definition of lucky labeling. For a graph G , let $c : V(G) \rightarrow \mathbb{N}$ be a vertex labeling of G . If k labels are used by c , then c is a k -labeling of G . If for every two adjacent vertices v and u of G , $\sum_{w \sim v} c(w) \neq \sum_{w \sim u} c(w)$, then c is called a sigma partitioning of G . The minimum number of labels required in a sigma partitioning is called the sigma number of G . Now, we show that the alternative definition is equivalent to the first definition. In any sigma partitioning of a graph G with $\sigma(G)$ labels, we can use the set of labels $\{s^i : 0 \leq i \leq \sigma(G) - 1\}$, where s is a sufficiently large number (it is enough to put $s \geq \Delta(G) + 1$). So the sigma number is the minimum number k such that the vertices of graph can be partitioned into k sets P_1, \dots, P_k such that for every edge uv , there is an index i that u and v have different numbers of neighbors in P_i .

Sigma partitioning and lucky labeling have been studied extensively by several authors, for instance see [2, 3, 5, 6, 7, 11, 20, 23]. Note that lucky labeling is also called an *additive labeling* of a graph [1, 16], and sigma partitioning is also called the *sigma chromatic number* of a graph [6].

The sigma number can also be thought of as a vertex version of the *detection number* of a graph. For a connected graph G of order $|V(G)| \geq 3$ and a k -labeling $c : E(G) \rightarrow \mathbb{N}_k$ of the edges of G , the code of a vertex v of G is the ordered k -tuple (l_1, l_2, \dots, l_k) , where l_i is the number of edges incident with v that are labeled i . The k -labeling c is detectable, if every two adjacent vertices of G have distinct codes. The minimum positive integer k for which G has a detectable k -labeling is the *detection number* of G . It was shown that it is NP-complete to decide if the detection number of a 3-regular graph is 2 [17]. Also, it was proved that the detection number of every bipartite graph of minimum degree at least 3 is at most 2 [17].

Remark 1 For a partition $\mathcal{P} = \cup_{i=1}^k P_i$ of the vertices of a graph G , consider the function $f_{\mathcal{P}} : V(G) \rightarrow (\mathbb{N} \cup \{0\})^k$ with the map $f_{\mathcal{P}}(x) = (N(x) \cap P_1, \dots, N(x) \cap P_k)$. The function \mathcal{P} is called sigma partitioning if and only if $f_{\mathcal{P}}$ is a proper coloring for G . For a given graph G , let \mathcal{P}_k be the set of partitions of the vertices such that each partition has at most k parts. Now, consider the following parameter:

$$\Omega_p(G, k) = \max_{\mathcal{P} \in \mathcal{P}_k} \min_{v, u \in E(G)} |f_{\mathcal{P}}(v) - f_{\mathcal{P}}(u)|_p,$$

where $p \geq 1$ and $|\cdot|_p$ is the norm p . Note that $p = 1$ is the natural case and $\sigma(G) = \min_k \Omega_p(G, k) \neq 0$ (for all p).

Remark 2 The difference between the sigma number and the lucky number of a graph can be arbitrarily large. In fact, there are graphs with $\sigma \leq 2$ and arbitrary large lucky numbers. For instance, for every k consider a complete graph with $\binom{k+2}{2} - 1$ vertices $\{v_{\alpha\beta} : \alpha \in \mathbb{N}_k \cup \{0\}, \beta \in \mathbb{N}_{\alpha+2}\}$; next join each vertex $v_{\alpha\beta}$ to α new isolated vertices $v_{\alpha\beta}^\gamma$ for $\gamma \in \mathbb{N}_\alpha$. We call this graph G . Note that G has $k(k+1)(2k+7)/6$ leaves. We show $\sigma(G) \leq 2$. To do this, put $A = \{v_{\alpha\beta} : \beta = 1\} \cup \{v_{\alpha\beta}^\gamma : \gamma \leq \beta - 2\}$ and let $\bar{A} = V(G) \setminus A$. In order to show that (A, \bar{A}) is a sigma partition one could mention that we only need to consider pairs of vertices of the same degree, such as v_{ia} and v_{ib} for some i and $a < b$, in which case v_{ib} has more neighbors in A than v_{ia} does. Thus (A, \bar{A}) is a sigma partitioning. On the other hand, let ℓ be a lucky labeling of G , for every two adjacent vertices $u, v \in K_{\binom{k+2}{2}-1}$, we have:

$$\sum_{\substack{w \sim v \\ w \text{ is a leaf}}} \ell(w) + \ell(u) \neq \sum_{\substack{w \sim u \\ w \text{ is a leaf}}} \ell(w) + \ell(v),$$

Therefore, we have:

$$\sum_{\substack{w \sim v \\ w \text{ is a leaf}}} \ell(w) - \ell(v) \neq \sum_{\substack{w \sim u \\ w \text{ is a leaf}}} \ell(w) - \ell(u).$$

$\sum_{\substack{w \sim v \\ w \text{ is a leaf}}} \ell(w) - \ell(v)$ is one of the $(k+1)\eta(G) - 1$ different numbers. Thus

$$(k+1)\eta(G) - 1 \geq \binom{k+2}{2} - 2.$$

Therefore, $\eta(G) \geq k/2$.

