

On certain non-unique solutions of the Stieltjes moment problem

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We construct explicit solutions of a number of Stieltjes moment problems based on moments of the form $\rho_1^{(r)}(n) = (2rn)!$ and $\rho_2^{(r)}(n) = [(rn)!]^2$, $r = 1, 2, \dots$, $n = 0, 1, 2, \dots$, *i.e.* we find functions $W_{1,2}^{(r)}(x) > 0$ satisfying $\int_0^\infty x^n W_{1,2}^{(r)}(x) dx = \rho_{1,2}^{(r)}(n)$. It is shown using criteria for uniqueness and non-uniqueness (Carleman, Krein, Berg, Pakes, Stoyanov) that for $r > 1$ both $\rho_{1,2}^{(r)}(n)$ give rise to non-unique solutions. Examples of such solutions are constructed using the technique of the inverse Mellin transform supplemented by a Mellin convolution. We outline a general method of generating non-unique solutions for moment problems generalizing $\rho_{1,2}^{(r)}(n)$, such as the product $\rho_1^{(r)}(n) \cdot \rho_2^{(r)}(n)$ and $[(rn)!]^p$, $p = 3, 4, \dots$

Keywords: Classical moment problem, Stieltjes moment problem, Mellin transform

1 Introduction

This paper concerns solutions of the Stieltjes moment problem [1, 2], *i.e.* positive functions $W(x)$ which satisfy the infinite set of equations

$$\int_0^\infty x^n W(x) dx = \rho(n), \quad n = 0, 1, 2, \dots \quad (1)$$

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We previously met this problem when considering the properties of the so-called *coherent states* (CS) of Quantum Mechanics [3]. These CS, as well as their many generalizations, should satisfy the property of *Resolution of Unity* which is essentially equivalent to Eq. (1) [4, 5]. Standard CS lead to $\rho(n) = n!$ and $W(x) = e^{-x}$. Generalized CS lead to $\rho(n)$'s other than $n!$ [6] and the solutions of Eq.(1), if any, must be studied in each individual case separately [7, 8, 9, 10]. It quickly occurred to us that an efficient approach to this problem was to use the inverse Mellin transform method which allows one to establish many solutions of Eq. (1), either by analytic methods [11] or by extensive use of available tables [12, 13]. As a byproduct of this method we have established that, for a large number of combinatorial sequences such as Bell and Catalan numbers, *etc.*, the corresponding sequences $\rho(n)$ are solutions of the moment problem Eq. (1) [14, 15]. Likewise, sequences arising in theory of ordering of differential operators [16] solve appropriate moment problems too. This admixture of quantum-mechanical, analytical and combinatorial features deserves a deeper study which we intend to pursue.

Our paper is partly expository in character, and has the following structure: first we establish a link between the Mellin transform and the moment problem; next, in Section 3, we provide principal solutions to two moment problems, termed toy models. In Section 4 we discuss criteria for the uniqueness of solutions of the moment problem. Section 5 is devoted to the explicit construction of non-unique solutions of the toy-models. In Section 5 some generalizations of model sequences, together with their solutions, are reviewed. Section 6 is devoted to a discussion and conclusions.

We dedicate this paper to Philippe Flajolet on the occasion of his 60th birthday. His pioneering applications of Mellin transform asymptotics to the analysis of combinatorial structures [17, 18, 19] have been a source of inspiration for us.

2 The Mellin transform *versus* moment problem

The Mellin transform of a function $f(x)$ of the real variable x is defined for complex s by the following relation

$$\mathcal{M}[f(x);s] = \int_0^{\infty} x^{s-1} f(x) dx \equiv f^*(s) \quad (2)$$

(in this definition $f^*(s)$ is not the complex conjugate of $f(s)$!), and its inverse \mathcal{M}^{-1} is defined by

$$\mathcal{M}^{-1}[f^*(s);x] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} ds. \quad (3)$$

See Ref. [20] for a discussion of the dependence of $f(x)$ on the real constant c . Among the many relations satisfied by the Mellin transform we shall mainly use the following

$$\mathcal{M}[x^b f(ax^h);s] = \frac{1}{h} a^{-\frac{s+b}{h}} f^*\left(\frac{s+b}{h}\right), \quad a, h > 0. \quad (4)$$

