On the maximum number of minimum total dominating sets in forests

Michael A. Henning$^1$* Elena Mohr$^2$ Dieter Rautenbach$^2$

$^1$ Department of Pure and Applied Mathematics, University of Johannesburg, South Africa
$^2$ Institute of Optimization and Operations Research, Ulm University, Germany


We propose the conjecture that every tree with order $n$ at least 2 and total domination number $\gamma_t$ has at most $(n - \frac{\gamma_t}{2})^{\frac{\gamma_t}{2}}$ minimum total dominating sets. As a relaxation of this conjecture, we show that every forest $F$ with order $n$, no isolated vertex, and total domination number $\gamma_t$ has at most

$$\min \left\{ (8\sqrt{e})^n \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\frac{\gamma_t}{2}}, (1 + \sqrt{2})^{n - \gamma_t}, 1.4865^n \right\}$$

minimum total dominating sets.

Keywords: Tree, forest, total domination, domination

1 Introduction

A set $D$ of vertices of a graph $G$ is a dominating set of $G$ if every vertex of $G$ that is not in $D$ has a neighbor in $D$, and $D$ is a total dominating set of $G$ if every vertex of $G$ has a neighbor in $D$. The minimum cardinalities of a dominating set of $G$ and a total dominating set of $G$ are the well studied $\gamma(G)$ and $\gamma_t(G)$, respectively. A (total) dominating set is minimal if no proper subset is a (total) dominating set. A dominating set of $G$ of cardinality $\gamma(G)$ is a minimum dominating set of $G$, and a total dominating set of $G$ of cardinality $\gamma_t(G)$ is a minimum total dominating set or $\gamma_t$-set of $G$. For a graph $G$, let $\sharp \gamma_t(G)$ be the number of minimum total dominating sets of $G$.

Providing a negative answer to a question of Fricke et al. [6], Bien [2] showed that trees with domination number $\gamma$ can have more than $2^\gamma$ minimum dominating sets. In fact, Bien's example allows to construct forests with domination number $\gamma$ that have up to $2.0598\gamma$ minimum dominating sets. In [1] Alvarado et al. showed that every forest with domination number $\gamma$ has at most $2.4606\gamma$ minimum dominating sets, and they conjectured that every tree with domination number $\gamma$ has $O\left(\frac{\gamma^2}{\ln \gamma}\right)$ minimum dominating sets.

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In the present paper we consider analogous problems for total domination, which turns out to behave quite differently. As shown by the star \(K_{1,n-1}\) which has total domination number 2 but \(n-1\) minimum total dominating sets, the number of minimum total dominating sets of a tree is not bounded in terms of its total domination number alone, but in terms of both the order and the total domination number. In Figure 1 we illustrate what we believe to be the structure of trees \(T\) with given order \(n\) at least 2 and total domination number \(\gamma_t\) that maximize \(\sharp\gamma_t(T)\).

If \(\gamma_t\) is even, say \(\gamma_t = 2k\), then the tree \(T_{\text{even}}\) in the left of Figure 1 satisfies
\[
\sharp\gamma_t(T_{\text{even}}) = \prod_{i=1}^{k} (\ell_i + 1) \leq \left(\frac{n - \frac{\gamma_t}{2}}{2}\right)^{\frac{\gamma_t}{2}},
\]
where we use that the geometric mean is at most the arithmetic mean. Similarly, if \(\gamma_t\) is odd, say \(\gamma_t = 2k + 1\), then the tree \(T_{\text{odd}}\) in the right of Figure 1 satisfies
\[
\sharp\gamma_t(T_{\text{odd}}) = \sum_{i=1}^{k} \left(\prod_{j=1}^{i-1} \ell_j \prod_{j=i+1}^{k} (\ell_j + 1) \right) \leq k \left(\frac{n - k - 4}{k - 1}\right)^{k-1} = \left(\frac{\gamma_t - 1}{2}\right) \left(\frac{n - (\gamma_t + 7)}{2}\right)^{\frac{\gamma_t - 3}{2}}.
\]
In view of these estimates, we pose the following.

**Conjecture 1.** If a tree \(T\) has order \(n\) at least 2 and total domination number \(\gamma_t\), then
\[
\sharp\gamma_t(T) \leq \left(\frac{n - \frac{\gamma_t}{2}}{2}\right)^{\frac{\gamma_t}{2}}.
\]

As our first result, we show that Conjecture 1 holds up to a constant factor for bounded values of \(\gamma_t\). More precisely, we show the following.

