On the maximum number of minimum total dominating sets in forests

Michael A. Henning\textsuperscript{1*} Elena Mohr\textsuperscript{2} Dieter Rautenbach\textsuperscript{2}

\textsuperscript{1} Department of Pure and Applied Mathematics, University of Johannesburg, South Africa
\textsuperscript{2} Institute of Optimization and Operations Research, Ulm University, Germany


We propose the conjecture that every tree with order $n$ at least 2 and total domination number $\gamma_t$ has at most
\[
\left(\frac{n - \frac{2}{\sqrt{\ln n}}}{\gamma_t}\right)^{\frac{2}{\gamma_t}}
\]
minimum total dominating sets. As a relaxation of this conjecture, we show that every forest $F$ with order $n$, no isolated vertex, and total domination number $\gamma_t$ has at most
\[
\min\left\{\left(8\sqrt{\pi}\right)^n \left(\frac{n - \frac{2}{\sqrt{\ln n}}}{\gamma_t}\right)^{\frac{2}{\gamma_t}} ; (1 + \sqrt{2})^{n - \gamma_t}, 1.4865^n \right\}
\]
minimum total dominating sets.

**Keywords:** Tree, forest, total domination, domination

\section{Introduction}

A set $D$ of vertices of a graph $G$ is a dominating set of $G$ if every vertex of $G$ that is not in $D$ has a neighbor in $D$, and $D$ is a total dominating set of $G$ if every vertex of $G$ has a neighbor in $D$. The minimum cardinalities of a dominating set of $G$ and a total dominating set of $G$ are the well studied \cite{7, 8} domination number $\gamma(G)$ of $G$ and the total domination number $\gamma_t(G)$ of $G$, respectively. A (total) dominating set is minimal if no proper subset is a (total) dominating set. A dominating set of $G$ of cardinality $\gamma(G)$ is a minimum dominating set of $G$, and a total dominating set of $G$ of cardinality $\gamma_t(G)$ is a minimum total dominating set or $\gamma_t$-set of $G$. For a graph $G$, let $\sharp \gamma_t(G)$ be the number of minimum total dominating sets of $G$.

Providing a negative answer to a question of Fricke et al. \cite{6}, Bień \cite{2} showed that trees with domination number $\gamma$ can have more than $2^\gamma$ minimum dominating sets. In fact, Bień’s example allows to construct forests with domination number $\gamma$ that have up to $2.0598^\gamma$ minimum dominating sets. In \cite{1} Alvarado et al. showed that every forest with domination number $\gamma$ has at most $2.4606^\gamma$ minimum dominating sets, and they conjectured that every tree with domination number $\gamma$ has $O\left(\frac{\gamma^2}{\ln \gamma}\right)$ minimum dominating sets.
In the present paper we consider analogous problems for total domination, which turns out to behave quite differently. As shown by the star $K_{1,n-1}$ which has total domination number 2 but $n-1$ minimum total dominating sets, the number of minimum total dominating sets of a tree is not bounded in terms of its total domination number alone, but in terms of both the order and the total domination number. In Figure 1 we illustrate what we believe to be the structure of trees $T$ with given order $n$ at least 2 and total domination number $\gamma_t$ that maximize $\sharp\gamma_t(T)$.

If $\gamma_t$ is even, say $\gamma_t = 2k$, then the tree $T_{\text{even}}$ in the left of Figure 1 satisfies

$$\sharp\gamma_t(T_{\text{even}}) = \prod_{i=1}^{k} \left( \ell_i + 1 \right) \leq \left( \frac{n - \frac{\gamma_t}{2}}{k} \right)^2.$$

where we use that the geometric mean is at most the arithmetic mean. Similarly, if $\gamma_t$ is odd, say $\gamma_t = 2k + 1$, then the tree $T_{\text{odd}}$ in the right of Figure 1 satisfies

$$\sharp\gamma_t(T_{\text{odd}}) = \sum_{i=1}^{k} \left( \prod_{j=1}^{i-1} \ell_j \prod_{j=i+1}^{k} \left( \ell_j + 1 \right) \right) \leq k \left( \frac{n - k - 4}{k - 1} \right)^{k-1} = \left( \frac{\gamma_t - 1}{2} \right) \left( \frac{n - \left( \frac{\gamma_t + 7}{2} \right)}{\gamma_t - 3} \right)^{2k - 3/2}.$$

In view of these estimates, we pose the following.

