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On the maximum number of minimum total dominating sets in forests

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We propose the conjecture that every tree with order n at least 2 and total domination number γ_t has at most $\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}$ minimum total dominating sets. As a relaxation of this conjecture, we show that every forest F with order n, no isolated vertex, and total domination number γ_t has at most

$$\min\left\{\left(8\sqrt{e}\right)^{\gamma_t}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}, (1+\sqrt{2})^{n-\gamma_t}, 1.4865^n\right\}$$

minimum total dominating sets.

Keywords: Tree, forest, total domination, domination

1 Introduction

A set D of vertices of a graph G is a *dominating set* of G if every vertex of G that is not in D has a neighbor in D, and D is a *total dominating set* of G if every vertex of G has a neighbor in D. The minimum cardinalities of a dominating set of G and a total dominating set of G are the well studied [7, 8] *domination number* $\gamma(G)$ of G and the *total domination number* $\gamma_t(G)$ of G, respectively. A (total) dominating set is *minimal* if no proper subset is a (total) dominating set. A dominating set of G of cardinality $\gamma(G)$ is a *minimum dominating set* of G, and a total dominating set of G of cardinality $\gamma_t(G)$ is a *minimum total dominating set* or γ_t*-set* of G. For a graph G, let $\sharp\gamma_t(G)$ be the number of minimum total dominating sets of G.

Providing a negative answer to a question of Fricke et al. [6], Bień [2] showed that trees with domination number γ can have more than 2^γ minimum dominating sets. In fact, Bień's example allows to construct forests with domination number γ that have up to 2.0598^γ minimum dominating sets. In [1] Alvarado et al. showed that every forest with domination number γ has at most 2.4606^γ minimum dominating sets, and they conjectured that every tree with domination number γ has $O\left(\frac{\gamma 2^\gamma}{\ln\gamma}\right)$ minimum dominating sets.

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In the present paper we consider analogous problems for total domination, which turns out to behave quite differently. As shown by the star $K_{1,n-1}$ which has total domination number 2 but $n-1$ minimum total dominating sets, the number of minimum total dominating sets of a tree is not bounded in terms of its total domination number alone, but in terms of both the order and the total domination number. In Figure 1 we illustrate what we believe to be the structure of trees T with given order n at least 2 and total domination number γ_t that maximize $\sharp\gamma_t(T)$.

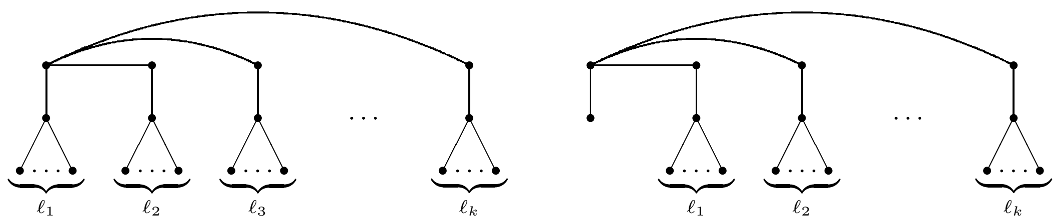

Fig. 1: For the tree $T_{\rm even}$ on the left, we have $k=\frac{\gamma_t}{2}$, $1\leq \ell_1,\ldots,\ell_k$, and $(\ell_1+1)+\ldots+(\ell_k+1)=n-k$, while for the tree $T_{\rm odd}$ on the right, we have $k=\frac{\gamma_t-1}{2}$, $1\leq \ell_1,\ldots,\ell_k$, and $(\ell_1+1)+\ldots+(\ell_k+1)=n-k-2$.

If γ_t is even, say $\gamma_t=2k$, then the tree $T_{\rm even}$ in the left of Figure 1 satisfies

$$\sharp\gamma_t(T_{\rm even})=\prod_{i=1}^{k}(\ell_i+1)\leq\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}},$$

where we use that the geometric mean is at most the arithmetic mean. Similarly, if γ_t is odd, say $\gamma_t=2k+1$, then the tree $T_{\rm odd}$ in the right of Figure 1 satisfies

$$\sharp\gamma_t(T_{\rm odd})=\sum_{i=1}^{k}\left(\prod_{j=1}^{i-1}\ell_j\prod_{j=i+1}^{k}(\ell_j+1)\right)\leq k\left(\frac{n-k-4}{k-1}\right)^{k-1}=\left(\frac{\gamma_t-1}{2}\right)\left(\frac{n-\left(\frac{\gamma_t+7}{2}\right)}{\frac{\gamma_t-3}{2}}\right)^{\frac{\gamma_t-3}{2}}.$$

In view of these estimates, we pose the following.

