Discrete Mathematics and Theoretical Computer Science DMTCS vol. **21**:3, 2019, #1

Binding Number, Toughness and General Matching Extendability in Graphs

Hongliang Lu[1*] Qinglin Yu[2,3†]

[1] *Department of Mathematics, Xi'an Jiaotong University, Xi'an, China*
[2] *School of Science, Xi'an Polytechnic University, Xi'an, China*
[3] *Department of Mathematics and Statistics, Thompson Rivers University, Kamloops, BC, Canada*

received 31st July 2018, accepted 13th Dec. 2018.

A connected graph G with at least $2m + 2n + 2$ vertices which contains a perfect matching is $E(m, n)$-*extendable*, if for any two sets of disjoint independent edges M and N with $|M| = m$ and $|N| = n$, there is a perfect matching F in G such that $M \subseteq F$ and $N \cap F = \emptyset$. Similarly, a connected graph with at least $n + 2k + 2$ vertices is called (n, k)-*extendable* if for any vertex set S of size n and any matching M of size k of $G - S$, $G - S - V(M)$ contains a perfect matching. Let ε be a small positive constant, $b(G)$ and $t(G)$ be the binding number and toughness of a graph G. The two main theorems of this paper are: for every graph G with sufficiently large order, 1) if $b(G) \geq 4/3 + \varepsilon$, then G is $E(m, n)$-extendable and also (n, k)-extendable; 2) if $t(G) \geq 1 + \varepsilon$ and G has a high connectivity, then G is $E(m, n)$-extendable and also (n, k)-extendable. It is worth to point out that the binding number and toughness conditions for the existence of the general matching extension properties are almost same as that for the existence of perfect matchings.

Keywords: Binding number, toughness, perfect matching, matching extendability

1 Introduction

In this paper, we only consider simple connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *matching* is a set of independent edges and we often refer a matching with k edges as a k-*matching*. For a matching M, we use $V(M)$ to denote the set of the endvertices of the edges in M and $|M|$ to denote the number of edges in M. A matching is called a *perfect matching* if it covers all vertices of graph G. For $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S and $G - S$ for $G[V(G)\backslash S]$. The number of odd components (i.e., components with odd order) and the number of components of G are denoted by $c_0(G)$ and $c(G)$, respectively. Let $N_G(S)$ denote the set of neighbors of a set S in a graph G, and $\kappa(G)$ denote the vertex connectivity of graph G.

*Supported by the National Natural Science Foundation of China, No. 11471257 and 11871391.
†Supported by the Discovery Grant from the Natural Sciences and Engineering Research Council of Canada, and the Shanxi Hundred-Talent Program of Shanxi Province.
Corresponding email: yu@tru.ca

ISSN 1365–8050

Let M be a matching of G. If there is a matching M' of G such that $M \subseteq M'$, we say that M can be extended to M' or M' is an *extension* of M. Suppose that G is a connected graph with perfect matchings. If each k-matching can be extended to a perfect matching in G, then G is called k-*extendable*. To avoid triviality, we require that $|V(G)| \geq 2k+2$ for k-extendable graphs. This family of graphs was introduced and studied first by Plummer (1980). A graph G is called n-*factor-critical* if after deleting any n vertices the remaining subgraph of G has a perfect matching, which was introduced in Yu (1993) and was a generalization of the notions of the well-known factor-critical graphs and bicritical graphs (the cases corresponding to $n = 1$ and 2, respectively). Note that every connected factor-critical graph is 2-edge-connected (see Yu (1993)).

Let G be a graph and let n, k be nonnegative integers such that $|V(G)| \geq n+2k+2$ and $|V(G)|-n \equiv 0 \pmod 2$. If deleting any n vertices from G the remaining subgraph of G contains a k-matching and moreover, each k-matching in the subgraph can be extended to a perfect matching, then G is called (n, k)-*extendable* (Liu and Yu (2001)). This term can be considered as a general framework to unify the concepts of n-factor-criticality and k-extendability. In particular, $(n, 0)$-extendable graphs are exactly n-factor-critical graphs and $(0, k)$-extendable graphs are the same as k-extendable graphs. A graph is called $E(m, n)$-*extendable* if deleting edges of any n-matching, the resulted graph is m-extendable (Porteous and Aldred (1996)). $E(m, 0)$-extendability is equivalent to m-extendability, and (n, k)-extendability and $E(m, n)$-extendability are referred as general matching extensions, which are widely studied in graph theory (see Plummer (1994, 1996, 2008)).