**Conjecture 1.** If a tree $T$ has order $n$ at least 2 and total domination number $\gamma_t$, then

$$\sharp\gamma_t(T) \leq \left( \frac{n - \frac{\gamma_t}{2}}{\gamma_t} \right)^{2k}.$$

As our first result, we show that Conjecture 1 holds up to a constant factor for bounded values of $\gamma_t$. More precisely, we show the following.

**Theorem 2.** If a forest $F$ has order $n$, no isolated vertex, and total domination number $\gamma_t$, then

$$\sharp\gamma_t(F) \leq \left( 8 \sqrt{e} \right)^{\gamma_t} \left( \frac{n - \frac{\gamma_t}{2}}{\gamma_t} \right)^{2k}.$$
The well known estimate $1 + x \leq e^x$ implies
\[
\left(\frac{n - \frac{n - \gamma_t}{2}}{\frac{n - \gamma_t}{2}}\right)^{\frac{\gamma_t}{2}} = \left(1 + \frac{n - \gamma_t}{\frac{n - \gamma_t}{2}}\right)^{\frac{\gamma_t}{2}} \leq e^{n - \gamma_t}.
\]

In the following theorem we can show an upper bound that is a little better. But since $1 + x << e^x$ for large $x$, the estimate is not good for fixed $\gamma_t$ and large values of $n$. In this case Theorems 2 and 4 give better upper bounds.

**Theorem 3.** If a forest $F$ has order $n$, no isolated vertex, and total domination number $\gamma_t$, then
\[
\sharp \gamma_t(F) \leq (1 + \sqrt{2})^{n - \gamma_t},
\]
with equality if and only if every component of $F$ is $K_2$.

Note that Theorem 3 is only tight for $\gamma_t = n$, which corresponds to the fact that $1 + x = e^x$ only for $x = 0$. For $n$ divisible by 5, the disjoint union of $\frac{n}{5}$ stars of order 5 yields a forest $F$ with $\sharp \gamma_t(F) = 4^{\frac{n}{5}} \approx 1.3195^n$. Our third result comes close to that value.

**Theorem 4.** If a forest $F$ has order $n$ and no isolated vertex, then $\sharp \gamma_t(F) \leq 1.4865^n$.

Before we proceed to the proofs of our results, we mention some related research. Connolly et al. [4] gave bounds on the maximum number of minimum dominating sets for general graphs. The maximum number of minimal dominating sets was studied by Fomin et al. [5], and the maximum number of general dominating sets by Wagner [12] and Skupień [11], and by Bród and Skupień [3] for trees. Krzywkowski and Wagner [9] study the maximum number of total dominating sets for general graphs and trees. For similar research concerning independent sets we refer to [10, 13, 14].

The next section contains the proofs of our results. We use standard graph theoretical terminology and notation. An endvertex is a vertex of degree at most 1, and a support vertex is a vertex that is adjacent to an endvertex.

### 2 Proofs

For the proof of Theorem 2, we need the following lemma.

**Lemma 5.** If $T$ is a tree of order $n$ at least 2, and $B$ is a set of vertices of $T$ such that

1. $|B \cap \{u, v\}| \leq 1$ for every $uv \in E(T)$, and
2. $|B \cap N_T(u)| \leq 1$ for every $u \in V(T)$,

then $|B| \leq \frac{n}{2}$.

**Proof:** The proof is by induction on $n$. If $T$ is a star, then (i) and (ii) imply $|B| \leq 1 \leq \frac{n}{2}$. Now, let $T$ be a tree that is not a star; in particular, $n \geq 4$. Let $uvw \ldots$ be a longest path in $T$. By (i) and (ii), we have $|B \cap (N_T[v] \setminus \{w\})| \leq 1$. By induction applied to the tree $T' = T - (N_T[v] \setminus \{w\})$ and the set $B' = B \cap V(T')$, we obtain $|B| \leq |B'| + |B \cap (N_T[v] \setminus \{w\})| \leq \frac{n(T')}{2} + 1 \leq \frac{n}{2}$. \[\square\]

We are now in a position to present the proof of Theorem 2.
Proof of Theorem 2: Let $F$ be a forest of order $n$ and total domination number $\gamma_t$ such that $\not\exists\gamma_t(F)$ is as large as possible. Let $D$ be a $\gamma_t$-set of $F$. Let $F'$ arise by removing from $F$ all endvertices of $F$ that do not belong to $D$. For every $u \in D$, let $L(u) = N_F(u) \setminus N_{F'}(u)$ and $\ell(u) = |L(u)|$, that is, $L(u)$ is the set of neighbors of $u$ in $D$ that are endvertices of $F$ that do not belong to $D$. We call a vertex $u$ in $D$ big if $\ell(u) \geq 2$, and we assume that – subject to the above conditions – the forest $F$ and the set $D$ are chosen such that the number $k$ of big vertices is as small as possible.