1.1 Complexity for 3-regular graphs

For a given graph G , we have $\sigma(G) = \eta(G) = 1$ if and only if every two adjacent vertices of G have different degrees. We know that $\sigma(G) \leq \chi(G)$ [6], on the other hand we have the following conjecture about the lucky number.

Conjecture 1 [Additive Coloring Conjecture [7]] For every graph G , $\eta(G) \leq \chi(G)$.

It is not known whether this conjecture is true even for bipartite graphs. Moreover, it is not even known if $\eta(G)$ is bounded for bipartite graphs. Recently, Grytczuk et al. [16] proved that $\eta(G) \leq 468$ for every planar graph G .

It was shown in [2] that it is NP-complete to decide for a given planar 3-colorable graph G , whether $\eta(G) = 2$. Here, we are interested in the following conjecture:

Conjecture 2 [2] It is NP-complete to decide whether $\eta(G) = 2$ for a given 3-regular graph G .

In this work, we prove Conjecture 2. In order to prove Conjecture 2; first, we prove Theorem 1. After that by this theorem we prove the conjecture.

Theorem 1 It is NP-complete to decide for a given 3-regular graph G , whether $\sigma(G) = 2$.

It is easy to check that for every regular graph G , $\sigma(G) = 2$ if and only if $\eta(G) = 2$. Therefore from Theorem 1 we have Theorem 2 and we can prove Conjecture 2.

Theorem 2 It is NP-complete to decide for a given 3-regular graph G , whether $\eta(G) = 2$.

In the proof of Theorem 1, for a given formula Ψ , we transform Ψ into a 3-regular graph G_Ψ such that $\sigma(G_\Psi) = 2$ if and only if Ψ has a Not-All-Equal (NAE) truth assignment, where a NAE assignment is an assignment such that each clause has at least one true literal and at least one false literal. On the other hand, G_Ψ has a triangle and is a 3-regular graph, so we have $\chi(G_\Psi) = 3$. Also, since G_Ψ is a 3-regular graph, $\sigma(G_\Psi) \geq 2$. Thus, $\sigma(G) = \chi(G)$ if and only if Ψ does not have any NAE truth assignment. Consequently, by Theorem 1, we have the following corollary.

Corollary 1 *It is NP-complete to determine whether $\sigma(G) = \chi(G)$, for a given 3-regular graph G .*

It was proven in [2] that for every $k \geq 2$, it is NP-complete to decide whether $\eta(G) = k$ for a given graph G . Here, we present the following result for sigma partitioning.

Theorem 3 *For every $k \geq 3$, it is NP-complete to decide whether $\sigma(G) = k$ for a given graph G .*

Consider the problem of partitioning the vertices of G into $\sigma(G)$ parts, such that this partitioning is a sigma partitioning and some parts have the smallest possible size. We show that for a given planar 3-regular graph with sigma number two the following problem is NP-complete.

Problem Ξ .

INSTANCE: A planar 3-regular graph G with $\sigma(G) = 2$ and a real number $0 < r < 1$.

QUESTION: Does G have a sigma partitioning c with $\sigma(G)$ parts, such that there is a part with at most $r|V(G)|$ vertices?

Theorem 4 *Problem Ξ is NP-complete.*

1.2 Some upper bounds

It was shown that $\sigma(G) = \mathcal{O}(\Delta^2)$ for every graph G [23]. We prove the following upper bound for triangle-free graphs.

Theorem 5 *Let $\mathcal{G} = \{G_i\}_{i \in \mathbb{N}}$ be a sequence of triangle-free graphs. Then for this family,*

(i) *if $\delta = \Omega(\Delta)$ then $\sigma = \mathcal{O}(1)$ for all graphs in \mathcal{G} , except a finite number of them;*

(ii) *if $\delta \geq 3.501(\ln \Delta)$, then $\sigma = \mathcal{O}(\delta)$.*

Next, we show that almost all graphs have a small sigma number. Let $G(n, p)$ be the notation of Erdős-Rényi random graph on n vertices.

Theorem 6 *For every constant p , $0 < p < 1$, $\sigma(G(n, p)) \leq 5$.*

1.3 Notation

We conclude the section by fixing some notation which is not defined here. When we say that f is a sigma partitioning for a graph G , we mean that $f : V(G) \rightarrow \mathbb{N}$ such that for every two adjacent vertices v and u of G , $\sum_{w \sim v} f(w) \neq \sum_{w \sim u} f(w)$. For a vertex v of G , let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). Let $N[v] = N(v) \cup \{v\}$ denote the closed neighborhood of v . Also, for every $v \in V(G)$, $d_G(v)$ denotes the degree of v (for simplicity we denote $d_G(v)$ by $d(v)$). For a natural number k , a graph G is called a k -regular graph if $d(v) = k$, for each $v \in V(G)$. We denote the maximum degree and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. For $k \in \mathbb{N}$, a proper vertex k -coloring of G is a function $c : V(G) \rightarrow \mathbb{N}_k$, such that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. The smallest integer k such that G has a proper vertex k -coloring is called the chromatic number of G and denoted by $\chi(G)$. We follow [24] for terminology and notation which are not defined here.