**Theorem 2.** If a forest \(F\) has order \(n\), no isolated vertex, and total domination number \(\gamma_t\), then
\[
\sharp\gamma_t(F) \leq \left(8\sqrt{e}\right)^{\gamma_t} \left(\frac{n - \frac{\gamma_t}{2}}{2}\right)^{\frac{\gamma_t}{2}}.
\]
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The well known estimate $1 + x \leq e^x$ implies

$$
\left( \frac{n - \gamma t}{\gamma t^2} \right)^{\frac{\gamma t}{2}} = \left( 1 + \frac{n - \gamma t}{\gamma t^2} \right)^{\frac{\gamma t}{2}} \leq e^{n-\gamma t}.
$$

In the following theorem we can show an upper bound that is a little better. But since $1 + x << e^x$ for large $x$, the estimate is not good for fixed $\gamma t$ and large values of $n$. In this case Theorems 2 and 4 give better upper bounds.

**Theorem 3.** If a forest $F$ has order $n$, no isolated vertex, and total domination number $\gamma t$, then

$$
\#\gamma t(F) \leq \left( 1 + \sqrt{2} \right)^{n-\gamma t},
$$

with equality if and only if every component of $F$ is $K_2$.

Note that Theorem 3 is only tight for $\gamma t = n$, which corresponds to the fact that $1 + x = e^x$ only for $x = 0$. For $n$ divisible by 5, the disjoint union of $\frac{n}{5}$ stars of order 5 yields a forest $F$ with $\#\gamma t(F) = 4 \frac{n}{5} \approx 1.3195^n$. Our third result comes close to that value.

**Theorem 4.** If a forest $F$ has order $n$ and no isolated vertex, then $\#\gamma t(F) \leq 1.4865^n$.

Before we proceed to the proofs of our results, we mention some related research. Connolly et al. [4] gave bounds on the maximum number of minimum dominating sets for general graphs. The maximum number of minimal dominating sets was studied by Fomin et al. [5], and the maximum number of general dominating sets by Wagner [12] and Skupień [11], and by Bród and Skupień [3] for trees. Krzywokowski and Wagner [9] study the maximum number of total dominating sets for general graphs and trees. For similar research concerning independent sets we refer to [10, 13, 14].

The next section contains the proofs of our results. We use standard graph theoretical terminology and notation. An *endvertex* is a vertex of degree at most 1, and a *support vertex* is a vertex that is adjacent to an endvertex.

## 2 Proofs

For the proof of Theorem 2, we need the following lemma.

**Lemma 5.** If $T$ is a tree of order $n$ at least 2, and $B$ is a set of vertices of $T$ such that

(i) $|B \cap \{ u, v \}| \leq 1$ for every $uv \in E(T)$, and

(ii) $|B \cap N_T(u)| \leq 1$ for every $u \in V(T)$,

then $|B| \leq \frac{n}{2}$.

**Proof:** The proof is by induction on $n$. If $T$ is a star, then (i) and (ii) imply $|B| \leq 1 \leq \frac{n}{2}$. Now, let $T$ be a tree that is not a star; in particular, $n \geq 4$. Let $uvw \ldots$ be a longest path in $T$. By (i) and (ii), we have $|B \cap (N_T[u] \setminus \{w\})| \leq 1$. By induction applied to the tree $T' = T - (N_T[v] \setminus \{w\})$ and the set $B' = B \cap V(T')$, we obtain $|B| \leq |B'| + |B \cap (N_T[v] \setminus \{w\})| \leq \frac{n(T')}{2} + 1 \leq \frac{n}{2}$. \hfill \Box

We are now in a position to present the proof of Theorem 2.
**Proof of Theorem 2:** Let $F$ be a forest of order $n$ and total domination number $\gamma_t$ such that $z\gamma_t(F)$ is as large as possible. Let $D$ be a $\gamma_t$-set of $F$. Let $F'$ arise by removing from $F$ all endvertices of $F$ that do not belong to $D$. For every $u \in D$, let $L(u) = N_F(u) \setminus N_{F'}(u)$ and $\ell(u) = |L(u)|$, that is, $L(u)$ is the set of neighbors of $u$ in $D$ that are endvertices of $F$ that do not belong to $D$. We call a vertex $u$ in $D$ big if $\ell(u) \geq 2$, and we assume that – subject to the above conditions – the forest $F$ and the set $D$ are chosen such that the number $k$ of big vertices is as small as possible.