Claim 1. No two big vertices are adjacent.

Proof of Claim 1: Suppose, for a contradiction, that $u$ and $v$ are adjacent big vertices. Let $L'$ be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$, that is, we shift $\ell(u) - 1$ neighbors of $u$ in $L(u)$ to $v$. Clearly, the vertices $u$ and $v$ both belong to every $\gamma_t$-set of $F$ and also to every $\gamma_t$-set of $F'$. This easily implies that a set of vertices of $F$ is a $\gamma_t$-set of $F$ if and only if it is a $\gamma_t$-set of $F'$. It follows that $D$ is a $\gamma_t$-set of $F'$ and that $\not\exists\gamma_t(F) = \not\exists\gamma_t(F')$. Since $F'$ and $D$ lead to less than $k$ big vertices, we obtain a contradiction to the choice of $F$ and $D$. \hfill \Box

Claim 2. No two big vertices have a common neighbor in $D$.

Proof of Claim 2: Suppose, for a contradiction, that $u$ and $w$ are big vertices with a common neighbor $v$ in $D$. Let

- $\#_u$ be the number of $\gamma_t$-sets of $F$ that contain a vertex from $L(u)$,
- $\#_w$ be the number of $\gamma_t$-sets of $F$ that contain a vertex from $L(w)$, and
- $\#_{u,w}$ be the number of $\gamma_t$-sets of $F$ that contain no vertex from $L(u) \cup L(w)$.

In view of $v$, no $\gamma_t$-set of $F$ contains a vertex from both sets $L(u)$ and $L(w)$, which implies

$$\not\exists\gamma_t(F) = \#_u + \#_w + \#_{u,w}.$$

Note that $\frac{\#_u}{\ell(u)}$ is the number of subsets of $V(F) \setminus L(u)$ that can be extended to a $\gamma_t$-set of $F$ by adding one vertex from $L(u)$. By symmetry, we may assume that $\frac{\#_u}{\ell(u)} \leq \frac{\#_w}{\ell(w)}$. Again, let $L'$ be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$. Similarly as before, the vertices $u$ and $w$ both belong to every $\gamma_t$-set of $F$ and also to every $\gamma_t$-set of $F'$. It follows that $D$ is a $\gamma_t$-set of $F'$, and that

$$\not\exists\gamma_t(F') = \frac{\#_u}{\ell(u)} + \frac{\#_w}{\ell(w)}(\ell(u) + \ell(w) - 1) + \#_{u,w} \geq \#_u + \#_w + \#_{u,w} = \not\exists\gamma_t(F).$$

Since $F'$ and $D$ lead to less than $k$ big vertices, this contradicts the choice of $F$ and $D$. \hfill \Box

Claim 3. $k \leq \frac{\gamma_t}{2}$.

Proof of Claim 3: This follows immediately by applying Lemma 5 to each component of $F[D]$, choosing $B$ as the set of big vertices in that component. \hfill \Box
On the maximum number of minimum total dominating sets in forests

Let \( n' = n(F') \), let \( V_1' \) be the set of endvertices of \( F' \), let \( n'_1 = |V_1'| \), and let \( m \) be the number of edges of \( F' \) between \( D \) and \( V(F') \setminus D \). Since the vertices in \( V_1' \) are either endvertices of \( F \) that belong to \( D \) or are adjacent to an endvertex of \( F \), we obtain that \( V_1' \subseteq D \). Since \( D \) is a total dominating set, we obtain

\[
n' - \gamma_t = |V(F') \setminus D| \leq m \leq \sum_{u \in D} (d_{F'}(u) - 1). \tag{1}
\]

Since \( F' \) is a forest with, say, \( \kappa \) components,

\[
n'_1 = 2\kappa + \sum_{u \in V(F') : d_{F'}(u) \geq 2} (d_{F'}(u) - 2)
\geq \sum_{u \in D : d_{F'}(u) \geq 2} (d_{F'}(u) - 2)
= \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u) - 2(\gamma_t - n'_1),
\]

which implies

\[
2\gamma_t - n'_1 \geq \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u). \tag{2}
\]

Now, we obtain

\[
n' \leq \sum_{u \in D} d_{F'}(u) = \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u) + n'_1 \leq 2\gamma_t. \tag{3}
\]