For a non-complete graph G, its *toughness* is defined by

$$t(G) = \min_{S \subset V(G)} \frac{|S|}{c(G-S)}$$

where S is taken over all cut-sets of G. The *binding number* $b(G)$ is defined to be the minimum, taken over all $S \subseteq V(G)$ with $S \neq \emptyset$ and $N_G(S) \neq V(G)$, of the ratios $\frac{|N_G(S)|}{|S|}$.

Toughness and binding number have been effective graphic parameters for studying factors and matching extensions in graphs. For instances, 1-tough graphs guarantee the existence of perfect matchings (see Chvátal (1973)) and graphs with $b(G) \geq \frac{4}{3}$ contain perfect matchings (see Woodall (1973)). There are sufficient conditions with respect to $t(G)$ and $b(G)$ in terms of m, n, k to ensure the existences of k-extendability, $E(m, n)$-extendability and other matching extensions (see Chen (1995); Liu and Yu (1998); Plummer (1988a, 2008)). Moreover, Robertshaw and Woodall (2002) proved a remarkable result that a graph with $b(G)$ slightly greater than $\frac{4}{3}$ ensure k-extendability if the order of G is sufficiently large. Recently, Plummer and Saito (2017) extended this result to $E(m, n)$-extendability. In this paper, we continue the study in this direction and prove that the essential bounds of $t(G)$ and $b(G)$ (i.e., 1 and $\frac{4}{3}$) which guarantee the existence of a perfect matching can also ensure the existence of all general matching extensions mentioned earlier.

Tutte (1947) gave a characterization for a graph to have a perfect matching.

Theorem 1.1 (Tutte (1947)) *Let G be a graph with even order. Then G contains a perfect matching if and only if for any $S \subseteq V(G)$*

$$c_0(G-S) \leq |S|.$$

The following result is an extension of Tutte's theorem and also a lean version of a comprehensive structure theorem for matchings, due to Gallai (1964) and Edmonds (1965). See Lovász and Plummer (1986) for a detailed statement and discussion of this theorem.

Theorem 1.2 (see Lovász and Plummer (1986)) *Let G be a graph with even order. Then G contains no perfect matchings if and only if there exists a set $S \subset V(G)$ such that*

$$fc(G-S) \geq |S| + 2,$$

where $fc(G-S)$ denotes the number of factor-critical components of $G-S$.

The proofs of the main theorems require the following two results as lemmas.

Theorem 1.3 (Liu and Yu (2001)) *If G is an (n,k)-extendable graph and $n \geq 1, k \geq 2$, then G is also $(n+2, k-2)$-extendable.*

Theorem 1.4 (Plummer (1988b)) *If a graph G is connected and k-extendable graph ($k \geq 1$), then $G-e$ is $(k-1)$-extendable for any edge e of G.*

2 Binding Number and Matching Extendability

Chen (1995) proved that a graph G of even order at least $2m+2$ is m-extendable if $b(G) > \max\{m, (7m+13)/12\}$. Robertshaw and Woodall (2002) proved a stronger result (in most cases). We state their result in a simpler but slightly weaker form below.

Theorem 2.1 (Robertshaw and Woodall (2002)) *For any positive real number ε and nonnegative integer m, there exists an integer $N = N(\varepsilon, m)$ such that every graph G of even order greater than N and $b(G) > 4/3 + \varepsilon$ is m-extendable.*

In this section, we extend the above result using a different proof technique.