**Claim 1.** No two big vertices are adjacent.

**Proof of Claim 1:** Suppose, for a contradiction, that $u$ and $v$ are adjacent big vertices. Let $L'$ be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$, that is, we shift $\ell(u) - 1$ neighbors of $u$ in $L(u)$ to $v$. Clearly, the vertices $u$ and $v$ both belong to every $\gamma_t$-set of $F$ and also to every $\gamma_t$-set of $F'$. This easily implies that a set of vertices of $F$ is a $\gamma_t$-set of $F$ if and only if it is a $\gamma_t$-set of $F'$. It follows that $D$ is a $\gamma_t$-set of $F'$ and that $z\gamma_t(F) = z\gamma_t(F')$. Since $F'$ and $D$ lead to less than $k$ big vertices, we obtain a contradiction to the choice of $F$ and $D$. \qed

**Claim 2.** No two big vertices have a common neighbor in $D$.

**Proof of Claim 2:** Suppose, for a contradiction, that $u$ and $w$ are big vertices with a common neighbor $v$ in $D$. Let

- $z_u$ be the number of $\gamma_t$-sets of $F$ that contain a vertex from $L(u)$,
- $z_w$ be the number of $\gamma_t$-sets of $F$ that contain a vertex from $L(w)$, and
- $z_{u,w}$ be the number of $\gamma_t$-sets of $F$ that contain no vertex from $L(u) \cup L(w)$.

In view of $v$, no $\gamma_t$-set of $F$ contains a vertex from both sets $L(u)$ and $L(w)$, which implies

$$z\gamma_t(F) = z_u + z_w + z_{u,w}.$$ 

Note that $\frac{z_u}{\ell(u)}$ is the number of subsets of $V(F) \setminus L(u)$ that can be extended to a $\gamma_t$-set of $F$ by adding one vertex from $L(u)$. By symmetry, we may assume that $\frac{z_u}{\ell(u)} \leq \frac{z_w}{\ell(w)}$. Again, let $L'$ be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$. Similarly as before, the vertices $u$ and $w$ both belong to every $\gamma_t$-set of $F$ and also to every $\gamma_t$-set of $F'$. It follows that $D$ is a $\gamma_t$-set of $F'$, and that

$$z\gamma_t(F') = \frac{z_u}{\ell(u)} + \frac{z_w}{\ell(w)}(\ell(u) + \ell(w) - 1) + z_{u,w} \geq z_u + z_w + z_{u,w} = z\gamma_t(F).$$

Since $F'$ and $D$ lead to less than $k$ big vertices, this contradicts the choice of $F$ and $D$. \qed

**Claim 3.** $k \leq \frac{\gamma_t}{2}$.

**Proof of Claim 3:** This follows immediately by applying Lemma 5 to each component of $F[D]$, choosing $B$ as the set of big vertices in that component. \qed
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Let \( n' = n(F') \), let \( V'_1 \) be the set of endvertices of \( F' \), let \( n'_1 = |V'_1| \), and let \( m \) be the number of edges of \( F' \) between \( D \) and \( V(F') \setminus D \). Since the vertices in \( V'_1 \) are either endvertices of \( F \) that belong to \( D \) or are adjacent to an endvertex of \( F \), we obtain that \( V'_1 \subseteq D \). Since \( D \) is a total dominating set, we obtain

\[
n' - \gamma_t = |V(F') \setminus D| \leq m \leq \sum_{u \in D} (d_{F'}(u) - 1). \tag{1}
\]

Since \( F' \) is a forest with, say, \( \kappa \) components,

\[
n'_1 = 2\kappa + \sum_{u \in V(F') : d_{F'}(u) \geq 2} (d_{F'}(u) - 2)
\geq \sum_{u \in D : d_{F'}(u) \geq 2} (d_{F'}(u) - 2)
= \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u) - 2(\gamma_t - n'_1),
\]

which implies

\[
2\gamma_t - n'_1 \geq \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u). \tag{2}
\]

Now, we obtain

\[
n' \overset{(1)}{\leq} \sum_{u \in D} d_{F'}(u) = \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u) + n'_1 \overset{(2)}{\leq} 2\gamma_t. \tag{3}
\]

Let \( u_1, \ldots, u_k \) be the big vertices. By (3), the forest \( F'' = F - \bigcup_{i=1}^k L(u_i) \) has order at most \( 3\gamma_t \). Let \( D'' \) be a set of vertices of \( F'' \) that is a subset of some \( \gamma_t \)-set \( D \) of \( F \). For every \( i \in \{1, \ldots, k\} \), if \( u_i \) has a neighbor in \( D'' \), then \( D \) contains no vertex from \( L(u_i) \), otherwise, the set \( D \) contains exactly one vertex from \( L(u_i) \). This implies that each of the \( 2^{n(F'')} \) subsets of \( V(F'') \) can be extended to a \( \gamma_t \)-set of \( F \) in at most \( \prod_{i=1}^k \ell(u_i) \) many ways.