Let \( u_1, \ldots, u_k \) be the big vertices. By (3), the forest \( F'' = F' - \bigcup_{i=1}^k L(u_i) \) has order at most \( 3\gamma_t \). Let \( D'' \) be a set of vertices of \( F'' \) that is a subset of some \( \gamma_t \)-set \( D \) of \( F \). For every \( i \in \{1, \ldots, k\} \), if \( u_i \) has a neighbor in \( D'' \), then \( D \) contains no vertex from \( L(u_i) \), otherwise, the set \( D \) contains exactly one vertex from \( L(u_i) \). This implies that each of the \( 2^n(F'') \) subsets of \( V(F'') \) can be extended to a \( \gamma_t \)-set of \( F \) in at most \( \prod_{i=1}^k \ell(u_i) \) many ways.

Since

(i) \( n(F'') \leq 3\gamma_t \),

(ii) the geometric mean is less or equal the arithmetic mean,

(iii) \( \sum_{i=1}^k \ell(u_i) = n - n(F'') \leq n - \gamma_t \leq n - \frac{\gamma_t}{2} \),

(iv) \( \left(1 + \frac{3\gamma_t}{k}\right)^{k} \leq e^{\frac{3\gamma_t}{2} - k} \leq e^{\frac{3\gamma_t}{2}} \), and

(v) \( \frac{\gamma_t}{n} \leq 1 \).
we obtain

\[ \sharp_{\gamma_t}(F) \leq 2^{n(F''')} \prod_{i=1}^{k} \ell(u_i) \]

\[ \leq 2^{3^{3^{\gamma_t}}} \prod_{i=1}^{k} \ell(u_i) \]

\[ \leq 2^{3^{\gamma_t}} \left( \frac{1}{k} \sum_{i=1}^{k} \ell(u_i) \right)^k \]

\[ \leq 2^{3^{\gamma_t}} \left( \frac{n - \frac{\gamma_t}{2}}{k} \right)^k \]

\[ \leq 2^{3^{\gamma_t}} e^{\frac{\gamma_t}{2}} \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\frac{3}{2}} \]

\[ \leq \left( 8\sqrt{e} \right)^{\gamma_t} \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\frac{3}{2}} . \]

which completes the proof. \( \square \)

There is clearly some room for lowering \( 8\sqrt{e} \) to a smaller constant. Since the dependence on \( \gamma_t \) would still be exponential, we did not exploit this for the sake of simplicity. It would be interesting to see whether the bound can be improved to

\[ \left( 1 + o \left( \frac{n}{\gamma_t} \right) \right) \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\frac{3}{2}} . \]

Note that Theorem 2 implies

\[ \sharp_{\gamma_t}(T) \leq \left( 1 + \sqrt{2} \right)^{n - \gamma_t} . \]

We proceed to our next proof.

**Proof of Theorem 3:** We proceed by induction on \( n \). If \( n = 2 \), then \( F = K_2 \), \( \gamma_t = 2 \), and \( \sharp_{\gamma_t}(F) = 1 = \left( 1 + \sqrt{2} \right)^0 = \left( 1 + \sqrt{2} \right)^{n - \gamma_t} \). Now, let \( n \geq 3 \).

**Claim 1.** If \( F \) contains a component \( T \) that is a star, then \( \sharp_{\gamma_t}(F) \leq \left( 1 + \sqrt{2} \right)^{n - \gamma_t} \), with strict inequality if \( T \) has order at least 3.

**Proof of Claim 1:** Suppose that \( F \) contains a component \( T \) that is a star. Thus, \( T = K_{1,t} \) for some \( t \geq 1 \). The forest \( F' = F - V(T) \) has order \( n' = n - t - 1 \), no isolated vertex, and total domination number
\( \gamma'_t = \gamma_t - 2 \). By induction, we obtain

\[
\sharp \gamma_t(F) = t \cdot \sharp \gamma_t(F') \leq t(1 + \sqrt{2})^{n' - \gamma'_t} = t(1 + \sqrt{2})^{n - t - 1 - (\gamma_t - 2)} \\
= (1 + \sqrt{2})^{n - \gamma_t} (t(1 + \sqrt{2})^{1 - t}) \leq (1 + \sqrt{2})^{n - \gamma_t},
\]

where we use \( t(1 + \sqrt{2})^{1 - t} \leq 1 \) for \( t = 1 \) and \( t \geq 2 \). Furthermore, if \( t \geq 2 \), then \( t(1 + \sqrt{2})^{1 - t} < 1 \), in which case \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t} \).

\[\square\]

Claim 2. If \( F \) contains a component \( T \) of diameter 3, then \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t} \).