Theorem 2.2 *Let k, g be two positive integers such that $g \geq 3$ and let $g_0 = 2\lfloor \frac{g}{2} \rfloor + 1$. For any positive real number $\varepsilon < \frac{1}{g_0}$, there exists $N = N(\varepsilon, k, g_0)$ such that for every graph G with order at least N and girth g, if $b(G) > \frac{g_0+1}{g_0} + \varepsilon$, then G is k-extendable.*

Proof: Suppose that the result does not hold. Then there exists a graph G with order at least N and $b(G) > \frac{g_0+1}{g_0} + \varepsilon$ such that G is not k-extendable. By the definition of k-extendable graphs, there exists a k-matching M such that $G - V(M)$ contains no perfect matchings. From Theorem 1.2, there exists $S \subset V(G) - V(M)$ such that

$$fc(G - V(M) - S) = s + q,$$

where $q \geq 2$ is even by parity and $s := |S|$. Let $C_1, \ldots, C_{s+q}$ denote these factor-critical components of $G - S - V(M)$ such that $|C_1| \leq \cdots \leq |C_{s+q}|$. Without loss of generality, we assume $|C_1| = \ldots = |C_l| = 1$. Note that $|C_i| \geq 3$ implies $g(C_i) \geq g$ as C_i is 2-edge-connected. Thus we have $|C_i| \geq g_0$ for $l+1 \leq i \leq s+q$. Write $U = \cup_{i=2}^{s+q} V(C_i)$ and $W = V(G) - U - S - V(M)$. Note that $V(C_1) \subseteq W$ and $s + q \geq 2$. So we have $U \neq \emptyset$ and $W \neq \emptyset$. One may see that $N(U) \cap W = \emptyset$ and $N(W) \cap U = \emptyset$. Hence $N(U) \neq V(G)$ and $N(W) \neq V(G)$. Denote $r = \max\{2, l+1\}$. Thus we have

$$\begin{aligned}
b(G) &\leq \min\{\frac{|N(U)|}{|U|}, \frac{|N(W)|}{|W|}\} \\
&\leq \min\{\frac{2k + s + \sum_{i=r}^{s+q} |C_i|}{r - 2 + \sum_{i=r}^{s+q} |C_i|}, \frac{|G| - \sum_{i=2}^{s+q} |C_i|}{|G| - 2k - s - \sum_{i=2}^{s+q} |C_i|}\} \\
&= \min\{f, h\}
\end{aligned}$$

where $f=\frac{2k+s+\sum_{i=r}^{s+q}|C_i|}{r-2+\sum_{i=r}^{s+q}|C_i|}$ and $h=\frac{|G|-\sum_{i=2}^{s+q}|C_i|}{|G|-2k-s-\sum_{i=2}^{s+q}|C_i|}$.

Claim 1. $2k+s>r-2$.

This claim is implied by the following inequality:

$$1<\frac{g_0+1}{g_0}+\varepsilon<b(G)\le f=\frac{2k+s+\sum_{i=r}^{s+q}|C_i|}{r-2+\sum_{i=r}^{s+q}|C_i|},$$

Claim 2. $\sum_{i=r}^{s+q}|C_i|<g_0(2k+s)$.

Suppose that $\sum_{i=r}^{s+q}|C_i|\ge g_0(2k+s)$. By Claim 1, we have

$$\begin{aligned}b(G)\le f&\le\frac{2k+s+g_0(2k+s)}{r-2+g_0(2k+s)}\\&\le\frac{2k+s+g_0(2k+s)}{g_0(2k+s)}\\&=\frac{g_0+1}{g_0},\end{aligned}$$

a contradiction.

Claim 3. $s<\max\{2(g_0-1)k,\frac{2k}{g_0\varepsilon}\}$.