Since

(i) \( n(F'') \leq 3\gamma_t \),

(ii) the geometric mean is less or equal the arithmetic mean,

(iii) \( \sum_{i=1}^k \ell(u_i) = n - n(F'') \leq n - \gamma_t \leq n - \frac{n}{3} \),

(iv) \( \left(1 + \frac{\gamma_t - k}{k}\right)^k \leq e^{\frac{\gamma_t}{2} - k} \leq e^{\frac{\gamma_t}{2}} \), and

(v) \( \frac{n}{\gamma_t} \leq 1 \),
we obtain
\[ \sharp_{\gamma_t}(F) \leq 2^{\gamma_t(F')} \prod_{i=1}^{k} \ell(u_i) \]
\[ \leq 2^{3\gamma_t} \prod_{i=1}^{k} \ell(u_i) \]
\[ \leq 2^{3\gamma_t} \left( \frac{1}{k} \sum_{i=1}^{k} \ell(u_i) \right)^k \]
\[ \leq 2^{3\gamma_t} \left( \frac{n - \frac{2\gamma_t}{k}}{k} \right)^k \]
\[ \leq 2^{3\gamma_t} \left( 1 + \frac{\gamma_t}{k} - \frac{k}{k} \left( \frac{n - \frac{2\gamma_t}{k}}{n - \frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} \left( \frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} \right) \]
\[ \leq 2^{3\gamma_t} e^{\frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}}} \left( \frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} \]
\[ \leq 2^{3\gamma_t} e^{\frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}}} \left( \frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} \]
\[ \leq \left( 8\sqrt{e} \right)^{\gamma_t} \left( \frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} \]
which completes the proof.

There is clearly some room for lowering $8\sqrt{e}$ to a smaller constant. Since the dependence on $\gamma_t$ would still be exponential, we did not exploit this for the sake of simplicity. It would be interesting to see whether the bound can be improved to
\[ \left( 1 + o \left( \frac{n}{\gamma_t} \right) \right) \left( \frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} \]

Note that Theorem 2 implies
\[ \sharp_{\gamma_t}(T) \leq \left( \frac{n - \frac{2\gamma_t}{k}}{\frac{2\gamma_t}{k}} \right) ^{\frac{k}{k}} + o(\frac{n}{\gamma_t}) \]

We proceed to our next proof.

**Proof of Theorem 3:** We proceed by induction on $n$. If $n = 2$, then $F = K_2$, $\gamma_t = 2$, and $\sharp_{\gamma_t}(F) = 1 = (1 + \sqrt{2})^0 = (1 + \sqrt{2})^{n - \gamma_t}$. Now, let $n \geq 3$.

**Claim 1.** If $F$ contains a component $T$ that is a star, then $\sharp_{\gamma_t}(F) \leq (1 + \sqrt{2})^{n - \gamma_t}$, with strict inequality if $T$ has order at least 3.

**Proof of Claim 1:** Suppose that $F$ contains a component $T$ that is a star. Thus, $T = K_{1,t}$ for some $t \geq 1$. The forest $F' = F - V(T)$ has order $n' = n - t - 1$, no isolated vertex, and total domination number
\[
gamma'_t = \gamma_t - 2. \text{ By induction, we obtain}
\]
\[
2 \gamma_t(F) = \gamma_t(F') \leq t(1 + \sqrt{2})^{n' - \gamma'_t} = t(1 + \sqrt{2})^{n - t - 1 - (\gamma_t - 2)}
\]
\[
= (1 + \sqrt{2})^{n - \gamma_t} \cdot (t(1 + \sqrt{2})^{1 - t}) \leq (1 + \sqrt{2})^{n - \gamma_t},
\]
where we use \((1 + \sqrt{2})^{1 - t} \leq 1\) for \(t = 1\) and \(t \geq 2\). Furthermore, if \(t \geq 2\), then \((1 + \sqrt{2})^{1 - t} < 1\), in which case \(\gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t} \). 