Proof of Claim 2: Suppose that \( F \) contains a component \( T \) of diameter 3. Note that \( T \) has a unique minimum total dominating set. The forest \( F' = F - V(T) \) has order \( n' \leq n - 4 \), no isolated vertex, and total domination number \( \gamma'_t = \gamma_t - 2 \). By induction, we obtain

\[
\sharp \gamma_t(F) = \sharp \gamma_t(F') \leq (1 + \sqrt{2})^{n' - \gamma'_t} \leq (1 + \sqrt{2})^{n - \gamma_t - 2} < (1 + \sqrt{2})^{n - \gamma_t}.
\]

\[\square\]

By Claim 1 and Claim 2, we may assume that there is a component of \( F \) that has diameter at least 4, for otherwise the desired result follows. Let \( T \) be such a component of \( F \). Let \( wvwx \ldots r \) be a longest path in \( T \), and consider \( T \) as rooted in \( r \). For a vertex \( z \) of \( T \), let \( V_z \) be the set that contains \( z \) and all its descendants.

Claim 3. If \( d_T(w) \geq 3 \), then \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t} \).

Proof of Claim 3: Suppose that \( d_T(w) \geq 3 \), which implies that \( w \) belongs to every \( \gamma_t \)-set of \( F \), because either \( w \) is a support vertex or \( w \) is the only neighbor of two support vertices, that is no leaf. Let \( v' \) be a child of \( w \) distinct from \( v \). Let \( F'' = F - V_{v'} \). If \( v' \) is an endvertex, then \( F'' \) has order \( n' \leq n - 4 \), no isolated vertex, and total domination number \( \gamma'_t = \gamma_t - 2 \). By induction, we obtain

\[
\sharp \gamma_t(F) \leq \sharp \gamma_t(F') \leq (1 + \sqrt{2})^{n' - \gamma'_t} = (1 + \sqrt{2})^{n - \gamma_t - 2} < (1 + \sqrt{2})^{n - \gamma_t}.
\]

If \( v' \) is not an endvertex, then \( F'' \) has order \( n' \leq n - 2 \), no isolated vertex, and total domination number \( \gamma'_t = \gamma_t - 1 \). Note that if \( T \) is a minimum total dominating set of \( F \), \( T - \{v\} \) is a total dominating set of \( F'' \), since \( v' \) is a support vertex and \( v \) and \( w \) are part of every minimum total dominating set of \( F \). By induction, we obtain

\[
\sharp \gamma_t(F) \leq \sharp \gamma_t(F') \leq (1 + \sqrt{2})^{n' - \gamma'_t} \leq (1 + \sqrt{2})^{n - \gamma_t - 1} < (1 + \sqrt{2})^{n - \gamma_t}.
\]

In both cases, \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t} \).

\[\square\]

By Claim 3, we may assume that \( d_T(w) = 2 \), for otherwise the desired result holds.

Claim 4. If \( d_T(v) \geq 3 \), then \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n - \gamma_t} \).

Proof of Claim 4: Suppose that \( \ell = d_T(v) - 1 \geq 2 \). Let \( F' = F - V_w \), \( F'' = F - (V_F(v) \setminus \{w\}) \), and \( F''' = F - (V_w \cup \{x\}) \). See Figure 2 for an illustration.
Fig. 2: The important details of the forests $F$, $F'$, $F''$, and $F'''$. 

- There are at most $\ell \cdot \sharp \gamma_t(F')$ many $\gamma_t$-sets of $F$ that contain $v$ and a child of $v$ but do not contain $w$. Furthermore, if such a $\gamma_t$-set exists, then $F'$ has order $n' = n - \ell - 2$, no isolated vertex, and total domination number $\gamma'_t = \gamma_t - 2$.

- There are at most $\sharp \gamma_t(F'')$ many $\gamma_t$-sets of $F$ that contain $v$, $w$, and $x$. Furthermore, if such a $\gamma_t$-set exists, then $F''$ has order $n'' = n - \ell$, no isolated vertex, and total domination number $\gamma''_t = \gamma_t - 1$.

- There are at most $\sharp \gamma_t(F''')$ many $\gamma_t$-sets of $F$ that contain both $v$ and $w$ but do not contain $x$. Furthermore, if such a $\gamma_t$-set exists, then $F'''$ has order $n''' = n - \ell - 3$, no isolated vertex, and total domination number $\gamma'''_t = \gamma_t - 2$.