Suppose that $s\ge\max\{2(g_0-1)k,\frac{2k}{g_0\varepsilon}\}$. Since $s\ge 2(g_0-1)k$, we infer that

$$\frac{s(g_0+1)+2k}{g_0s}\le\frac{g_0}{g_0-1}.\tag{1}$$

If

$$\frac{g_0+1}{g_0}+\varepsilon<\frac{(g_0+1)s+2k}{g_0s},\tag{2}$$

then $s<\frac{2k}{g_0\varepsilon}$, a contradiction. So it is enough for us to show (2). Consider $q<r-1$. Then we infer that

$$\begin{aligned}\frac{g_0+1}{g_0}+\varepsilon<f&\le\frac{2k+s+g_0(s+q-r+1)}{r-2+g_0(s+q-r+1)}\qquad\text{(by Claim 1 and }\textstyle\sum_{i=r}^{s+q}|C_i|\ge g_0(s+q-r+1))\\&=\frac{s(g_0+1)+2k+g_0(q-r+1)}{g_0s+g_0(q-r+1)+r-2}\\&<\frac{s(g_0+1)+2k+g_0(q-r+1)}{g_0s+g_0(q-r+1)+r-1-q}\\&=\frac{s(g_0+1)+2k-g_0(r-1-q)}{g_0s-(g_0-1)(r-1-q)}\\&\le\frac{(g_0+1)s+2k}{g_0s}.\qquad\text{(by (1) and }g_0s+g_0(q-r+1)>q-r+1)\end{aligned}$$

Next, we consider $q \geq r-1$, then

$$\begin{aligned}
\frac{g_0+1}{g_0}+\varepsilon < f &\leq \frac{2k+s+g_0(s+q-r+1)}{r-2+g_0(s+q-r+1)} && \text{(by Claim 1 and } \textstyle\sum_{i=r}^{s+q}|C_i| \geq g_0(s+q-r+1)) \\
&\leq \frac{2k+s+g_0(s+q'-r+1)}{r-2+g_0(s+q'-r+1)} && \text{(for any } q' \text{ satisfying } q \geq q' \geq r-1) \\
&= \frac{s(g_0+1)+2k}{g_0 s+r-2} \\
&\leq \frac{(g_0+1)s+2k}{g_0 s}.
\end{aligned}$$

This completes the proof of Claim 3.

Claim 4. $l < \max\{2g_0k+1, \frac{2k}{g_0\varepsilon}+1\}$.

Suppose that $l \geq \max\{2g_0k+1, \frac{2k}{g_0\varepsilon}+1\}$. From Claim 3, we have

$$s < \max\{2(g_0-1)k, \frac{2k}{g_0\varepsilon}\}. \tag{3}$$

From (3), we see $l \geq s+1$ and thus

$$\begin{aligned}
\frac{g_0+1}{g_0}+\varepsilon < \ f &= \frac{2k+s+\sum_{i=r}^{s+q}|C_i|}{r-2+\sum_{i=r}^{s+q}|C_i|} \\
&= \frac{2k+s+\sum_{i=r}^{s+q}|C_i|}{l-1+\sum_{i=r}^{s+q}|C_i|} \\
&\leq \frac{2k+s}{l-1} && \text{(by Claim 1)} \\
&\leq \frac{2k+l-1}{l-1} \\
&\leq \frac{g_0+1}{g_0}, && \text{(since } l \geq 2g_0k+1)
\end{aligned}$$

a contradiction.

From Claim 2, we have

$$\sum_{i=r}^{s+q}|C_i| < g_0(2k+s). \tag{4}$$

Thus

$$\begin{aligned}
\frac{g_0+1}{g_0}+\varepsilon < h &= \frac{|G|-\sum_{i=2}^{s+q}|C_i|}{|G|-2k-s-\sum_{i=2}^{s+q}|C_i|}\\
&= \frac{|G|-(r-2)-\sum_{i=r}^{s+q}|C_i|}{|G|-2k-s-(r-2)-\sum_{i=r}^{s+q}|C_i|}\\
&\le \frac{|G|-(r-2)-g_0(2k+s)}{|G|-2k-s-(r-2)-g_0(2k+s)} \quad \text{(by (4))}\\
&\le \frac{|G|-l-g_0(2k+s)}{|G|-2k-s-l-g_0(2k+s)} \quad (\text{since } r=\max\{2,l+1\}\le l+2)\\
&= \frac{|G|-2kg_0-g_0s-l}{|G|-2k-2kg_0-(g_0+1)s-l},
\end{aligned}$$

i.e.,

$$\frac{g_0+1}{g_1}+\varepsilon < \frac{|G|-2kg_0-g_0s-l}{|G|-2k-2kg_0-(g_0+1)s-l}. \tag{5}$$