2 Complexity results

Proof of Theorem 1. Here, we prove that for a given 3-regular graph G , it is **NP**-complete to decide whether $\sigma(G) = 2$. Let Ψ be a 3-SAT formula with clauses $C = \{c_1, \dots, c_k\}$ and variables $X = \{x_1, \dots, x_n\}$. The following problem is **NP**-complete [14].

Not-All-Equal (NAE) 3-SAT.

INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $|c| = 3$.

QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?

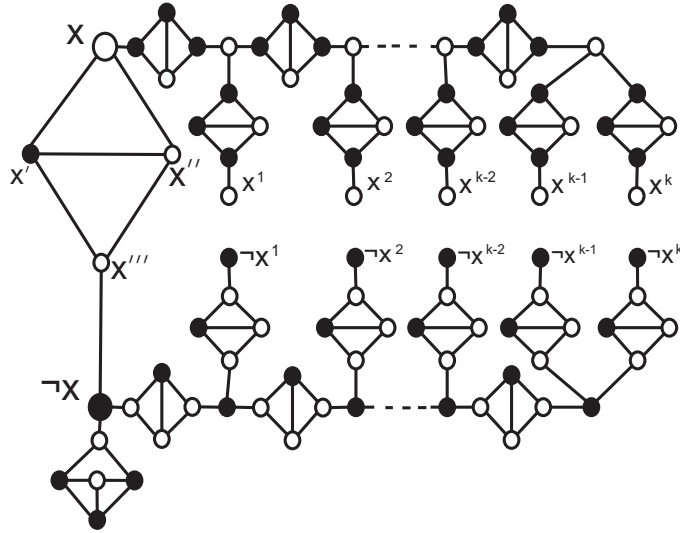


Fig. 1: The graph H_x . In every lucky labeling $\ell : V(H_x) \rightarrow \mathbb{N}_2$, the set of black vertices has the same label and also the set of white vertices has the same label and these two labels are different (with respect to symmetry).

Since for every regular graph G we have that $\sigma(G) = 2$ if and only if $\eta(G) = 2$, it suffices to prove the theorem for the lucky labeling.

We reduce *NAE 3-SAT* to our problem in polynomial time. Consider an instance Ψ with the set of variables X and the set of clauses C . We transform this into a 3-regular graph G_Ψ such that $\eta(G_\Psi) = 2$ if and only if Ψ has an NAE truth assignment. We use three auxiliary graphs H_x , I_{c_j} and T . The gadgets H_x and T are shown in Figure 1 and Figure 2. Also, the gadget I_{c_j} is a triangle $c_j^1 c_j^2 c_j^3$. The graph G_Ψ has a copy of H_x for each variable $x \in X$ and a copy of I_{c_j} for each clause $c_j \in C$. For each clause $c_j = y \vee z \vee w$, where $y, w, z \in X \cup \neg X$, add the edges $c_j^1 y^j$, $c_j^2 z^j$ and $c_j^3 w^j$. Finally, for every vertex v with $d(v) < 3$, put a copy of T and add edge between v and t . Repeat this procedure one more time to obtain a 3-regular graph G_Ψ .

We next discuss basic properties of the graph G_Ψ . Assume that $\eta(G_\Psi) = 2$ and $\ell : V(G_\Psi) \rightarrow \mathbb{N}_2$ is a lucky labeling.

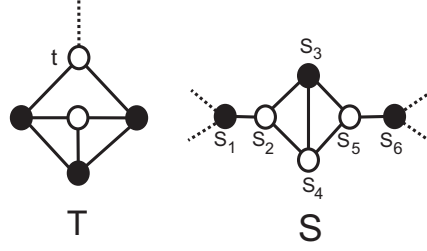


Fig. 2: Two auxiliary graphs T and S . For every lucky labeling of T with the labels \mathbb{N}_2 , the set of black vertices has a same label and also the set of white vertices has a same label and these two labels are different (with respect to symmetry). This fact is also true for S .

Fact 1 For every $x \in X$, in the subgraph H_x we have, $\ell(x) \neq \ell(\neg x)$.

Proof. For a contradiction, suppose that $\ell(x) = \ell(\neg x)$. Consequently, $\ell \upharpoonright_{\{x', x'', x'''\}}$ is a lucky labeling for the odd cycle $x'x''x'''$. But the lucky number of every odd cycle is 3, a contradiction. ♠

If u and v are two vertices in the graph G_Ψ such that $N[u] = N[v]$, then $\ell(u) \neq \ell(v)$. As a consequence, we have the following fact.

Fact 2 In every copy of S and T (see Figure 2), the set of black vertices has the same label and also the set of white vertices has the same label and these two labels are different (with respect to symmetry).