Since all $\gamma_t$-sets of $F$ are of one of the three considered types, we obtain, by induction,

$$\sharp \gamma_t(F) \leq \ell \cdot \sharp \gamma_t(F') + \sharp \gamma_t(F'') + \sharp \gamma_t(F''')$$

$$\leq \ell (1 + \sqrt{2})^{n-\ell-(\gamma_t-2)} + (1 + \sqrt{2})^{n-\ell-(\gamma_t-1)} + (1 + \sqrt{2})^{n-\ell-3-(\gamma_t-2)}$$

$$= (1 + \sqrt{2})^{n-\gamma_t}(1 + \sqrt{2})^{-\ell-1}(\ell(1 + \sqrt{2}) + (1 + \sqrt{2})^2 + 1)$$

$$< (1 + \sqrt{2})^{n-\gamma_t},$$

where we use $\ell(1 + \sqrt{2}) + (1 + \sqrt{2})^2 + 1 < (1 + \sqrt{2})^{\ell+1}$ for all $\ell \geq 2$. \hfill \Box

By Claim 4, we may assume that $d_F(v) = 2$, for otherwise the desired result holds.

Claim 5. If $x$ is a support vertex, then $\sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}$.

Proof of Claim 5: Suppose that $x$ is a support vertex, which implies that $v$ and $x$ belong to every $\gamma_t$-set of $F$. Let $F' = F - V_w$ and $F'' = F - (N_F[v] \cup N_F[x])$. 

• There are at most \(2^{q} \gamma_{t}(F')\) many \(\gamma_{t}\)-sets of \(F\) that contain \(w\) but do not contain \(w\). Furthermore, if such a \(\gamma_{t}\)-set exists, then \(F'\) has order \(n' = n - 3\), no isolated vertex, and total domination number \(\gamma_{t}' = \gamma_{t} - 2\).

• There are at most \(2^{q} \gamma_{t}(F'')\) many \(\gamma_{t}\)-sets of \(F\) that contain \(w\) and at least one other neighbour of \(x\). Furthermore, if such a \(\gamma_{t}\)-set exists, then \(F''\) has order \(n'' = n - 3\), no isolated vertex, and total domination number \(\gamma_{t}'' = \gamma_{t} - 2\).

• There are at most \(2^{q} \gamma_{t}(F''')\) many \(\gamma_{t}\)-sets of \(F\) that contain \(w\) and no other neighbour of \(x\). Furthermore, if such a \(\gamma_{t}\)-set exists, then \(F'''\) has order \(n''' \leq n - 5\), no isolated vertex, and total domination number \(\gamma_{t}''' = \gamma_{t} - 3\).

Since all \(\gamma_{t}\)-sets of \(F\) are of one of the three considered types, we obtain, by induction,

\[
2^{q} \gamma_{t}(F') \leq 2^{q} \gamma_{t}(F'') < 2(1 + \sqrt{2})^{n-3-(\gamma_{t}-2)} + (1 + \sqrt{2})^{n-5-(\gamma_{t}-3)}
\]

\[
= (1 + \sqrt{2})^{n-\gamma_{t}}(1 + \sqrt{2})^{-3}(2(1 + \sqrt{2}) + 1) = (1 + \sqrt{2})^{n-\gamma_{t}},
\]

where we use \(2(1 + \sqrt{2}) + 1 = (1 + \sqrt{2})^{2}\). Note that in \(F'\) there is a component that contains a path of length two, in particular not every component of \(F''\) is a \(K_{2}\).

By Claim 5, we may assume that \(x\) is not a support vertex, for otherwise the desired result holds.

Claim 6. If \(x\) has a child that is a support vertex, then \(2^{q} \gamma_{t}(F) < (1 + \sqrt{2})^{n-\gamma_{t}}\).

Proof of Claim 6: Suppose that \(x\) has a child \(w'\) that is a support vertex. Clearly, the vertex \(w'\) is distinct from \(w\) and belongs to every \(\gamma_{t}\)-set of \(F\). The forest \(F' = F - V_{w}\) has order \(n' = n - 3\), no isolated vertex, and total domination number \(\gamma_{t}' = \gamma_{t} - 2\). By induction, we obtain

\[
2^{q} \gamma_{t}(F') \leq 2(1 + \sqrt{2})^{n-3-(\gamma_{t}-2)} = (1 + \sqrt{2})^{n-\gamma_{t}}2(1 + \sqrt{2})^{-3} < (1 + \sqrt{2})^{n-\gamma_{t}},
\]

where we use \(2 < (1 + \sqrt{2})\).