Claims 2 and 3 imply that s, l are bounded, therefore

$$\lim_{|G|\to\infty}\frac{|G|-2kg_0-g_0s-l}{|G|-2k-2kg_0-(g_0+1)s-l}=1.$$

For a large N, (5) leads to a contradiction when $|G|>N$. This completes the proof. □

Clearly, Theorem 2.2 is a generalization of Theorem 2.1. For connected graphs G, the girth g of G is at least three. Setting $g_0=3$, we obtain the following results regarding the general matching extensions (i.e., stronger properties).

Corollary 2.3 *Let n,k be two positive integers. For any $\varepsilon<1/3$, there exists $N=N(\varepsilon,n,k)$ such that if $b(G)>\frac{4}{3}+\varepsilon$ and the order of G is at least N, then G is (n,k)-extendable.*

Proof: Since $b(G)>\frac{4}{3}+\varepsilon$, by Theorem 2.1, for a sufficiently large $|G|$, G is $(k+2n)$-extendable or $(0,k+2n)$-extendable. By Theorem 1.3, G is (n,k)-extendable. □

With similar discussion as in Corollary 2.3, we can deduce $E(m,n)$-extendability with the same conditions, which is a result proved in Plummer and Saito (2017) but here we gave a much shorter proof.

Corollary 2.4 *Let m,n be two positive integers. For any $\varepsilon<\frac{1}{3}$, there exists $N=N(\varepsilon,m,n)$ such that for every graph G with order at least N, if $b(G)>\frac{4}{3}+\varepsilon$, then G is $E(m,n)$-extendable.*

Proof: Since $b(G)>\frac{4}{3}+\varepsilon$, by Theorem 2.1, for a sufficiently large $|G|$, G is $(m+n)$-extendable. Let $M=\{e_1,e_2,\ldots,e_n\}$ be any n-matching. By Theorem 1.4, $G_1=G-e_1$ is $(m+n-1)$-extendable. Applying Theorem 1.4 recursively, we conclude that $G_n=G-\{e_1,e_2,\ldots,e_n\}$ is m-extendable, that is, G is $E(m,n)$-extendable. □

Remark: Clearly, Corollaries 2.3 and 2.4 can be easily stated in terms of the more general condition $b(G)>\frac{g_0+1}{g_0}+\varepsilon$. However, without the parameter g, the results look more neatly.

3 Toughness and Matching Extendability

It is not hard to construct examples with any given large toughness, but do not have (n,k)-extendability or $E(m,n)$-extendability. Therefore toughness alone is insufficient to guarantee the general matching extension properties. However, with an additional condition in terms of connectivity, it only requires slightly large than 1-toughness to deduce the desired matching extendability.

Theorem 3.1 *Let n be a positive integer, ε be a small positive constant and G be a graph with $t(G) \geq 1+\varepsilon$ and $|V(G)| \equiv n \pmod 2$. If $\kappa(G) > \frac{(n-2)(1+\varepsilon)}{\varepsilon}$, then G is n-factor-critical.*

Proof: Suppose that G is not n-factor-critical. By the definition of n-factor-critical, there exists a subset S of order n such that $G-S$ contains no perfect matchings. By Theorem 1.1, there exists $T \subseteq V(G)-S$ such that

$$q = c_0(G-S-T) \geq |T|+2.$$

Note that $q \geq 2$. So

$$\begin{aligned} 1+\varepsilon \leq t(G) &\leq \frac{|S|+|T|}{|T|+2} \\ &\leq \frac{\kappa}{\kappa-n+2}, \qquad \text{(since } \kappa \leq n+|T|\text{)} \end{aligned}$$