Proof. In the subgraphs S , first assume that $\ell(s_1) = 2$. Since $N[s_3] = N[s_4]$, we have that $\ell(s_3) \neq \ell(s_4)$. Without loss of generality suppose that $\ell(s_3) = 2$ and $\ell(s_4) = 1$. If $\ell(s_5) = 2$, then $\ell \upharpoonright_{\{s_2, s_3, s_4\}}$ is a lucky labeling for the odd cycle $s_2s_3s_4$, but it is a contradiction, hence $\ell(s_5) = 1$. If $\ell(s_2) = 2$, then $\sum_{w \sim s_2} \ell(w) = \sum_{w \sim s_4} \ell(w)$, so $\ell(s_2) = 1$. Also if $\ell(s_6) = 1$, then $\sum_{w \sim s_4} \ell(w) = \sum_{w \sim s_5} \ell(w)$, so $\ell(s_6) = 2$. By a similar argument if $\ell(s_1) = 1$, then $\ell(s_6) = 1$. Therefore, the set of black vertices has the same label and also the set of white vertices has the same label and these two labels are different. For the subgraph T we have a similar argument. ♠

By Fact 1, Fact 2 and since S is a subgraph of H_x , without loss of generality, for every $x \in X$, in the subgraph H_x , the set of black vertices has the same label and also the set of white vertices has the same label and these two labels are different. In other words, $\ell(x) = \ell(x^1) = \dots = \ell(x^k) \neq \ell(\neg x) = \ell(\neg x^1) = \dots = \ell(\neg x^k)$. For an arbitrary clause $c_j = y \vee z \vee w$, where $y, w, z \in X \cup \neg X$, assume that $\ell(y^j) = \ell(z^j) = \ell(w^j)$, consequently $\ell \upharpoonright_{\{c_j^1, c_j^2, c_j^3\}}$ is a lucky labeling for the odd cycle $c_j^1c_j^2c_j^3$, but the lucky number of an odd cycle is 3. This is a contradiction. Hence, we have the following fact:

Fact 3 For every clause $c_j = y \vee z \vee w$, where $y, z, w \in X \cup \neg X$, we have $\{\ell(y^j), \ell(z^j), \ell(w^j)\} = \mathbb{N}_2$.

First, assume that $\eta(G_\Psi) = 2$ and let $\ell : V(G_\Psi) \rightarrow \mathbb{N}_2$ be a lucky labeling. We present an NAE satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$ for Ψ . Now put $\Gamma(x_i) = \text{true}$ if and only if $\ell(x_i) = 1$. By Fact 1, for every x_i we have $\ell(x_i) \neq \ell(\neg x_i)$ so it is impossible that both $\ell(x_i)$ and $\ell(\neg x_i)$ are 1. For every $c_j = y \vee z \vee w$ by Fact 3, $|\{\ell(y^j), \ell(z^j), \ell(w^j)\}| = 2$; so at least one of the literals y, z, w is true and at least one of the literals is false. On the other hand, suppose that Ψ is satisfiable with the satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$. We present the lucky labeling ℓ for G_Ψ from the set $\{1, 2\}$. By attention to Figure 1 and Figure 2, in each copy of H_x , if $\Gamma(x) = \text{true}$, label the set of black vertices

by 1 and label the set of white vertices by 2. Also if $\Gamma(x) = \text{false}$, label the set of black vertices by 2 and label the set of white vertices by 1. For every copy of T , label the vertex t with the label different from $\ell(t')$, where t' is the unique neighbor of t which is not in T . This determines the labels of all other vertices of T . By Fact 3, for every $c_j \in C$, we can determine the labels of the vertices c_j^1, c_j^2, c_j^3 . Finally, by straightforward counting one can see that ℓ is a lucky labeling. This completes the proof. \square

Proof of Theorem 3. Here, we prove that for every $k \geq 3$, for a given graph G it is **NP**-complete to decide whether $\sigma(G) = k$. It was shown in [8] that the following problem is NP-complete. We reduce it to our problem in polynomial time.

The 3-colorability of 4-regular graphs.

INSTANCE: A 4-regular graph G .

QUESTION: Is $\chi(G) \leq 3$?

For a given 4-regular graph G , we construct a regular graph G^* such that $\chi(G^*) \leq k$ if and only if G is 3-colorable (step 1). Next, we construct the graph G^{**} such that $\sigma(G^{**}) \leq k$ if and only if $\chi(G^*) \leq k$ (step 2).

Step 1. If $k = 3$ we can assume that $G^* \cong G$, otherwise do the following. Consider a copy of the graph G and a copy of the complete graph K_{k-3} . Join each vertex of G to each vertex of K_{k-3} and call the resultant graph G' . One can see that $\chi(G') = \chi(G) + k - 3$ and $\Delta(G') = k - 4 + |V(G)|$. Now, consider a copy of the graph G' and for each vertex $v \in G'$, put $\Delta(G') - d_{G'}(v)$ new isolated vertices and join them to the vertex v . Also, put $\Delta(G')$ copies of the K_2 's. Call the resultant graph G'' . It is easy to check that $\delta(G'') = 1$, $\Delta(G'') = \Delta(G')$ and $\chi(G'') = \chi(G')$. Let $S = \{v \mid d_{G''}(v) = 1\}$. Now, consider two copies of G'' and put $\Delta(G'') - 1$ distinct perfect matching between the set of vertices S in the first copy of G'' and the set of vertices S in the second copy of G'' . Call the resultant graph G^* . One can see that G^* is a regular with $\chi(G^*) = \chi(G) + k - 3$.