Conjecture 1. *If a tree T has order n at least 2 and total domination number γ_t, then*

$$\sharp\gamma_t(T)\leq\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}.$$

As our first result, we show that Conjecture 1 holds up to a constant factor for bounded values of γ_t. More precisely, we show the following.

Theorem 2. *If a forest F has order n, no isolated vertex, and total domination number γ_t, then*

$$\sharp\gamma_t(F)\leq\left(8\sqrt{e}\right)^{\gamma_t}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}.$$

The well known estimate $1 + x \leq e^x$ implies

$$\left(\frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}} = \left(1 + \frac{n - \gamma_t}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}} \leq e^{n-\gamma_t}.$$

In the following theorem we can show an upper bound that is a little better. But since $1 + x << e^x$ for large x, the estimate is not good for fixed γ_t and large values of n. In this case Theorems 2 and 4 give better upper bounds.

Theorem 3. *If a forest F has order n, no isolated vertex, and total domination number γ_t, then*

$$\sharp\gamma_t(F) \leq (1 + \sqrt{2})^{n-\gamma_t},$$

with equality if and only if every component of F is K_2.

Note that Theorem 3 is only tight for $\gamma_t = n$, which corresponds to the fact that $1 + x = e^x$ only for $x = 0$. For n divisible by 5, the disjoint union of $\frac{n}{5}$ stars of order 5 yields a forest F with $\sharp\gamma_t(F) = 4^{\frac{n}{5}} \approx 1.3195^n$. Our third result comes close to that value.

Theorem 4. *If a forest F has order n and no isolated vertex, then $\sharp\gamma_t(F) \leq 1.4865^n$.*

Before we proceed to the proofs of our results, we mention some related research. Connolly et al. [4] gave bounds on the maximum number of minimum dominating sets for general graphs. The maximum number of minimal dominating sets was studied by Fomin et al. [5], and the maximum number of general dominating sets by Wagner [12] and Skupień [11], and by Bród and Skupień [3] for trees. Krzywkowski and Wagner [9] study the maximum number of total dominating sets for general graphs and trees. For similar research concerning independent sets we refer to [10, 13, 14].

The next section contains the proofs of our results. We use standard graph theoretical terminology and notation. An *endvertex* is a vertex of degree at most 1, and a *support vertex* is a vertex that is adjacent to an endvertex.

2 Proofs

For the proof of Theorem 2, we need the following lemma.

Lemma 5. *If T is a tree of order n at least 2, and B is a set of vertices of T such that*

(i) $|B \cap \{u, v\}| \leq 1$ *for every $uv \in E(T)$, and*

(ii) $|B \cap N_T(u)\}| \leq 1$ *for every $u \in V(T)$,*

then $|B| \leq \frac{n}{2}$.

Proof: The proof is by induction on n. If T is a star, then (i) and (ii) imply $|B| \leq 1 \leq \frac{n}{2}$. Now, let T be a tree that is not a star; in particular, $n \geq 4$. Let $uvw \ldots$ be a longest path in T. By (i) and (ii), we have $|B \cap (N_T[v] \setminus \{w\})| \leq 1$. By induction applied to the tree $T' = T - (N_T[v] \setminus \{w\})$ and the set $B' = B \cap V(T')$, we obtain $|B| \leq |B'| + \Big|B \cap (N_T[v] \setminus \{w\})\Big| \leq \frac{n(T')}{2} + 1 \leq \frac{n}{2}$. $\square$

We are now in a position to present the proof of Theorem 2.

Proof of Theorem 2: Let F be a forest of order n and total domination number γ_t such that $\sharp\gamma_t(F)$ is as large as possible. Let D be a γ_t-set of F. Let F' arise by removing from F all endvertices of F that do not belong to D. For every $u \in D$, let $L(u) = N_F(u) \setminus N_{F'}(u)$ and $\ell(u) = |L(u)|$, that is, $L(u)$ is the set of neighbors of u in D that are endvertices of F that do not belong to D. We call a vertex u in D *big* if $\ell(u) \geq 2$, and we assume that – subject to the above conditions – the forest F and the set D are chosen such that the number k of big vertices is as small as possible.