If $\mathcal{M}[f(x);s] = f^*(s)$ and $\mathcal{M}[g(x);s] = g^*(s)$, then

$$\mathcal{M}^{-1}[f^*(s)g^*(s);x] = \int_0^{\infty} f\left(\frac{x}{t}\right) g(t) \frac{dt}{t} = \int_0^{\infty} g\left(\frac{x}{t}\right) f(t) \frac{dt}{t} \quad (5)$$

which is called the Mellin convolution property. Note that if in Eq. (5) both $f(x)$ and $g(x)$ are positive for $x > 0$ then $\int_0^{\infty} f(\frac{x}{t})g(t)\frac{dt}{t}$ is also positive for $x > 0$. This means that the Mellin convolution preserves positivity, an essential property when considering the moment problem.

Eq.(1), if rewritten for $n = s - 1$ as

$$W(x) = \mathcal{M}^{-1}[\rho(s-1);x], \quad (6)$$

is, if $W(x) > 0$, a solution of a Stieltjes moment problem. Thus according to Eq. (6) one can solve the Stieltjes moment problem by performing the inverse Mellin transform on the moment sequence and checking if the resulting function is positive. All the solutions in the sequel have been obtained using Eqs. (6), (4) and (5). Note that $W(x)$ obtained *via* Eq. (6) may not be the only solution of Eq.(1). We call $W(x) > 0$ obtained *via* Eq. (6) from Eq. (1) the principal solution.

3 Principal solutions of the moment problems

Let us consider two sequences of integers given by

$$\rho_1^{(r)}(n) = (2rn)!, \quad n = 0, 1, 2, \dots, \quad r = 1, 2, \dots \quad (7)$$

$$\rho_2^{(r)}(n) = [(rn)!]^2, \quad n = 0, 1, 2, \dots, \quad r = 1, 2, \dots \quad (8)$$

In the following we shall obtain the solutions of the Stieltjes moment problem for the moment sequences given by Eqs. (7) and (8), *i.e.* the functions $W_{1,2}^{(r)}(x) > 0$ satisfying

$$\int_0^{\infty} x^n W_{1,2}^{(r)}(x) dx = \rho_{1,2}^{(r)}(n). \quad (9)$$

From now on we shall refer to the model problems $\rho_{1,2}^{(r)}(n)$ as *toy models TM1* and *TM2*, respectively.

a) **TM1**: we begin by obtaining $W_1^{(1)}(x)$. Note that $(2n)! = \Gamma(2n+1) = \Gamma(2(s-\frac{1}{2}))$. We now apply Eq.(4) with $a = 1$, $b = -\frac{1}{2}$ and $h = \frac{1}{2}$. We subsequently use $\mathcal{M}^{-1}[\Gamma(s);x] = e^{-x}$ which gives

$$\int_0^{\infty} x^n W_1^{(1)}(x) dx = \int_0^{\infty} x^n \left[\frac{e^{-\sqrt{x}}}{2\sqrt{x}} \right] dx = (2n)!, \quad n = 0, 1, 2, \dots \quad (10)$$

In the same spirit we observe that $(2rn)! = \Gamma(2r(n+\frac{1}{2r})) = \Gamma(2r(s-\frac{2r-1}{2r}))$. Upon using Eq. (4) but now with $a = 1$, $b = -\frac{2r-1}{2r}$ and $h = \frac{1}{2r}$ one obtains

$$\int_0^{\infty} x^n W_1^{(r)}(x) dx = \int_0^{\infty} x^n \left[\frac{1}{2rx^{\frac{2r-1}{2r}}} e^{-x^{\frac{1}{2r}}} \right] dx = (2rn)!, \quad n = 0, 1, 2, \dots \quad (11)$$

We call $W_1^{(r)}(x) > 0$ the *principal solution* of the moment problem Eq. (11).

b) **TM2**: we begin by deriving $W_2^{(1)}(x) = \mathcal{M}^{-1}[\Gamma^2(s); x]$ and employ the Mellin convolution Eq. (5). By using the Sommerfeld representation of the modified Bessel function of second kind $K_0(x)$ [21] one obtains

$$\int_0^{\infty} x^n W_2^{(1)}(x) dx = \int_0^{\infty} x^n [2K_0(2x^{\frac{1}{2}})] dx = (n!)^2, \quad n = 0, 1, 2, \dots \quad (12)$$