which implies

$$\kappa \leq \frac{(n-2)(1+\varepsilon)}{\varepsilon},$$

a contradiction. This completes the proof. □

Remark: The connectivity condition in the theorem is sharp. Let n, t be two positive integers and ε be a small constant such that $n+t < \frac{(n-2)(1+\varepsilon)}{\varepsilon}$. Let $G_1 = K_{n+t}$, $G_2 = (t+1)K_1$, and $G_3 = K_r$ (r is any positive integer). Define $G = G_1 + (G_2 \cup G_3)$, that is, G is a graph obtained by connecting each vertex in G_1 to each vertex in G_2 and G_3. Let $S = V(G_1)$. Then S is a cut set of G and thus $\kappa \leq n+t \leq \frac{(n-2)(1+\varepsilon)}{\varepsilon}$. It is easy to verify that

$$t(G) = \frac{|S|}{c(G-S)} = \frac{n+t}{t+2} \geq 1+\varepsilon.$$

However, for any set R of n vertices in S, $G-R$ has no perfect matchings. So G is not n-factor-critical.

From Theorem 3.1, it is easy to see the following.

Corollary 3.2 *Let n, k be two positive integers. Let ε be a positive constant and G be a graph with $t(G) \geq 1+\varepsilon$. If $\kappa(G) > \frac{(2k-2)(1+\varepsilon)}{\varepsilon}$, then G is k-extendable.*

With the same arguments as in the proof of Corollary 2.4, Theorem 3.1 implies the following.

Corollary 3.3 *Let m, n be two positive integers. Let ε be a positive constant and G be a graph with $t(G) \geq 1+\varepsilon$. If $\kappa(G) > \frac{(2m+2n-2)(1+\varepsilon)}{\varepsilon}$, then G is $E(m,n)$-extendable.*

Acknowledgements

The authors are grateful to an anonymous referee for his/her useful suggestions.

References

C. P. Chen. Binding number and toughness for matching extension. *Discrete Math.*, 146:303–306, 1995.

V. Chvátal. Tough graphs and hamiltonian circuits. *Discrete Math.*, 5:215–228, 1973.

G. Liu and Q. Yu. Toughness and perfect matchings in graphs. *Ars Combin.*, 48:129–134, 1998.

G. Liu and Q. Yu. Generalization of matching extensions in graphs. *Discrete Math.*, 231:311–320, 2001.

L. Lovász and M. D. Plummer. Matching theory. volume 29 of *Annals of Discrete Mathematics*. North-Holland, 1986.

M. D. Plummer. On n-extendable graphs. *Discrete Math.*, 31:201–210, 1980.

M. D. Plummer. Toughness and matching extension in graphs. *Discrete Math.*, 72:311–320, 1988a.

M. D. Plummer. Matching extension and connectivity in graphs. *Congress. Numer.*, 63:147–160, 1988b.

M. D. Plummer. Extending matching in graphs: a survey. *Discrete Math.*, 127:277–292, 1994.

M. D. Plummer. Extending matching in graphs: an update. *Congress. Numer.*, 116:3–32, 1996.

M. D. Plummer. Recent progress in matching extension, building bridges. volume 19 of *Bolyai Soc. Math. Stud.*, pages 427–454. Springer, Berlin, 2008.

M. D. Plummer and A. Saito. Toughness, binding number and restricted matching extension in a graph. *Discrete Math.*, 340:2665–2672, 2017.

M. Porteous and R. E. L. Aldred. Matching extensions with prescribed and forbidden edges. *Australas. J. Combin.*, 13:163–174, 1996.

A. M. Robertshaw and D. R. Woodall. Binding number conditions for matching extension. *Discrete Math.*, 248:169–179, 2002.

W. T. Tutte. The factorization of linear graphs. *J. Lond. Math. Soc.*, 22:107–111, 1947.

D. R. Woodall. The binding number of a graph and its anderson number. *J. Combin. Theory Ser. B*, 15: 225–255, 1973.

Q. Yu. Characterizations of various matching extensions in graphs. *Australas. J. Combin.*, 7:55–64, 1993.