Claim 1. *No two big vertices are adjacent.*

Proof of Claim 1: Suppose, for a contradiction, that u and v are adjacent big vertices. Let L' be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$, that is, we shift $\ell(u) - 1$ neighbors of u in $L(u)$ to v. Clearly, the vertices u and v both belong to every γ_t-set of F and also to every γ_t-set of F'. This easily implies that a set of vertices of F is a γ_t-set of F if and only if it is a γ_t-set of F'. It follows that D is a γ_t-set of F' and that $\sharp\gamma_t(F) = \sharp\gamma_t(F')$. Since F' and D lead to less than k big vertices, we obtain a contradiction to the choice of F and D. $\square$

Claim 2. *No two big vertices have a common neighbor in D.*

Proof of Claim 2: Suppose, for a contradiction, that u and w are big vertices with a common neighbor v in D. Let

- $\sharp_u$ be the number of γ_t-sets of F that contain a vertex from $L(u)$,
- $\sharp_w$ be the number of γ_t-sets of F that contain a vertex from $L(w)$, and
- $\sharp_{\bar{u},\bar{w}}$ be the number of γ_t-sets of F that contain no vertex from $L(u) \cup L(w)$.

In view of v, no γ_t-set of F contains a vertex from both sets $L(u)$ and $L(w)$, which implies

$$\sharp\gamma_t(F) = \sharp_u + \sharp_w + \sharp_{\bar{u},\bar{w}}.$$

Note that $\frac{\sharp_u}{\ell(u)}$ is the number of subsets of $V(F) \setminus L(u)$ that can be extended to a γ_t-set of F by adding one vertex from $L(u)$. By symmetry, we may assume that $\frac{\sharp_u}{\ell(u)} \leq \frac{\sharp_w}{\ell(w)}$. Again, let L' be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{wx : x \in L'\}$. Similarly as before, the vertices u and w both belong to every γ_t-set of F and also to every γ_t-set of F'. It follows that D is a γ_t-set of F', and that

$$\sharp\gamma_t(F') = \frac{\sharp_u}{\ell(u)} + \frac{\sharp_w}{\ell(w)}(\ell(u) + \ell(w) - 1) + \sharp_{\bar{u},\bar{w}} \geq \sharp_u + \sharp_w + \sharp_{\bar{u},\bar{w}} = \sharp\gamma_t(F).$$

Since F' and D lead to less than k big vertices, this contradicts the choice of F and D. $\square$

Claim 3. $k \leq \frac{\gamma_t}{2}$.

Proof of Claim 3: This follows immediately by applying Lemma 5 to each component of $F[D]$, choosing B as the set of big vertices in that component. $\square$

Let $n' = n(F')$, let V_1' be the set of endvertices of F', let $n_1' = |V_1'|$, and let m be the number of edges of F' between D and $V(F') \setminus D$. Since the vertices in V_1' are either endvertices of F that belong to D or are adjacent to an endvertex of F, we obtain that $V_1' \subseteq D$. Since D is a total dominating set, we obtain

$$n' - \gamma_t = |V(F') \setminus D| \leq m \leq \sum_{u \in D} (d_{F'}(u) - 1). \tag{1}$$

Since F' is a forest with, say, κ components,

$$\begin{aligned} n_1' &= 2\kappa + \sum_{u \in V(F'): d_{F'}(u) \geq 2} (d_{F'}(u) - 2) \\ &\geq \sum_{u \in D: d_{F'}(u) \geq 2} (d_{F'}(u) - 2) \\ &= \sum_{u \in D: d_{F'}(u) \geq 2} d_{F'}(u) - 2(\gamma_t - n_1'), \end{aligned}$$

which implies

$$2\gamma_t - n_1' \geq \sum_{u \in D: d_{F'}(u) \geq 2} d_{F'}(u). \tag{2}$$

Now, we obtain

$$n' \overset{(1)}{\leq} \sum_{u \in D} d_{F'}(u) = \sum_{u \in D: d_{F'}(u) \geq 2} d_{F'}(u) + n_1' \overset{(2)}{\leq} 2\gamma_t. \tag{3}$$