Subsequently note that $[(rn)!]^2 = [\Gamma(r(n + \frac{1}{r}))]^2 = [\Gamma(r(s - \frac{r-1}{r}))]^2$ and again apply Eq.(4) with $a = 1$, $b = -\frac{r-1}{r}$ and $h = \frac{1}{r}$. The result is

$$\int_0^{\infty} x^n W_2^{(r)}(x) dx = \int_0^{\infty} x^n \left[\frac{2}{rx^{\frac{r-1}{r}}} K_0(2x^{\frac{1}{r}}) \right] dx = [(rn)!]^2, \quad n = 0, 1, 2, \dots \quad (13)$$

Since $K_0(t) > 0$ for $t > 0$, $W_2^{(r)}(x) > 0$ is the principal solution of the moment problem Eq. (13).

4 Criteria of uniqueness and non-uniqueness of the Stieltjes moment problem

It was realized from the very beginning of the history of the moment problem that its solutions may not be unique; *i.e.* for a given moment sequence there may exist more than one solution. Stieltjes himself gave an example of a non-unique solution of a problem leading to what turned out to be the lognormal distribution [22]. This example, being about the only one available, was quoted repeatedly in the literature. As recently as twenty years ago new non-unique solutions of other types of problems have been constructed. Of special interest were Stieltjes moment problems arising in probability theory, related to investigation of probability distributions not determined by their moments. Consequently, the subject was further developed and systematized, largely due to the comprehensive work of Berg [23], Berg and Pedersen [24], Lin [25], Stoyanov [26, 27], Pakes [28], Gut [29] and others. From the practical point of view one needs criteria to decide whether the pursuit of non-unique solutions is reasonable. Such criteria are either based on the $\rho(n)$'s alone or on the solution $W(x)$ alone, or on both $\rho(n)$ and $W(x)$, see below. From now on we assume that all the $\rho(n)$'s are finite and that $W(x)$ is continuous. We now give, in a somewhat condensed form, a list of such criteria.

C1 Carleman uniqueness criterion (T. Carleman, 1922, [1]) This is based on the properties of the $\rho(n)$'s alone and is:

If $S = \sum_{n=1}^{\infty} [\rho(n)]^{-\frac{1}{2n}} = \infty$, then the solution is unique.

This criterion does not imply that if $S < \infty$ then the solution is non-unique. In fact it is possible to construct models for which $S < \infty$ and solutions are still unique [26, 30].

C2 Krein's non-uniqueness criterion (M. G. Krein, around 1950, [31])

This is based entirely on the solution $W(x)$ and does not involve the moments.

*If $\int_0^{\infty} \frac{-\ln[W(x^2)]}{1+x^2} dx < \infty$, *i.e.*, the so-called Krein integral exists, then the solution is non-unique.*

C3 Converse Carleman criterion for non-uniqueness (A. Pakes, 2001, [28], A. Gut, 2002, [29])

This is based on $\rho(n)$'s and $W(x)$.

If there exists $x' \geq 0$ such that for $x > x'$, $0 < W(x) < \infty$ and $\psi(y) = -\ln[W(e^y)]$ is convex in (y', ∞) , where $y' = \ln(x')$, and if, in addition, $S = \sum_{n=1}^{\infty} [\rho(n)]^{-\frac{1}{2n}} < \infty$, then the solution $W(x)$ is non-unique.

See [23, 24, 26, 28, 29] for various refinements of these criteria.

5 Construction of non-unique solutions for TM1 and TM2

We first apply the criterion C1 to sequences $\rho_{1,2}^{(r)}(n)$ (the logarithmic test of divergence of the series S is conclusive) and conclude that for $r = 1$ both solutions $W_1^{(1)}(x) = \frac{e^{-\sqrt{x}}}{2\sqrt{x}}$ and $W_2^{(1)}(x) = 2K_0(2x^{\frac{1}{2}})$ are unique.

For $r > 1$ we find that $\sum_{n=1}^{\infty} [\rho_{1,2}^{(r)}(n)]^{-\frac{1}{2n}}$ is convergent. Application of criterion C2 gives convergence of the Krein integral, showing that the solutions of **TM1** and **TM2** are non-unique. These findings are confirmed by use of criterion C3: convexity of $\psi_{1,2}^{(r)}(x) = -\ln[W_{1,2}^{(r)}(e^x)]$ is proved as it is equivalent to $e^{x/r} > 0$ for $x > 0$, (**TM1**) and $K_1(2e^{x/r}) - K_0(2e^{x/r}) > 0$ for $x > 0$, (**TM2**). The quest for non-unique solutions for $r > 1$ is then well founded.