**Claim 2.** If \(F\) contains a component \(T\) of diameter 3, then \(2 \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t}\).

**Proof of Claim 2:** Suppose that \(F\) contains a component \(T\) of diameter 3. Note that \(T\) has a unique minimum total dominating set. The forest \(F' = F - V(T)\) has order \(n' \leq n - 4\), no isolated vertex, and total domination number \(\gamma'_t = \gamma_t - 2\). By induction, we obtain
\[
2 \gamma_t(F) = 2 \gamma_t(F') \leq (1 + \sqrt{2})^{n' - \gamma'_t} \leq (1 + \sqrt{2})^{n - \gamma_t - 2} < (1 + \sqrt{2})^{n - \gamma_t}.
\]

By Claim 1 and Claim 2, we may assume that there is a component of \(F\) that has diameter at least 4, for otherwise the desired result follows. Let \(T\) be such a component of \(F\). Let \(uvwxy \ldots r\) be a longest path in \(T\), and consider \(T\) as rooted in \(r\). For a vertex \(z\) of \(T\), let \(V_z\) be the set that contains \(z\) and all its descendants.

**Claim 3.** If \(d_F(w) \geq 3\), then \(2 \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t}\).

**Proof of Claim 3:** Suppose that \(d_F(w) \geq 3\), which implies that \(w\) belongs to every \(\gamma_t\)-set of \(F\), because either \(w\) is a support vertex or \(w\) is the only neighbor of two support vertices, that is no leaf. Let \(v'\) be a child of \(w\) distinct from \(v\). Let \(F' = F - V_{v'}\). If \(v'\) is an endvertex, then \(F'\) has order \(n' = n - 1\), no isolated vertex, and total domination number \(\gamma'_t = \gamma_t - 1\). By induction, we obtain
\[
2 \gamma_t(F) \leq 2 \gamma_t(F') \leq (1 + \sqrt{2})^{n' - \gamma'_t} = (1 + \sqrt{2})^{n - \gamma_t - 1} < (1 + \sqrt{2})^{n - \gamma_t}.
\]

If \(v'\) is not an endvertex, then \(F'\) has order \(n' \leq n - 2\), no isolated vertex, and total domination number \(\gamma'_t = \gamma_t - 1\). Note that if \(T\) is a minimum total dominating set of \(F\), \(T - \{v\}\) is a total dominating set of \(F'\), since \(v'\) is a support vertex and \(v\) and \(w\) are part of every minimum total dominating set of \(F\). By induction, we obtain
\[
2 \gamma_t(F) \leq 2 \gamma_t(F') \leq (1 + \sqrt{2})^{n' - \gamma'_t} \leq (1 + \sqrt{2})^{n - \gamma_t - 1} < (1 + \sqrt{2})^{n - \gamma_t}.
\]

In both cases, \(2 \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t}\).

By Claim 3, we may assume that \(d_F(w) = 2\), for otherwise the desired result holds.

**Claim 4.** If \(d_F(v) \geq 3\), then \(2 \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t}\).

**Proof of Claim 4:** Suppose that \(\ell = d_F(v) - 1 \geq 2\). Let \(F' = F - V_w, F'' = F - (N_F(v) \setminus \{v\})\), and \(F''' = F - (V_w \cup \{x\})\). See Figure 2 for an illustration.
The forest $F$  

The forest $F'$  

The forest $F''$  

The forest $F'''$

\textbf{Fig. 2:} The important details of the forests $F$, $F'$, $F''$ and $F'''$.

- There are at most $\ell \cdot \sharp\gamma_t(F')$ many $\gamma_t$-sets of $F$ that contain $v$ and a child of $v$ but do not contain $w$. Furthermore, if such a $\gamma_t$-set exists, then $F'$ has order $n' = n - \ell - 2$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$.

- There are at most $\sharp\gamma_t(F'')$ many $\gamma_t$-sets of $F$ that contain $v$, $w$, and $x$. Furthermore, if such a $\gamma_t$-set exists, then $F''$ has order $n'' = n - \ell$, no isolated vertex, and total domination number $\gamma_t'' = \gamma_t - 1$.

- There are at most $\sharp\gamma_t(F''')$ many $\gamma_t$-sets of $F$ that contain both $v$ and $w$ but do not contain $x$. Furthermore, if such a $\gamma_t$-set exists, then $F'''$ has order $n''' = n - \ell - 3$, no isolated vertex, and total domination number $\gamma_t''' = \gamma_t - 2$.