Let $u_1, \ldots, u_k$ be the big vertices. By (3), the forest $F'' = F - \bigcup_{i=1}^{k} L(u_i)$ has order at most $3\gamma_t$. Let D'' be a set of vertices of F'' that is a subset of some γ_t-set D of F. For every $i \in \{1, \ldots, k\}$, if u_i has a neighbor in D'', then D contains no vertex from $L(u_i)$, otherwise, the set D contains exactly one vertex from $L(u_i)$. This implies that each of the $2^{n(F'')}$ subsets of $V(F'')$ can be extended to a γ_t-set of F in at most $\prod_{i=1}^{k} \ell(u_i)$ many ways.

Since

(i) $n(F'') \leq 3\gamma_t$,

(ii) the geometric mean is less or equal the arithmetic mean,

(iii) $\sum_{i=1}^{k} \ell(u_i) = n - n(F'') \leq n - \gamma_t \leq n - \frac{\gamma_t}{2}$,

(iv) $\left(1 + \frac{\frac{\gamma_t}{2} - k}{k}\right)^k \leq e^{\frac{\gamma_t}{2} - k} \leq e^{\frac{\gamma_t}{2}}$, and

(v) $\frac{\frac{\gamma_t}{2}}{n - \frac{\gamma_t}{2}} \leq 1$,

we obtain

$$\begin{aligned}
\sharp\gamma_t(F) &\leq 2^{n(F'')}\prod_{i=1}^{k}\ell(u_i)\\
&\overset{(i)}{\leq} 2^{3\gamma_t}\prod_{i=1}^{k}\ell(u_i)\\
&\overset{(ii)}{\leq} 2^{3\gamma_t}\left(\frac{1}{k}\sum_{i=1}^{k}\ell(u_i)\right)^k\\
&\overset{(iii)}{\leq} 2^{3\gamma_t}\left(\frac{n-\frac{\gamma_t}{2}}{k}\right)^k\\
&= 2^{3\gamma_t}\left(1+\frac{\frac{\gamma_t}{2}-k}{k}\right)^k\left(\frac{\frac{\gamma_t}{2}}{n-\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}-k}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}\\
&\overset{(iv)}{\leq} 2^{3\gamma_t}e^{\frac{\gamma_t}{2}}\left(\frac{\frac{\gamma_t}{2}}{n-\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}-k}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}\\
&\overset{\text{Claim 3, }(v)}{\leq} 2^{3\gamma_t}e^{\frac{\gamma_t}{2}}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}\\
&= \left(8\sqrt{e}\right)^{\gamma_t}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}},
\end{aligned}$$

which completes the proof. □

There is clearly some room for lowering $8\sqrt{e}$ to a smaller constant. Since the dependence on γ_t would still be exponential, we did not exploit this for the sake of simplicity. It would be interesting to see whether the bound can be improved to

$$\left(1+o\left(\frac{n}{\gamma_t}\right)\right)\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}.$$

Note that Theorem 2 implies

$$\sharp\gamma_t(T)\leq\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}+o\left(\frac{n}{\gamma_t}\right)}.$$

We proceed to our next proof.

Proof of Theorem 3: We proceed by induction on n. If $n=2$, then $F=K_2$, $\gamma_t=2$, and $\sharp\gamma_t(F)=1=(1+\sqrt{2})^0=(1+\sqrt{2})^{n-\gamma_t}$. Now, let $n\geq 3$.

Claim 1. *If F contains a component T that is a star, then $\sharp\gamma_t(F)\leq(1+\sqrt{2})^{n-\gamma_t}$, with strict inequality if T has order at least 3.*

Proof of Claim 1: Suppose that F contains a component T that is a star. Thus, $T=K_{1,t}$ for some $t\geq 1$. The forest $F'=F-V(T)$ has order $n'=n-t-1$, no isolated vertex, and total domination number