By Claim 6, we may assume that no child of \(x\) is a support vertex, for otherwise the desired result holds. Together with Claims 3 and 4, we may assume that the subforest of \(F\) induced by \(V_{x}\) arises from a star \(K_{1,q}\) for some \(q \geq 1\) by subdividing every edge twice. Let \(F' = F - V_{x}\), \(F'' = F - (V_{x} \cup \{y\})\), and \(F''' = F - (V_{x} \cup N_{F}[y])\).

• There are at most \(2^{q} 2^{q} \gamma_{t}(F')\) many \(\gamma_{t}\)-sets of \(F\) that do not contain \(x\). Furthermore, if such a \(\gamma_{t}\)-set exists, then \(F'\) has order \(n' = n - 3q - 1\), no isolated vertex, and total domination number \(\gamma_{t}' = \gamma_{t} - 2q\).

• There are at most \((2^{q} - 1) 2^{q} \gamma_{t}(F'')\) many \(\gamma_{t}\)-sets of \(F\) that contain \(x\) but do not contain \(y\). Furthermore, if such a \(\gamma_{t}\)-set exists, then \(F''\) has order \(n'' = n - 3q - 2\), no isolated vertex, and total domination number \(\gamma_{t}'' = \gamma_{t} - 2q - 1\).

• There are at most \(2^{q} \gamma_{t}(F''')\) many \(\gamma_{t}\)-sets of \(F\) that contain both \(x\) and \(y\). Furthermore, if such a \(\gamma_{t}\)-set exists, then \(F'''\) has order \(n''' = n - 3q - 3\), no isolated vertex, and total domination number \(\gamma_{t}''' = \gamma_{t} - 2q - 2\).
Since all $\gamma_t$-sets of $F$ are of one of the three considered types, we obtain, by induction,

$$
\sharp \gamma_t(F) \leq 2^q \sharp \gamma_t(F') + (2^q - 1) \sharp \gamma_t(F'') + 2^q \sharp \gamma_t(F''') \\
\leq 2^q(1 + \sqrt{2})^{n-3q-1-(\gamma_t-2q)} + (2^q - 1)(1 + \sqrt{2})^{n-3q-2-(\gamma_t-2q-1)} + 2^q(1 + \sqrt{2})^{n-3q-3-(\gamma_t-2q-2)} \\
= (1 + \sqrt{2})^{n-\gamma_t} + (1 + \sqrt{2})^{-\gamma_t-1}(2^q + 2^q - 1 + 2^q) \\
< (1 + \sqrt{2})^{n-\gamma_t},
$$

where we use $3 \cdot 2^q - 1 < (1 + \sqrt{2})^{q+1}$ for all $q \geq 1$. This completes the proof of Theorem 3.

We proceed to the proof of Theorem 4, which uses exactly the same approach as Theorem 3.

**Proof of Theorem 4:** By induction on $n$, we show that $\sharp \gamma_t(F) \leq \beta^n$, where $\beta$ is the unique positive real solution of the equation $2\beta + \beta^3 + 1 = \beta^5$, that is, $\beta \approx 1.4865$. If $n = 2$, then $F = K_2$ and $\sharp \gamma_t(F) = 1 < \beta^0$. Now, let $n \geq 3$.

**Claim 1.** If $F$ contains a component $T$ that is a star, then $\sharp \gamma_t(F) \leq \beta^n$.

**Proof of Claim 1:** Suppose that $F$ contains a component $T$ that is a star. Thus, $T = K_{1,t}$ for some $t \geq 1$. The forest $F' = F - V(T)$ has order $n' = n - t - 1$ and no isolated vertex. By induction, we obtain $\sharp \gamma_t(F) = t \cdot \sharp \gamma_t(F') \leq t\beta^{n-t-1} \leq \beta^n$, where we use $t \leq \beta^{t+1}$.

**Claim 2.** If $F$ contains a component $T$ of diameter 3, then $\sharp \gamma_t(F) \leq \beta^n$.

**Proof of Claim 2:** Suppose that $F$ contains a component $T$ of diameter 3. The forest $F' = F - V(T)$ has order $n' = n - 4$ and no isolated vertex. By induction, we obtain $\sharp \gamma_t(F) = \sharp \gamma_t(F') \leq \beta^{n'} < \beta^n$.