Since all $\gamma_t$-sets of $F$ are of one of the three considered types, we obtain, by induction,

\[
\sharp\gamma_t(F) \leq \ell \cdot \sharp\gamma_t(F') + \sharp\gamma_t(F'') + \sharp\gamma_t(F''') \\
\leq \ell(1 + \sqrt{2})^{n-\ell-2-(\gamma_t-2)} + (1 + \sqrt{2})^{n-\ell-(\gamma_t-1)} + (1 + \sqrt{2})^{n-\ell-3-(\gamma_t-2)} \\
= (1 + \sqrt{2})^{n-\gamma_t}(1 + \sqrt{2})^{-\ell-1}(\ell(1 + \sqrt{2}) + (1 + \sqrt{2})^2 + 1) \\
< (1 + \sqrt{2})^{n-\gamma_t},
\]

where we use $\ell(1 + \sqrt{2}) + (1 + \sqrt{2})^2 + 1 < (1 + \sqrt{2})^{\ell+1}$ for all $\ell \geq 2$. \qed

By Claim 4, we may assume that $d_F(v) = 2$, for otherwise the desired result holds.

\textbf{Claim 5.} If $x$ is a support vertex, then $\sharp\gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}$.

\textbf{Proof of Claim 5:} Suppose that $x$ is a support vertex, which implies that $v$ and $x$ belong to every $\gamma_t$-set of $F$. Let $F' = F - V_w$ and $F'' = F - (N_F[v] \cup N_F[x])$. 

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- There are at most \( \gamma_{t}(F') \) many \( \gamma_{t} \)-sets of \( F \) that contain \( u \) but do not contain \( w \). Furthermore, if such a \( \gamma_{t} \)-set exists, then \( F' \) has order \( n' = n - 3 \), no isolated vertex, and total domination number \( \gamma'_{t} = \gamma_{t} - 2 \).
- There are at most \( \gamma_{t}(F') \) many \( \gamma_{t} \)-sets of \( F \) that contain \( w \) and at least one other neighbour of \( x \). Furthermore, if such a \( \gamma_{t} \)-set exists, then \( F' \) has order \( n' = n - 3 \), no isolated vertex, and total domination number \( \gamma'_{t} = \gamma_{t} - 2 \).
- There are at most \( \gamma_{t}(F'') \) many \( \gamma_{t} \)-sets of \( F \) that contain \( w \) and no other neighbour of \( x \). Furthermore, if such a \( \gamma_{t} \)-set exists, then \( F'' \) has order \( n'' = n - 5 \), no isolated vertex, and total domination number \( \gamma''_{t} = \gamma_{t} - 3 \).

Since all \( \gamma_{t} \)-sets of \( F \) are of one of the three considered types, we obtain, by induction,

\[
\gamma_{t}(F) \leq 2\gamma_{t}(F') + \gamma_{t}(F'') < 2(1 + \sqrt{2})^{n-3-(\gamma-2)} + (1 + \sqrt{2})^{n-5-(\gamma-3)} = (1 + \sqrt{2})^{n-\gamma-2} + (1 + \sqrt{2})^{n-\gamma-3},
\]

where we use \( 2(1 + \sqrt{2}) + 1 = (1 + \sqrt{2})^2 \). Note that in \( F' \) there is a component that contains a path of length two, in particular not every component of \( F' \) is a \( K_2 \). \( \square \)

By Claim 5, we may assume that \( x \) is not a support vertex, for otherwise the desired result holds.

**Claim 6.** If \( x \) has a child that is a support vertex, then \( \gamma_{t}(F) < (1 + \sqrt{2})^{n-\gamma} \).

**Proof of Claim 6:** Suppose that \( x \) has a child \( w' \) that is a support vertex. Clearly, the vertex \( w' \) is distinct from \( w \) and belongs to every \( \gamma_{t} \)-set of \( F \). The forest \( F' = F - V_w \) has order \( n' = n - 3 \), no isolated vertex, and total domination number \( \gamma'_{t} = \gamma_{t} - 2 \). By induction, we obtain

\[
\gamma_{t}(F) = 2\gamma_{t}(F') \leq 2(1 + \sqrt{2})^{n-3-(\gamma-2)} = (1 + \sqrt{2})^{n-\gamma-2}2(1 + \sqrt{2})^{-1} < (1 + \sqrt{2})^{n-\gamma},
\]

where we use \( 2 < (1 + \sqrt{2}) \). \( \square \)

By Claim 6, we may assume that no child of \( x \) is a support vertex, for otherwise the desired result holds. Together with Claims 3 and 4, we may assume that the subforest of \( F \) induced by \( V_x \) arises from a star \( K_{1,q} \) for some \( q \geq 1 \) by subdividing every edge twice. Let \( F' = F - V_x \), \( F'' = F - (V_x \cup \{y\}) \), and \( F''' = F - (V_x \cup N_F[y]) \).