Step 2. Suppose that G^* is a regular graph with n vertices. For every α , $1 \leq \alpha \leq n$ consider a copy of a complete graph $K_{k^2-k+1}^{(\alpha)}$, with the vertices $\{y_{\beta\gamma}^\alpha : \beta, \gamma \in \mathbb{N}_{k-1}\} \cup \{y_{k\gamma}^\alpha : \gamma \in \mathbb{N}_k\}$. Next consider $k - 1$ isolated vertices v_1, \dots, v_{k-1} and join every $y_{\beta\gamma}^\alpha$ to v_β , also consider a copy of G^* with the vertices x_1, \dots, x_n and join every $y_{k\gamma}^\alpha$ to x_α . Finally put n isolated vertices z_1, \dots, z_n and join every x_α to z_α . Name the constructed graph G^{**} .

First, note that $\sigma(G^{**}) \geq k$, since in every sigma partitioning of G^{**} the labels of $y_{k1}^1, y_{k2}^1, \dots, y_{kk}^1$ are different. Let $c : V(G^{**}) \rightarrow \{a_1, \dots, a_k\}$ be a sigma partitioning. For every α and β , the labels of $y_{\beta 1}^\alpha, \dots, y_{\beta(k-1)}^\alpha$ are different. We claim that the labels of v_1, \dots, v_{k-1} are different (Property A). To the contrary suppose that $c(v_\beta) = c(v_{\beta'})$, then the labels of the vertices in $\{y_{\beta\gamma}^1 : \gamma \in \mathbb{N}_{k-1}\} \cup \{y_{\beta\gamma'}^1 : \gamma' \in \mathbb{N}_{k-1}\}$ must be different, so we need at least $2(k-1)$ labels. Similarly, for every i and γ , $i \in \mathbb{N}_n$, $\gamma \in \mathbb{N}_{k-1}$, we can see that $c(v_\beta) \neq c(x_i)$ (Property B). First, suppose that $\sigma(G^{**}) = k$, by Properties A and B, we have $c(x_1) = \dots = c(x_n)$. Consider the set of vertices $\{x_i : i \in \mathbb{N}_n\}$, since G^* is regular and for every α we have $|\{c(y_{k\gamma}^\alpha) : \gamma \in \mathbb{N}_k\}| = k$, therefore, the function $c' : \{x_i : i \in \mathbb{N}_n\} \rightarrow \mathbb{N}$, where $c'(x_\alpha) = c(z_\alpha)$, is a proper vertex coloring of G^* , so $\chi(G^*) \leq k$. On the other hand, let $\chi(G^*) \leq k$ and $c' : \{x_i : i \in \mathbb{N}_n\} \rightarrow \mathbb{N}_k$ be a proper vertex coloring G^* . Suppose that s is a sufficiently large number, we give a sigma partitioning c for G^{**} with the labels $\{s^i : i \in \mathbb{N}_k\}$, that is:

$$c(x_\alpha) = s^k, \quad c(v_\beta) = s^\beta, \quad c(y_{k\gamma}^\alpha) = s^\gamma, \quad c(z_\alpha) = s^{c'(x_\alpha)},$$

and for every α, β with $\beta \neq k$, label the vertices $y_{\beta\gamma}^\alpha$ such that $\{c(y_{\beta\gamma}^\alpha) : \gamma \in \mathbb{N}_k\} = \{s^i : i \in \mathbb{N}_k\} \setminus \{s^\beta\}$. It is not hard to check that c is a sigma partitioning of G^{**} . This completes the proof. \square

Proof of Theorem 4. Here, we prove that for a given planar 3-regular graph G with $\sigma(G) = 2$ and a real number $0 < r < 1$, determining whether G has a sigma partitioning with $\sigma(G)$ parts, such that there is a part with at most $r|V(G)|$ vertices, is NP-complete.

First consider the following problem.

Planar NAE 3-SAT.

INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $|c| = 3$ and the following graph obtained from 3-SAT is planar. The graph has one vertex for each variable, one vertex for each clause; all variable vertices are connected in a simple cycle and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause.

QUESTION: Is there an NAE truth assignment for X ?

Moret [22] proved that *Planar NAE 3-SAT* is in \mathbf{P} by a reduction to a known problem in \mathbf{P} , namely Planar MaxCut (for more information see [10, 12]). Moret's reduction used only local replacement. Given an instance of *Planar NAE 3-SAT* with k clauses, they transformed it in polynomial time into an instance Q of MaxCut with $9k$ vertices as follows. For each variable x forming a total of n_x literals in the k clauses, they put a cycle of $2n_x$ vertices. Alternating vertices represent complemented and uncomplemented literals. For each clause, they put a copy of the complete graph K_3 , where each vertex of the K_3 is connected by an edge to the (complement of the) corresponding literal vertex. They proved that $11k$ is the maximum attainable cut sum if and only if *Planar NAE 3-SAT* is satisfied. Note that the cycle that contains all variables does not have any role in the construction of Q . So, if we replace an instance of *Planar NAE 3-SAT* with an instance of the following problem; the proof remains correct. Therefore, the following problem is in \mathbf{P} .