By Claim 1 and Claim 2, we may assume that every component of $F$ has diameter at least 4, for otherwise the desired result follows. Let $T$ be an arbitrary component of $F$. Let $uvwxy \ldots r$ be a longest path in $T$, and consider $T$ as rooted in $r$. For a vertex $z$ of $T$, let $V_z$ be the set that contains $z$ and all its descendants.

**Claim 3.** If $d_F(w) \geq 3$, then $\sharp \gamma_t(F) \leq \beta^n$.

**Proof of Claim 3:** Suppose that $d_F(w) \geq 3$, which implies that $w$ belongs to every $\gamma_t$-set of $F$. Let $v'$ be a child of $w$ distinct from $v$. The forest $F' = F - V_v$ has order $n' = n - 4$ and no isolated vertex. Since $\sharp \gamma_t(F) \leq \sharp \gamma_t(F')$, we obtain, by induction, $\sharp \gamma_t(F) \leq \sharp \gamma_t(F') \leq \beta^{n'} < \beta^n$.

By Claim 3, we may assume that $d_F(w) = 2$, for otherwise the desired result holds.

**Claim 4.** If $d_F(w) \geq 3$, then $\sharp \gamma_t(F) \leq \beta^n$.

**Proof of Claim 4:** Suppose that $\ell = d_F(v) \geq 3$. Arguing exactly as in the proof of Claim 4 in the proof of Theorem 3 using the forests $F'$, $F''$, and $F'''$, we obtain, by induction,

$$
\sharp \gamma_t(F) \leq \ell \cdot \sharp \gamma_t(F') + \sharp \gamma_t(F'') + \sharp \gamma_t(F''') \\
\leq \ell \beta^{n-\ell-2} + \beta^{n-\ell} + \beta^{n-\ell-3} \\
= \beta^n \beta^{-\ell-3} (\ell \beta + \beta^3 + 1) \\
\leq \beta^n,
$$
where we use $\ell \beta + \beta^3 + 1 \leq \beta^{\ell+3}$ for all $\ell \geq 2$; in fact, this inequality is the reason for the specific choice of $\beta$.

By Claim 4, we may assume that $d_F(v) = 2$, for otherwise the desired result holds.

**Claim 5.** If $x$ is a support vertex, then $\sharp_{\gamma_t}(F) \leq \beta^n$.

**Proof of Claim 5:** Suppose that $x$ is a support vertex. Arguing exactly as in the proof of Claim 5 in the proof of Theorem 3 using the forests $F'$ and $F''$, we obtain, by induction,

$$\sharp_{\gamma_t}(F) \leq 2\sharp_{\gamma_t}(F') + \sharp_{\gamma_t}(F'') \leq 2\beta^{n-3} + \beta^{n-3} + 1 \leq \beta^n,$$

where we use $2\beta^3 + 1 \leq \beta^5$.

By Claim 5, we may assume that $x$ is not a support vertex, for otherwise the desired result holds.

**Claim 6.** If $x$ has a child that is a support vertex, then $\sharp_{\gamma_t}(F) \leq \beta^n$.

**Proof of Claim 6:** Suppose that $x$ has a child $w'$ that is a support vertex. Arguing exactly as in the proof of Claim 6 in the proof of Theorem 3 using the forest $F'$, we obtain, by induction,

$$\sharp_{\gamma_t}(F) = 2\sharp_{\gamma_t}(F') \leq 2\beta^{n-3} = \beta^n \beta^{-3} \leq \beta^n,$$

where we use $2 < \beta^3$.

Now, arguing exactly as at the end of the proof of Theorem 3 using the forests $F'$, $F''$, and $F'''$, we obtain, by induction,

$$\sharp_{\gamma_t}(F) \leq 2^q \sharp_{\gamma_t}(F') + (2^q - 1) \sharp_{\gamma_t}(F'') + 2^q \sharp_{\gamma_t}(F''')$$

$$\leq 2^q \beta^{n-3q-1} + (2^q - 1) \beta^{n-3q-2} + 2^q \beta^{n-3q-3}$$

$$= \beta^n \beta^{-3q-3} \left(2^q \beta^2 + (2^q - 1) \beta + 2^q\right)$$

$$\leq \beta^n \beta^{-3q-3} 2^q \left(\beta^2 + \beta + 1\right)$$

$$\leq \beta^n,$$

where we use $2^q \left(\beta^2 + \beta + 1\right) \leq \beta^{3q+3}$ for all $q \geq 1$.

**References**


[2] A. Bień, Properties of gamma graphs of trees, presentation at the 17th Workshop on Graph Theory Colourings, Independence and Domination (CID 2017), Piechowice, Poland.