- There are at most \( 2^{q}\gamma_{t}(F') \) many \( \gamma_{t} \)-sets of \( F \) that do not contain \( x \). Furthermore, if such a \( \gamma_{t} \)-set exists, then \( F' \) has order \( n' = n - 3q - 1 \), no isolated vertex, and total domination number \( \gamma'_{t} = \gamma_{t} - 2q \).
- There are at most \( (2^q - 1)\gamma_{t}(F'') \) many \( \gamma_{t} \)-sets of \( F \) that contain \( x \) but do not contain \( y \). Furthermore, if such a \( \gamma_{t} \)-set exists, then \( F'' \) has order \( n'' = n - 3q - 2 \), no isolated vertex, and total domination number \( \gamma''_{t} = \gamma_{t} - 2q - 1 \).
- There are at most \( 2^{q}\gamma_{t}(F''') \) many \( \gamma_{t} \)-sets of \( F \) that contain both \( x \) and \( y \). Furthermore, if such a \( \gamma_{t} \)-set exists, then \( F''' \) has order \( n''' \leq n - 3q - 3 \), no isolated vertex, and total domination number \( \gamma'''_{t} = \gamma_{t} - 2q - 2 \).
Since all \( \gamma_t \)-sets of \( F \) are of one of the three considered types, we obtain, by induction,

\[
\sharp \gamma_t(F) \leq 2^q \sharp \gamma_t(F') + (2^q - 1) \sharp \gamma_t(F'') + 2^q \sharp \gamma_t(F''')
\]

\[
\leq 2^q(1 + \sqrt{2})^{n-3q-1-\gamma_t} + (2^q - 1)(1 + \sqrt{2})^{n-3q-2-\gamma_t} + 2^q(1 + \sqrt{2})^{n-3q-3-\gamma_t}
\]

\[
= (1 + \sqrt{2})^{n-\gamma_t} (1 + \sqrt{2})^{-q-1}(2^q + 2^q - 1 + 2^q)
\]

\[
< (1 + \sqrt{2})^{n-\gamma_t},
\]

where we use \( 3 \cdot 2^q - 1 < (1 + \sqrt{2})^{q+1} \) for all \( q \geq 1 \). This completes the proof of Theorem 3.

We proceed to the proof of Theorem 4, which uses exactly the same approach as Theorem 3.

**Proof of Theorem 4:** By induction on \( n \), we show that \( \sharp \gamma_t(F) \leq \beta^n \), where \( \beta \) is the unique positive real solution of the equation \( 2\beta + \beta^3 + 1 = \beta^5 \), that is, \( \beta \approx 1.4865 \). If \( n = 2 \), then \( F = K_2 \) and \( \sharp \gamma_t(F) = 1 < \beta^2 \). Now, let \( n \geq 3 \).

**Claim 1.** If \( F \) contains a component \( T \) that is a star, then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 1:** Suppose that \( F \) contains a component \( T \) that is a star. Thus, \( T = K_{1,t} \) for some \( t \geq 1 \). The forest \( F' = F - V(T) \) has order \( n' = n - t - 1 \) and no isolated vertex. By induction, we obtain \( \sharp \gamma_t(F) = t \cdot \sharp \gamma_t(F') \leq t \beta^{n-1} \leq \beta^n \), where we use \( t \leq \beta^{t+1} \).

**Claim 2.** If \( F \) contains a component \( T \) of diameter 3, then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 2:** Suppose that \( F \) contains a component \( T \) of diameter 3. The forest \( F' = F - V(T) \) has order \( n' \leq n - 4 \) and no isolated vertex. By induction, we obtain \( \sharp \gamma_t(F) = \sharp \gamma_t(F') \leq \beta^{n'} \leq \beta^n \).