Planar NAE 3-SAT Type 2.

INSTANCE: Let X be the set of variables and let C be the set of clauses such that each clause $c \in C$ has $|c| = 3$ and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

QUESTION: Is there an NAE truth assignment for X ?

On the other hand, Moore and Robson [21] proved that the following 1-In-3 SAT problem is NP-complete.

Cubic Planar 1-In-3 3-SAT.

INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $|c| = 3$ and every variable appears in exactly three clauses, there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar. Note that $|C| = |X|$.

QUESTION: Is there a truth assignment for X such that each clause in C has exactly one true literal?

We reduce *Cubic Planar 1-In-3 3-SAT* to our problem in polynomial time. Assume that H is a dummy 3-regular graph with $\sigma(H) = 2$. Also, without loss of generality, suppose that H does not have any sigma

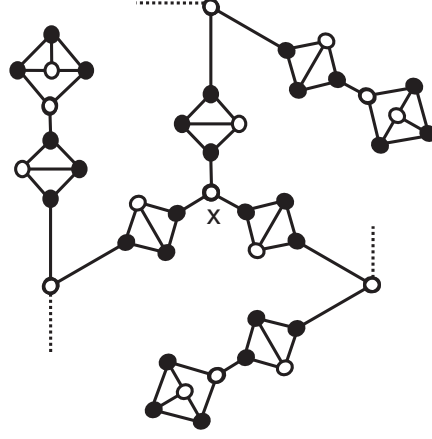


Fig. 3: The auxiliary graph A_x . For every sigma partitioning of A_x with the labels $\{\alpha, \beta\}$, the set of black vertices have the same label and also the set of white vertices have the same label and these two labels are different (with respect to the symmetry).

partitioning with two parts, such that there is a part with at most $\frac{62}{138}|V(H)|$ vertices. Now, consider an instance Φ with the set of $3n$ variables X and the set of $3n$ clauses C . By the polynomial time algorithm which was presented in [22] (they proved that *Planar NAE 3-SAT Type 2* is in \mathbf{P}), we check whether Φ has an NAE truth assignment. If Φ does not have any NAE truth assignment, then it does not have a 1-In-3 truth assignment. In this case we return H . Otherwise, we construct G_Φ . We use two auxiliary graphs A_x and B_c . The gadget A_x is shown in Figure 3 and the gadget B_c is a complete graph K_3 . Let H_Φ be the graph obtained by linking a variable and a clause if and only if the variable appears in the clause. Replace each variable $x \in X$ by A_x and replace each clause $c \in C$ by B_c . Call this planar 3-regular graph G_Φ . This graph has $138n$ vertices.

First we show that $\sigma(G_\Phi) = 2$. By assumption Φ has an NAE truth assignment $\Gamma : X \rightarrow \{true, false\}$. We present a sigma partitioning c for G_Φ , with labels α and β . Put $c(x) = \beta$ if and only if $\Gamma(x) = true$. This determines the labels of the remaining vertices in A_x . Let $c \in C$ be an arbitrary clause and without loss of generality, assume that $v_1 v_2 v_3$ is the complete graph corresponds to the clause $c = (x \vee y \vee z)$ in the graph. Also, assume that v_1 has a neighbor in A_x with label α , v_2 has a neighbor in A_y with label β and v_3 has a neighbor in A_z with label β . Put $c(v_2) = \alpha$ and $c(v_1) = c(v_3) = \beta$. Since Γ is an NAE truth assignment, one can label the vertices of B_c by the above method.

We prove that for every sigma partitioning with $\sigma(G_\Phi)$ labels, for every part P we have $|P|/|V(G_\Phi)| \geq \frac{62}{138}$ and the equality holds for a part, if and only if Φ has a truth assignment such that each clause in C has exactly one true literal. Assume that $\ell : V(G_\Phi) \rightarrow \{\alpha, \beta\}$ is a sigma partitioning. Consider the following fact:

Fact 4 Let G be a regular graph, $\sigma(G) = 2$ and $\ell : V(G) \rightarrow \{\alpha, \beta\}$ be a sigma partitioning. Also, let α' and β' be two arbitrary number such that $\alpha' \neq \beta'$. Define $\ell' : V(G) \rightarrow \{\alpha', \beta'\}$ such that $\ell'(v) = \alpha'$ if and only if $\ell(v) = \alpha$. The function ℓ' is a sigma partitioning for the graph G .