$\gamma_t' = \gamma_t - 2$. By induction, we obtain

$$\begin{aligned}\sharp\gamma_t(F) &= t \cdot \sharp\gamma_t(F') \leq t(1+\sqrt{2})^{n'-\gamma_t'} = t(1+\sqrt{2})^{n-t-1-(\gamma_t-2)} \\ &= (1+\sqrt{2})^{n-\gamma_t}(t(1+\sqrt{2})^{1-t}) \leq (1+\sqrt{2})^{n-\gamma_t},\end{aligned}$$

where we use $t(1+\sqrt{2})^{1-t} \leq 1$ for $t = 1$ and $t \geq 2$. Furthermore, if $t \geq 2$, then $t(1+\sqrt{2})^{1-t} < 1$, in which case $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$. □

Claim 2. *If F contains a component T of diameter 3, then $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$.*

Proof of Claim 2: Suppose that F contains a component T of diameter 3. Note that T has a unique minimum total dominating set. The forest $F' = F - V(T)$ has order $n' \leq n-4$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$. By induction, we obtain

$$\sharp\gamma_t(F) = \sharp\gamma_t(F') \leq (1+\sqrt{2})^{n'-\gamma_t'} \leq (1+\sqrt{2})^{n-\gamma_t-2} < (1+\sqrt{2})^{n-\gamma_t}.$$

□

By Claim 1 and Claim 2, we may assume that there is a component of F that has diameter at least 4, for otherwise the desired result follows. Let T be such a component of F. Let $uvwxy\ldots r$ be a longest path in T, and consider T as rooted in r. For a vertex z of T, let V_z be the set that contains z and all its descendants.

Claim 3. *If $d_F(w) \geq 3$, then $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$.*

Proof of Claim 3: Suppose that $d_F(w) \geq 3$, which implies that w belongs to every γ_t-set of F, because either w is a support vertex or w is the only neighbor of two support vertices, that is no leaf. Let v' be a child of w distinct from v. Let $F' = F - V_{v'}$. If v' is an endvertex, then F' has order $n' = n-1$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t$. By induction, we obtain

$$\sharp\gamma_t(F) \leq \sharp\gamma_t(F') \leq (1+\sqrt{2})^{n'-\gamma_t'} = (1+\sqrt{2})^{n-\gamma_t-1} < (1+\sqrt{2})^{n-\gamma_t}.$$

If v' is not an endvertex, then F' has order $n' \leq n-2$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 1$. Note that if T is a minimum total dominating set of F, $T - \{v\}$ is a total dominating set of F', since v' is a support vertex and v and w are part of every minimum total dominating set of F. By induction, we obtain

$$\sharp\gamma_t(F) \leq \sharp\gamma_t(F') \leq (1+\sqrt{2})^{n'-\gamma_t'} \leq (1+\sqrt{2})^{n-\gamma_t-1} < (1+\sqrt{2})^{n-\gamma_t}.$$

In both cases, $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$. □

By Claim 3, we may assume that $d_F(w) = 2$, for otherwise the desired result holds.

Claim 4. *If $d_F(v) \geq 3$, then $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$.*

Proof of Claim 4: Suppose that $\ell = d_F(v) - 1 \geq 2$. Let $F' = F - V_w$, $F'' = F - (N_F(v) \setminus \{w\})$, and $F''' = F - (V_w \cup \{x\})$. See Figure 2 for an illustration.

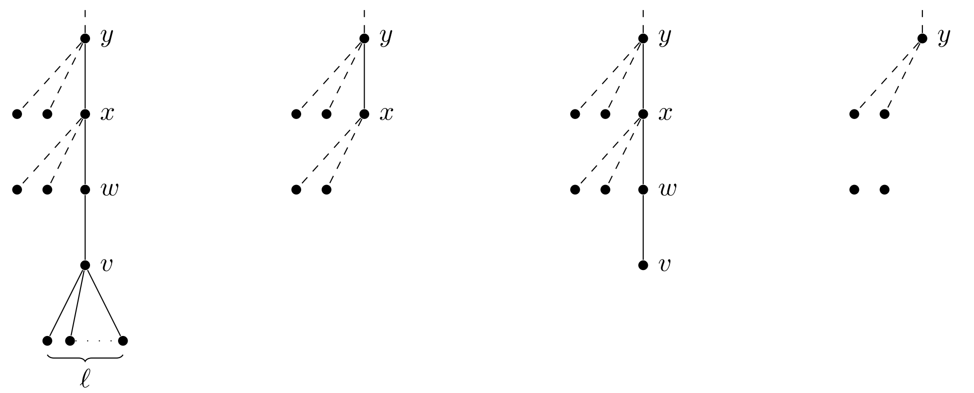

Fig. 2: The important details of the forests F, F', F'' and F'''.