We know of no general method to construct such solutions. However we have proposed a procedure based on the application of inverse Mellin transform which generates the required solutions [8, 32] which we now briefly expose.

The first step is to construct, within the framework of a given set of $\rho(n)$'s, a family of functions $\omega_k(x)$, parametrized by a constant k (to be defined below), such that *all* their moments vanish, *i.e.*

$$\int_0^{\infty} x^n \omega_k(x) dx = \int_0^{\infty} x^{s-1} \omega_k(x) dx = 0, \quad n = 0, 1, 2, \dots, \quad s = 1, 2, \dots \quad (14)$$

The Mellin transform of $\omega_k(x)$, $\omega_k^*(s)$, vanishes for $s = 1, 2, \dots$. Such functions are orthogonal to all polynomials and play an important role in the study of integral transforms [33]. For our purposes we choose a particular method of producing the functions $\omega_k^*(x)$:

$$\int_0^{\infty} x^n \omega_k(x) dx = \rho_{1,2}^{(r)}(n) \cdot h_k(n), \quad (15)$$

or equivalently

$$\int_0^{\infty} x^{s-1} \omega_k(x) dx = \rho_{1,2}^{(r)}(s-1) \cdot h_k(s-1), \quad (16)$$

where $h_k(s)$ is any holomorphic function vanishing for $s = 1, 2, \dots$. Among an infinity of possible choices the simplest one is $h_k(s) = \sin(\pi k(s+1))$ and it defines a discrete parameter $k = \pm 1, \pm 2, \pm 3, \dots$. The function $\omega_k(x)$ acquires new parameters now and is formally obtained by calculating the inverse Mellin transform

$$\omega_{1,2,k}^{(r)}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \rho_{1,2}^{(r)}(s-1) \sin(\pi k s) x^{-s} ds, \quad k = \pm 1, \pm 2, \pm 3, \dots \quad (17)$$

It turns out that for both $\rho_{1,2}^{(r)}(s)$ the integration in Eq.(17) can be performed:

a) **TM1**: for $\rho_1^{(r)}(n) = (2rn)!$, $\omega_{1,k}^{(r)}(x)$ is a special case of earlier evaluation [8] and reads:

$$\begin{aligned}\omega_{1,k}^{(r)}(x) &= \frac{1}{2rx^{\frac{2r-1}{2r}}} e^{-x^{\frac{1}{2r}}} \sin \left[k\pi \left(\frac{2r-1}{2r} \right) + x^{1/2r} \tan \left(\frac{k\pi}{2r} \right) \right], \quad (r > |k|) \\ &= W_1^{(r)}(x) \sin \left[k\pi \left(\frac{2r-1}{2r} \right) + x^{1/2r} \tan \left(\frac{k\pi}{2r} \right) \right].\end{aligned}\quad (18)$$

In Eq.(18) we notice a pleasant factorization of $W_1^{(r)}(x)$.

b) **TM2**: for $\rho_2^{(r)}(n) = [(rn)!]^2$ the corresponding function $\omega_{2,k}^{(r)}(x)$ has to be, in the first place, represented as

$$\omega_{2,k}^{(r)}(x) = \mathcal{M}^{-1} \underbrace{[\Gamma(rs - s + 1)]}_I \underbrace{[\Gamma(rs - s + 1) \sin(\pi ks)]}_II; x \quad (19)$$

which can be conceived as another case of Mellin convolution. A little thought gives the two partners to be convoluted as

$$I \rightarrow W_1^{(r/2)}(x) = \frac{1}{rx^{\frac{r-1}{r}}} e^{-x^{\frac{1}{r}}}, \quad (20)$$

(see Eq.(18)) and

$$II \rightarrow \omega_{1,k}^{(r/2)}(x) = \frac{1}{rx^{\frac{r-1}{r}}} e^{-x^{\frac{1}{r}}} \sin \left[k\pi \left(\frac{r-1}{r} \right) + x^{1/r} \tan \left(\frac{k\pi}{r} \right) \right], \quad (21)$$