Proof. Since the graph G is regular, the proof is clear. ♠

By Fact 4 and Fact 2, in the subgraph A_x , the label of x forces the labels of all other vertices; therefore, the labels of exactly 16 vertices including x are equal to $\ell(x)$. Also, by the structure of B_c , the function Γ is an NAE satisfying assignment. So, in every B_c , at least one vertex has label β , at least one vertex has label α , also at least one vertex of their neighbors in $\cup_{x \in X} A_x$ has label β and at least one vertex of their neighbors in $\cup_{x \in X} A_x$ has label α . As $|C| = |X| = 3n$ and Φ is satisfied, at least $\frac{1}{3}$ of the variables are true and at most $\frac{2}{3}$ of the variables are false. Thus for α , we have:

$$\begin{aligned} |\ell^{-1}(\alpha)| &= |\ell^{-1}(\alpha) \cap \bigcup_c B_c| + |\ell^{-1}(\alpha) \cap \bigcup_{\ell(x)=\alpha} A_x| + |\ell^{-1}(\alpha) \cap \bigcup_{\ell(x)=\beta} A_x| \\ &\geq (1 \times 3n) + (16 \times 2n) + (27 \times n) \\ &= 62n \end{aligned}$$

If the equality holds, then Φ has a 1-In-3 satisfying assignment with the satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$, when $\Gamma(x) = \text{true}$ if and only is $\ell(x) = \beta$. Next, suppose that Φ has 1-In-3 satisfying assignment with the satisfying assignment $\Gamma : X \rightarrow \{\text{true}, \text{false}\}$. We present the sigma partitioning σ for G_Φ from the set $\{\alpha, \beta\}$ such that exactly $62n$ vertices have the label β . Put $\ell(x) = \beta$ if and only if $\Gamma(x) = \text{true}$, the labels of other vertices of A_x are determined. Now, for each $c \in C$, exactly one of the vertices of B_c has a neighbor in $\cup_x A_x$ with the label β . Call this vertex of B_c by v_c . Label v_c by α and label the two other vertices of B_c by different labels. One can see that σ is a sigma partitioning. This completes the proof. \square

3 Random graphs and upper bounds

Proof of Theorem 5. (i) Let $\mathcal{G} = \{G_i\}_{i \in \mathbb{N}}$ be a sequence of triangle-free graphs. Then for this family we prove that if $\delta = \Omega(\Delta)$ then $\sigma = \mathcal{O}(1)$, for all graphs in \mathcal{G} , except a finite number of them. We will use the probabilistic method to prove the theorem. The following tool of the probabilistic method will be used several times.

Lemma A [The Local Lemma [4]] *Suppose A_1, \dots, A_n is a set of random events such that for each i , $\Pr(A_i) \leq p$ and A_i is mutually independent of the set of all but at most d other events. If $ep(d+1) < 1$, then with positive probability, none of the events occur.*

We will also use the following well known inequalities (Stirling's approximation).

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{e^2 n} \left(\frac{n}{e}\right)^n.$$

Let G be a given graph. We will use the Local Lemma in order to prove the Theorem. Color each vertex of G by a random color from the set \mathbb{N}_k such that each color is chosen with probability $\frac{1}{k}$ and the color of every vertex is independent from the colors of other vertices. The value of k will be determined later. For an edge $e = uv$, let B_e be the event “for each $i, i \in \mathbb{N}_k$, the vertex u and the vertex v have the same number of neighbors in part i ”. Note that if B_e occurs, then $d(u) = d(v)$. For every edge e , the event B_e is dependent on less than $2\Delta^3$ events. Since the maximum of $\Pr(B_e)$ is when $d(v) = d(u) = \delta(G)$ and G is triangle-free, we have:

$$\Pr(B_e) \leq \sum_{\sum_i t_i = \delta} \left(\frac{\binom{\delta}{t_1, \dots, t_k}}{k^\delta} \right)^2 \quad (1)$$

$$\leq \max_{\sum_i t_i = \delta} \frac{\binom{\delta}{t_1, \dots, t_k}}{k^\delta} \quad (2)$$

$$\leq \frac{\binom{\delta}{\frac{\delta}{k}, \dots, \frac{\delta}{k}}}{k^\delta} \quad (3)$$

$$= \frac{\delta!}{\left(\left(\frac{\delta}{k}\right)!\right)^k} \times k^{-\delta} \quad (4)$$

By Stirling's approximation and (4), we have

$$\Pr(B_e) \leq \frac{e\sqrt{\delta}(\delta/e)^\delta}{\left(\sqrt{\frac{2\pi\delta}{k}}\left(\frac{\delta}{ek}\right)^{\frac{\delta}{k}}\right)^k} \times k^{-\delta} \quad (5)$$

$$= \frac{e\sqrt{\delta}}{\left(\sqrt{\frac{2\pi\delta}{k}}\right)^k}. \quad (6)$$

It was shown in [23] that for every graph G , we have $\sigma(G) = \mathcal{O}(\Delta^2)$. So for a given graph G if $\Delta(G)$ is a constant number then $\sigma(G) = \mathcal{O}(1)$. Thus, we can assume that for a given graph G , Δ is a sufficiently large number. Since $\delta = \Omega(\Delta)$, there are constant numbers n_0 and c such that for each $i \geq n_0$, for the graph G_i , we have $\delta \geq c\Delta$. Put $k = 8$. For each $i \geq n_0$, for the graph G_i with a sufficiently large Δ , we have $ep(d+1) \leq e \frac{e\sqrt{\delta}}{\left(\sqrt{\frac{2\pi\delta}{8}}\right)^8} \Delta^3 < 1$. Thus by the Local Lemma there is a sigma partitioning with $\mathcal{O}(1)$

colors for all graphs in \mathcal{G} , except a finite number of them. This completes the proof of the part (i).