- There are at most $\ell \cdot \sharp\gamma_t(F')$ many γ_t-sets of F that contain v and a child of v but do not contain w. Furthermore, if such a γ_t-set exists, then F' has order $n' = n - \ell - 2$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$.
- There are at most $\sharp\gamma_t(F'')$ many γ_t-sets of F that contain v, w, and x. Furthermore, if such a γ_t-set exists, then F'' has order $n'' = n - \ell$, no isolated vertex, and total domination number $\gamma_t'' = \gamma_t - 1$.
- There are at most $\sharp\gamma_t(F''')$ many γ_t-sets of F that contain both v and w but do not contain x. Furthermore, if such a γ_t-set exists, then F''' has order $n''' = n - \ell - 3$, no isolated vertex, and total domination number $\gamma_t''' = \gamma_t - 2$.

Since all γ_t-sets of F are of one of the three considered types, we obtain, by induction,

$$\begin{aligned}
\sharp\gamma_t(F) &\leq \ell \cdot \sharp\gamma_t(F') + \sharp\gamma_t(F'') + \sharp\gamma_t(F''') \\
&\leq \ell(1+\sqrt{2})^{n-\ell-2-(\gamma_t-2)} + (1+\sqrt{2})^{n-\ell-(\gamma_t-1)} + (1+\sqrt{2})^{n-\ell-3-(\gamma_t-2)} \\
&= (1+\sqrt{2})^{n-\gamma_t}(1+\sqrt{2})^{-\ell-1}(\ell(1+\sqrt{2}) + (1+\sqrt{2})^2 + 1) \\
&< (1+\sqrt{2})^{n-\gamma_t},
\end{aligned}$$

where we use $\ell(1+\sqrt{2}) + (1+\sqrt{2})^2 + 1 < (1+\sqrt{2})^{\ell+1}$ for all $\ell \geq 2$. □

By Claim 4, we may assume that $d_F(v) = 2$, for otherwise the desired result holds.

Claim 5. *If x is a support vertex, then $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$.*

Proof of Claim 5: Suppose that x is a support vertex, which implies that v and x belong to every γ_t-set of F. Let $F' = F - V_w$ and $F'' = F - (N_F[v] \cup N_F[x])$.

- There are at most $\sharp\gamma_t(F')$ many γ_t-sets of F that contain u but do not contain w. Furthermore, if such a γ_t-set exists, then F' has order $n' = n-3$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$.
- There are at most $\sharp\gamma_t(F')$ many γ_t-sets of F that contain w and at least one other neighbour of x. Furthermore, if such a γ_t-set exists, then F' has order $n' = n-3$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$.
- There are at most $\sharp\gamma_t(F'')$ many γ_t-sets of F that contain w and no other neighbour of x. Furthermore, if such a γ_t-set exists, then F'' has order $n'' \leq n-5$, no isolated vertex, and total domination number $\gamma_t'' = \gamma_t - 3$.

Since all γ_t-sets of F are of one of the three considered types, we obtain, by induction,

$$\begin{aligned} \sharp\gamma_t(F) &\leq 2\sharp\gamma_t(F') + \sharp\gamma_t(F'') < 2(1+\sqrt{2})^{n-3-(\gamma_t-2)} + (1+\sqrt{2})^{n-5-(\gamma_t-3)} \\ &= (1+\sqrt{2})^{n-\gamma_t}(1+\sqrt{2})^{-2}(2(1+\sqrt{2})+1) = (1+\sqrt{2})^{n-\gamma_t}, \end{aligned}$$

where we use $2(1+\sqrt{2})+1 = (1+\sqrt{2})^2$. Note that in F' there is a component that contains a path of length two, in particular not every component of F' is a K_2. □

By Claim 5, we may assume that x is not a support vertex, for otherwise the desired result holds.