(compare Eq.(18)). Thus, the integral form of $\omega_{2,k}^{(r)}(x)$ is

$$\omega_{2,k}^{(r)}(x) = \int_0^\infty W_1^{(r/2)} \left(\frac{x}{t} \right) \omega_{1,k}^{(r/2)}(t) \frac{dt}{t} \quad (22)$$

whose evaluation requires a number of changes of variables as well as the use of formula 2.5.37.2, p. 453 of vol.1 Ref.[12], but is essentially elementary. The final result is

$$\begin{aligned}\omega_{2,k}^{(r)}(x) &= \frac{2}{rx^{\frac{r-1}{r}}} \operatorname{Re} \left[e^{i\pi \left(\frac{1}{2} - k \frac{r-1}{r} \right)} K_0 \left(2x^{1/2r} \left(1 + i \tan \left(\frac{\pi k}{r} \right) \right)^{1/2} \right) \right] \\ &\equiv \frac{2}{rx^{\frac{r-1}{r}}} V_k^{(r)}(x)\end{aligned}\quad (23)$$

where we note a ‘‘near’’ factorization of $W_2^{(r)}(x)$.

Armed with explicit forms for $\omega_{1,k}^{(r)}(x)$ and $\omega_{2,k}^{(r)}(x)$ we are in position now to write down the families of non-unique solutions. Their structure has the form: principal solution + const· $\omega_k(x)$. More precisely:

TM1:

$$\tilde{W}_1^{(r)}(\varepsilon, k, x) = W_1^{(r)}(x) \left[1 + \varepsilon \sin \left(k\pi \left(\frac{2r-1}{2r} \right) + x^{1/2r} \tan \left(\frac{k\pi}{2r} \right) \right) \right] \quad (24)$$

for real ε , $|\varepsilon| < 1$.

TM2:

$$\tilde{W}_2^{(r)}(\gamma, k, x) = W_2^{(r)}(x) \left[1 + \gamma \frac{V_k^{(r)}(x)}{K_0(2x^{1/2r})} \right] \quad (25)$$

for $r > 2|k|$.

As $\left[1 + \gamma \frac{V_k^{(r)}(x)}{K_0(2x^{1/2r})} \right]$ is an oscillating function of bounded variation, a constant $\gamma = \gamma(k, r)$ can be always found to assure the overall positivity of $\tilde{W}_2^{(r)}(\gamma, k, x)$. The above technique for obtaining non-unique solutions can be readily extended to moment sequences more general than $\rho_{1,2}^{(r)}(n)$. We shall simply mention two such extensions without entering into details.

For $\rho_3^{(r)}(n) = [(rn)!]^3$ we begin with the sequence $(n!)^3$ for which the solution is

$$\int_0^\infty x^n \text{MeijerG} \left(\left[\begin{matrix} \\ \end{matrix} \right], \left[\begin{matrix} \\ \end{matrix} \right], \left[\begin{matrix} 0, 0, 0 \\ \end{matrix} \right], x \right) dx = (n!)^3, \quad (26)$$

where we use a convenient and self-explanatory notation for Meijer's G-function borrowed from that of computer algebra systems. The extension, *via* Eq. (4), leads to the principal solution

$$\int_0^\infty x^n \left[\frac{1}{rx^{\frac{r-1}{r}}} \text{MeijerG} \left(\left[\begin{matrix} \\ \end{matrix} \right], \left[\begin{matrix} \\ \end{matrix} \right], \left[\begin{matrix} 0, 0, 0 \\ \end{matrix} \right], x^{1/r} \right) \right] dx = [(rn)!]^3. \quad (27)$$

The integrand in Eq. (27) is a positive function for $x > 0$ which cannot be represented by any other known special function. It possesses an infinite series representation in terms of polygamma functions, which we will not quote here. The Carleman sum S is convergent but the criterion C2 is not conclusive. Only the criterion C3 permits to ascertain the non-uniqueness. The corresponding function $\omega_{3,k}^{(r)}(x)$ is defined as

$$\omega_{3,k}^{(r)}(x) = \mathcal{M}^{-1} \left[\underbrace{\Gamma^2(rs - s + 1)}_{\text{I}} \underbrace{\Gamma(rs - s + 1) \sin(\pi ks)}_{\text{II}}; x \right] \quad (28)$$

which can be calculated as Mellin convolution of $\text{I} \rightarrow W_2^{(r)}(x)$ and $\text{II} \rightarrow \omega_{1,k}^{(r/2)}(x)$, see Eqs. (18) and (23), respectively.