(ii) Let $\mathcal{G} = \{G_i\}_{i \in \mathbb{N}}$ be a sequence of triangle-free graphs. Then for this family we prove that if $\delta \geq 3.501(\ln \Delta)$, then $\sigma = \mathcal{O}(\delta)$. Put $k = \delta$. We use the Local Lemma to prove the theorem. From the previous calculation we have:

$$\Pr(B_e) \leq \frac{\binom{\delta}{\frac{\delta}{k}, \dots, \frac{\delta}{k}}}{k^\delta} \quad (7)$$

Since $k = \delta$, by (7), we have

$$\Pr(B_e) \leq \frac{\binom{\delta}{1, \dots, 1}}{\delta^\delta} \quad (8)$$

$$= \frac{\delta!}{\delta^\delta} \quad (9)$$

By Stirling's approximation and (9), we have

$$\Pr(B_e) \leq \frac{e\sqrt{\delta}}{e^\delta} \quad (10)$$

It was shown in [23] that for every graph G , we have $\sigma(G) = \mathcal{O}(\Delta^2)$. So for a given graph G if $\Delta(G)$ is a constant number then $\sigma(G) = \mathcal{O}(1)$. Thus, we can assume that for a given graph G , Δ is a sufficiently large number. Since $\delta \geq 3.501(\ln \Delta)$ and Δ is a sufficiently large number, we have $ep(d+1) \leq e \frac{e\sqrt{\delta}}{\Delta^{3.501}} \Delta^3 < 1$. Thus by the Local Lemma there is a sigma partitioning. This completes the proof. \square

Proof of Theorem 6. Here, we prove that for every constant p , $0 < p < 1$, $\sigma(G(n, p)) \leq 5$. The proof is similar to the proof of Theorem 4 in [9]. Suppose that $\{\mathcal{P}_1, \dots, \mathcal{P}_5\}$ is a partition for n vertices such that the size of each part is $\lfloor n/5 \rfloor$ or $\lceil n/5 \rceil$. Next consider the graph $G(n, p)$ on these vertices. For every vertex v denote the number of neighbors of the vertex v in \mathcal{P}_i by \mathcal{A}_v^i . This partition is a sigma partitioning if for every two adjacent vertices v and u , there is an index i , such that $\mathcal{A}_v^i \neq \mathcal{A}_u^i$. First, we calculate $\Pr(\mathcal{A}_v^i = \mathcal{A}_u^i)$. There is a constant C such that

$$\Pr(\mathcal{A}_v^i = \mathcal{A}_u^i) \leq C \left(\sum_{t=0}^{\frac{n}{5}} \binom{\frac{n}{5}}{t} p^t (1-p)^{\frac{n}{5}-t} \right)^2 \quad (11)$$

By (11), there is a constant C' such that

$$\Pr(\mathcal{A}_v^i = \mathcal{A}_u^i) \leq C' \left(\max_{0 \leq t \leq n/5} \binom{\frac{n}{5}}{t} p^t (1-p)^{\frac{n}{5}-t} \right) \quad (12)$$

On the other hand, by Stirling's approximation we have

$$\Theta \left(\max_{0 \leq t \leq n/5} \binom{\frac{n}{5}}{t} p^t (1-p)^{\frac{n}{5}-t} \right) = \Theta \left(\binom{\frac{n}{5}}{\frac{pn}{5}} p^{\frac{pn}{5}} (1-p)^{\frac{(1-p)n}{5}} \right) = \Theta(n^{-\frac{1}{2}}) \quad (13)$$

Thus, by (12) and (13), we have $\Pr(\forall i \mathcal{A}_v^i = \mathcal{A}_u^i) = \Theta(n^{-\frac{5}{2}})$, therefore $\Pr(\exists vu \forall i \mathcal{A}_v^i = \mathcal{A}_u^i) = \Theta(n^2) \Theta(n^{-\frac{5}{2}}) = o(1)$. This completes the proof. \square

4 Concluding remarks

- A hypergraph \mathcal{H} is a pair (X, Y) , where X is the set of vertices and Y is a set of non-empty subsets of X , called edges. A k -coloring of \mathcal{H} is a coloring $\ell : X \rightarrow \mathbb{N}_k$ such that, for every edge e with $|e| > 1$,

there exist $v, u \in e$ such that $\ell(u) \neq \ell(v)$ (for more information about hypergraphs, see [15]). We say that two vertices v and u are adjacent if there is an edge like e such that $v, u \in e$. A sigma partitioning for a hypergraph \mathcal{H} is a partition for the vertices such that every edge e has two adjacent vertices v and u , such that v and u have different numbers of neighbors in some parts. Investigating the property of sigma partitioning in hypergraphs can be interesting as a future work.

• In Theorem 1, we used *NAE 3-SAT*. The planar version of *NAE 3-SAT* is in \mathbf{P} [22], so the computational complexity of σ for planar 3-regular graphs remains unsolved.

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