Claim 6. *If x has a child that is a support vertex, then $\sharp\gamma_t(F) < (1+\sqrt{2})^{n-\gamma_t}$.*

Proof of Claim 6: Suppose that x has a child w' that is a support vertex. Clearly, the vertex w' is distinct from w and belongs to every γ_t-set of F. The forest $F' = F - V_w$ has order $n' = n-3$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$. By induction, we obtain

$$\sharp\gamma_t(F) = 2\sharp\gamma_t(F') \leq 2(1+\sqrt{2})^{n-3-(\gamma_t-2)} = (1+\sqrt{2})^{n-\gamma_t} 2(1+\sqrt{2})^{-1} < (1+\sqrt{2})^{n-\gamma_t},$$

where we use $2 < (1+\sqrt{2})$. □

By Claim 6, we may assume that no child of x is a support vertex, for otherwise the desired result holds. Together with Claims 3 and 4, we may assume that the subforest of F induced by V_x arises from a star $K_{1,q}$ for some $q \geq 1$ by subdividing every edge twice. Let $F' = F - V_x$, $F'' = F - (V_x \cup \{y\})$, and $F''' = F - (V_x \cup N_F[y])$.

- There are at most $2^q\sharp\gamma_t(F')$ many γ_t-sets of F that do not contain x. Furthermore, if such a γ_t-set exists, then F' has order $n' = n-3q-1$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2q$.
- There are at most $(2^q-1)\sharp\gamma_t(F'')$ many γ_t-sets of F that contain x but do not contain y. Furthermore, if such a γ_t-set exists, then F'' has order $n'' = n-3q-2$, no isolated vertex, and total domination number $\gamma_t'' = \gamma_t - 2q - 1$.
- There are at most $2^q\sharp\gamma_t(F''')$ many γ_t-sets of F that contain both x and y. Furthermore, if such a γ_t-set exists, then F''' has order $n''' \leq n-3q-3$, no isolated vertex, and total domination number $\gamma_t''' = \gamma_t - 2q - 2$.

Since all γ_t-sets of F are of one of the three considered types, we obtain, by induction,

$$\begin{array}{rcl} \sharp\gamma_t(F) & \leq & 2^q\sharp\gamma_t(F') + (2^q-1)\sharp\gamma_t(F'') + 2^q\sharp\gamma_t(F''') \\ & \leq & 2^q(1+\sqrt{2})^{n-3q-1-(\gamma_t-2q)} \\ & & +(2^q-1)(1+\sqrt{2})^{n-3q-2-(\gamma_t-2q-1)} \\ & & +2^q(1+\sqrt{2})^{n-3q-3-(\gamma_t-2q-2)} \\ & = & (1+\sqrt{2})^{n-\gamma_t}(1+\sqrt{2})^{-q-1}(2^q+2^q-1+2^q) \\ & < & (1+\sqrt{2})^{n-\gamma_t}, \end{array}$$

where we use $3\cdot 2^q - 1 < (1+\sqrt{2})^{q+1}$ for all $q \geq 1$. This completes the proof of Theorem 3. □

We proceed to the proof of Theorem 4, which uses exactly the same approach as Theorem 3.

Proof of Theorem 4: By induction on n, we show that $\sharp\gamma_t(F) \leq \beta^n$, where β is the unique positive real solution of the equation $2\beta + \beta^3 + 1 = \beta^5$, that is, $\beta \approx 1.4865$. If $n = 2$, then $F = K_2$ and $\sharp\gamma_t(F) = 1 < \beta^2$. Now, let $n \geq 3$.

Claim 1. *If F contains a component T that is a star, then $\sharp\gamma_t(F) \leq \beta^n$.*

Proof of Claim 1: Suppose that F contains a component T that is a star. Thus, $T = K_{1,t}$ for some $t \geq 1$. The forest $F' = F - V(T)$ has order $n' = n - t - 1$ and no isolated vertex. By induction, we obtain $\sharp\gamma_t(F) = t \cdot \sharp\gamma_t(F') \leq t\beta^{n-t-1} \leq \beta^n$, where we use $t \leq \beta^{t+1}$. □

Claim 2. *If F contains a component T of diameter 3, then $\sharp\gamma_t(F) \leq \beta^n$.*

Proof of Claim 2: Suppose that F contains a component T of diameter 3. The forest $F' = F - V(T)$ has order $n' \leq n - 4$ and no isolated vertex. By induction, we obtain $\sharp\gamma_t(F) = \sharp\gamma_t(F') \leq \beta^{n'} < \beta^n$. □

By Claim 1 and Claim 2, we may assume that every component of F has diameter at least 4, for otherwise the desired result follows. Let T be an arbitrary component of F. Let $uvwxy\ldots r$ be a longest path in T, and consider T as rooted in r. For a vertex z of T, let V_z be the set that contains z and all its descendants.