By Claim 1 and Claim 2, we may assume that every component of \( F \) has diameter at least 4, for otherwise the desired result follows. Let \( T \) be an arbitrary component of \( F \). Let \( uvwx \ldots r \) be a longest path in \( T \), and consider \( T \) as rooted in \( r \). For a vertex \( z \) of \( T \), let \( V_z \) be the set that contains \( z \) and all its descendants.

**Claim 3.** If \( d_F(w) \geq 3 \), then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 3:** Suppose that \( d_F(w) \geq 3 \), which implies that \( w \) belongs to every \( \gamma_t \)-set of \( F \). Let \( v' \) be a child of \( w \) distinct from \( v \). The forest \( F' = F - V_v \) has order \( n' < n \) and no isolated vertex. Since \( \sharp \gamma_t(F) \leq \sharp \gamma_t(F') \), we obtain, by induction, \( \sharp \gamma_t(F) \leq \beta^{n'} \leq \beta^n \).

By Claim 3, we may assume that \( d_F(w) = 2 \), for otherwise the desired result holds.

**Claim 4.** If \( d_F(v) \geq 3 \), then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 4:** Suppose that \( \ell = d_F(v) - 1 \geq 2 \). Arguing exactly as in the proof of Claim 4 in the proof of Theorem 3 using the forests \( F', F'' \), and \( F''' \), we obtain, by induction,

\[
\sharp \gamma_t(F) \leq \ell \cdot \sharp \gamma_t(F') + \sharp \gamma_t(F'') + \sharp \gamma_t(F''') 
\]

\[
\leq \ell \beta^{n-\ell-2} + \beta^n t + \beta^n t - 3
\]

\[
= \beta^n \beta^{-\ell-3} (\ell \beta + \beta^3 + 1)
\]

\[
\leq \beta^n,
\]
On the maximum number of minimum total dominating sets in forests

where we use \( \ell \beta + \beta^3 + 1 \leq \beta^{\ell+3} \) for all \( \ell \geq 2 \); in fact, this inequality is the reason for the specific choice of \( \beta \). \( \square \)

By Claim 4, we may assume that \( d_F(v) = 2 \), for otherwise the desired result holds.

**Claim 5.** If \( x \) is a support vertex, then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 5:** Suppose that \( x \) is a support vertex. Arguing exactly as in the proof of Claim 5 in the proof of Theorem 3 using the forests \( F' \) and \( F'' \), we obtain, by induction,

\[
\sharp \gamma_t(F) \leq 2\sharp \gamma_t(F') + \sharp \gamma_t(F'') \leq 2\beta^{n-3} + \beta^{n-5} = \beta^n \beta^{-5}(2\beta^2 + 1) \leq \beta^n,
\]

where we use \( 2\beta^2 + 1 \leq \beta^5 \). \( \square \)

By Claim 5, we may assume that \( x \) is not a support vertex, for otherwise the desired result holds.

**Claim 6.** If \( x \) has a child that is a support vertex, then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 6:** Suppose that \( x \) has a child \( w' \) that is a support vertex. Arguing exactly as in the proof of Claim 6 in the proof of Theorem 3 using the forest \( F' \), we obtain, by induction,

\[
\sharp \gamma_t(F) = 2\sharp \gamma_t(F') \leq 2\beta^{n-3} = \beta^n 2\beta^{-3} \leq \beta^n,
\]

where we use \( 2 < \beta^3 \). \( \square \)

Now, arguing exactly as at the end of the proof of Theorem 3 using the forests \( F' \), \( F'' \), and \( F''' \), we obtain, by induction,

\[
\sharp \gamma_t(F) \leq 2^q \sharp \gamma_t(F') + (2^q - 1)\sharp \gamma_t(F'') + 2^q \sharp \gamma_t(F''')
\]
\[
\leq 2^q \beta^{n-3q-1} + (2^q - 1)\beta^{n-3q-2} + 2^q \beta^{n-3q-3}
\]
\[
= \beta^n \beta^{-3q-3}(2^q \beta^2 + (2^q - 1)\beta + 2^q)
\]
\[
\leq \beta^n \beta^{-3q-3}2^q (\beta^2 + \beta + 1)
\]
\[
\leq \beta^n,
\]

where we use \( 2^q (\beta^2 + \beta + 1) \leq \beta^{3q+3} \) for all \( q \geq 1 \). \( \square \)

**References**


[2] A. Bien, Properties of gamma graphs of trees, presentation at the 17th Workshop on Graph Theory Colourings, Independence and Domination (CID 2017), Piechowice, Poland.