As a final example consider the sequence $\rho_4^{(r)}(n) = \rho_1^{(r)}(n)\rho_2^{(r)}(n)$. The corresponding Mellin convolution of two principal solutions $W_{1,2}^{(r)}(x)$, see Eqs. (24) and (25), yields directly the principal solution for $\rho_4^{(r)}(n)$:

$$\int_0^\infty x^n \left[\frac{4^{r-1}}{r\sqrt{\pi x}^{\frac{2r-1}{r}}} \text{MeijerG} \left(\left[\begin{matrix} \\ \end{matrix} \right], \left[\begin{matrix} \\ \end{matrix} \right], \left[\begin{matrix} r - \frac{1}{2}, r, r, r \\ \end{matrix} \right], \frac{1}{4}x^{1/r} \right) \right] dx = (2rn)! [(rn)!]^2, \quad (29)$$

$$n = 0, 1, 2, \dots, \quad r = 1, 2, \dots,$$

which is non-unique by C3 only, as C2 remains inconclusive.

6 Discussion and Conclusion

We have demonstrated a methodology for obtaining unique and non-unique solutions of the Stieltjes moment problem using the Mellin convolution method. Although the initial moment sequences were simple and classical, one is rapidly forced to leave the realm of standard special functions, as the resulting solutions are special cases of Meijer G-functions. In most cases they resist the check for non-uniqueness *via* both the Carleman criterion C1 and the Krein criterion C2, and criterion C3 appears to be the only tool to decide this question. We have generated parametrized families of non-unique solutions exemplified here by Eqs. (18) and (23). Such functions are now called *Stieltjes classes* [34, 35] and their properties are investigated.

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References

- [1] N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, Edinburgh and London, 1963).
- [2] B. Simon, “The Classical Moment Problem as a Self-Adjoint Finite Difference Operator”, *Adv.Math.* **137** (1998) 82–203.
- [3] Coherent states were originally proposed by Erwin Schroedinger in the early days of Quantum Mechanics to describe wave packets obeying the time evolution close to the classical motion (E. Schroedinger, “Der Stetige Ubergang von der Mikro-zur Makromechanik”, *Naturwissenschaften* **14** (1926) 664–666). They were reintroduced more than half a century ago in seminal papers by R. J. Glauber and others: R. J. Glauber, “The quantum theory of optical coherence”, *Phys. Rev.* **130** (1963) 2529–2539, E. C. G. Sudarshan, “Equivalence of semiclassical and quantum mechanical description of statistical light beams”, *Phys. Rev. Lett.* **10** (1963) 277–279, and J. R. Klauder, “Continuous-Representation Theory. I. Postulates of Continuous-Representation Theory”, *J. Math. Phys.* **4** (1963) 1055–1058; “Continuous-Representation Theory. II. Generalized Relation between Quantum and Classical Dynamics”, *J. Math. Phys.* **4** (1963) 1058–1073. Such states are nowadays an important tool widely used in quantum optics and in general investigations of quantization.
- [4] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).
- [5] J. R. Klauder and B-S. Skagerstam, *Coherent States. Application in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
- [6] A. O. Barut and L. Girardello, “New ‘Coherent States’ Associated with Non-Compact Groups”, *Comm. Math. Phys.* **21** (1971) 41–55.