Claim 3. *If $d_F(w) \geq 3$, then $\sharp\gamma_t(F) \leq \beta^n$.*

Proof of Claim 3: Suppose that $d_F(w) \geq 3$, which implies that w belongs to every γ_t-set of F. Let v' be a child of w distinct from v. The forest $F' = F - V_{v'}$ has order $n' < n$ and no isolated vertex. Since $\sharp\gamma_t(F) \leq \sharp\gamma_t(F')$, we obtain, by induction, $\sharp\gamma_t(F) \leq \sharp\gamma_t(F') \leq \beta^{n'} < \beta^n$. □

By Claim 3, we may assume that $d_F(w) = 2$, for otherwise the desired result holds.

Claim 4. *If $d_F(v) \geq 3$, then $\sharp\gamma_t(F) \leq \beta^n$.*

Proof of Claim 4: Suppose that $\ell = d_F(v) - 1 \geq 2$. Arguing exactly as in the proof of Claim 4 in the proof of Theorem 3 using the forests F', F'', and F''', we obtain, by induction,

$$\begin{array}{rcl} \sharp\gamma_t(F) & \leq & \ell \cdot \sharp\gamma_t(F') + \sharp\gamma_t(F'') + \sharp\gamma_t(F''') \\ & \leq & \ell\beta^{n-\ell-2} + \beta^{n-\ell} + \beta^{n-\ell-3} \\ & = & \beta^n\beta^{-\ell-3}\left(\ell\beta + \beta^3 + 1\right) \\ & \leq & \beta^n, \end{array}$$

where we use $\ell\beta + \beta^3 + 1 \leq \beta^{\ell+3}$ for all $\ell \geq 2$; in fact, this inequality is the reason for the specific choice of β. □

By Claim 4, we may assume that $d_F(v) = 2$, for otherwise the desired result holds.

Claim 5. *If x is a support vertex, then $\sharp\gamma_t(F) \leq \beta^n$.*

Proof of Claim 5: Suppose that x is a support vertex. Arguing exactly as in the proof of Claim 5 in the proof of Theorem 3 using the forests F' and F'', we obtain, by induction,

$$\sharp\gamma_t(F) \quad \leq \quad 2\sharp\gamma_t(F') + \sharp\gamma_t(F'') \leq 2\beta^{n-3} + \beta^{n-5} = \beta^n\beta^{-5}(2\beta^2+1) \leq \beta^n,$$

where we use $2\beta^2 + 1 \leq \beta^5$. □

By Claim 5, we may assume that x is not a support vertex, for otherwise the desired result holds.

Claim 6. *If x has a child that is a support vertex, then $\sharp\gamma_t(F) \leq \beta^n$.*

Proof of Claim 6: Suppose that x has a child w' that is a support vertex. Arguing exactly as in the proof of Claim 6 in the proof of Theorem 3 using the forest F', we obtain, by induction,

$$\sharp\gamma_t(F) \quad = \quad 2\sharp\gamma_t(F') \leq 2\beta^{n-3} = \beta^n 2\beta^{-3} \leq \beta^n,$$

where we use $2 < \beta^3$. □

Now, arguing exactly as at the end of the proof of Theorem 3 using the forests F', F'', and F''', we obtain, by induction,

$$\begin{aligned}
\sharp\gamma_t(F) &\leq 2^q\sharp\gamma_t(F') + (2^q-1)\sharp\gamma_t(F'') + 2^q\sharp\gamma_t(F''') \\
&\leq 2^q\beta^{n-3q-1} + (2^q-1)\beta^{n-3q-2} + 2^q\beta^{n-3q-3} \\
&= \beta^n\beta^{-3q-3}\left(2^q\beta^2 + (2^q-1)\beta + 2^q\right) \\
&\leq \beta^n\beta^{-3q-3}2^q\left(\beta^2+\beta+1\right) \\
&\leq \beta^n,
\end{aligned}$$

where we use $2^q\left(\beta^2+\beta+1\right) \leq \beta^{3q+3}$ for all $q \geq 1$. □

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