- [7] K. A. Penson and A. I. Solomon, “New generalized coherent states”, *J. Math. Phys.* **40** (1999) 2354–2363.
- [8] J.-M. Sixdeniers, K. A. Penson and A. I. Solomon, “Mittag-Leffler coherent states”, *J. Phys. A: Math. Gen.* **32** (1999) 7543–7563.
- [9] J. R. Klauder, K. A. Penson and J.-M. Sixdeniers, “Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems”, *Phys. Rev. A* **64** (2001) 013817.
- [10] K. A. Penson and A. I. Solomon, “Coherent states from combinatorial sequences”, in *Proceedings 2nd Int. Symposium 'Quantum Theory and Symmetries', Kraków, Poland, 18-22 July 2001*, pp.527–530 (World Scientific, Singapore, 2002), arXiv:quant-ph/0111151.
- [11] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables* (Ellis Horwood, New York, 1983).
- [12] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series, vol.1, 'Elementary functions', vol.2, 'Special functions', vol.3, 'More special functions'* (Gordon and Breach, New York, 1998).
- [13] F. Oberhettinger, *Tables of Mellin Transforms* (Springer Verlag, Berlin, 1974).
- [14] K. A. Penson, P. Blasiak, G. Duchamp, A. Horzela and A.I. Solomon, “Hierarchical Dobiński relations via substitutions and the moment problem”, *J. Phys. A: Math. Gen.* **35** (2004) 3475–3487.
- [15] P. Blasiak, A. Horzela, K. A. Penson and A. I. Solomon, “Dobiński-type relations: Some properties and applications”, *J. Phys. A: Math. Gen.* **37** (2006) 4999–5006.
- [16] K. A. Penson, P. Blasiak, A. Horzela, G.H.E. Duchamp and A.I. Solomon, “Laguerre-type derivatives: Dobiński relations and combinatorial identities”, *J. Math. Phys.* **50** (2009) 083512.
- [17] P. Flajolet, X. Gourdon and P. Dumas, “Mellin transforms and asymptotics: Harmonic sums”, *Theor. Comp. Sc.* **144** (1995) 3–58.
- [18] P. Flajolet and R. Sedgewick, “Mellin transforms and asymptotics: Finite differences and Rice’s integral”, *Theor. Comp. Sc.* **144** (1995) 101–124.
- [19] P. Flajolet and R. Sedgewick, *Analytic Combinatorics* (Cambridge University Press, Cambridge UK, 2009), Appendices B and C.
- [20] I. N. Sneddon, *The Use of Integral Transforms* (Tata McGraw, New Delhi, 1972).
- [21] N. N. Lebedev, *Special Functions and Their Applications* (Dover, New York, 1972).
- [22] T. J. Stieltjes, “Recherches sur les fractions continues”, *Anns. Fac. Sci. Univ. Toulouse* **8** (1894) J1-J122, **9** (1895) A5-A47.
- [23] C. Berg, “Indeterminate moment problems and the theory of entire functions”, *J. Comp. Appl. Math.*, **65** (1995) 27–55.

- [24] C. Berg and H. L. Pedersen, “Logarithmic order and type of indeterminate moment problems”, in *Difference Equations, Special Functions and Orthogonal Polynomials, Proceedings of the International Conference, Munich, Germany, 25-30 July 2005*, pp. 51–80, (World Scientific, Singapore, 2007).
- [25] G. D. Lin, “On the moment problem”, *Stat. and Prob. Lett.* **35** (1997) 85–90.
- [26] J. Stoyanov, *Counterexamples in Probability*, 2nd ed., Section 11 (Wiley, New York, 1997).
- [27] J. Stoyanov, “Krein condition in probabilistic moment problems”, *Bernoulli* **6** (2000) 939–949.
- [28] A. G. Pakes, “Remarks on converse Carleman and Krein criteria for the classical moment problem”, *J. Austral. Math. Soc.* **71** (2001) 81–104, and references therein.
- [29] A. Gut, “On the moment problem”, *Bernoulli* **8** (2002) 407–421, and references therein.
- [30] see Ref. [2], remark on p.89, Corollary 4.21 and Theorem 6.2.
- [31] M. G. Krein formulated logarithmic integral criteria for the Hamburger moment problem in 40’ and 50’ of the XXth century. Extensions of Krein’s criteria to the Stieltjes moment problem were found much later.
- [32] J.-M. Sixdeniers, “Constructions de nouveaux états cohérents à l’aide de solutions des problèmes des moments”, Thèse de Doctorat de l’Université Paris 6 (2001), unpublished.
- [33] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications* (Gordon and Breach, New York, 1993).
- [34] J. Stoyanov and L. Tolmatz, “Method for constructing Stieltjes classes for M-indeterminate probability distributions”, *Appl. Math. Comp.* **165** (2005) 669–685.
- [35] A. G. Pakes, “Structure of Stieltjes classes of moment-equivalent probability laws”, *J. Math. Anal. Appl.* **326** (2007) 1268–1290.