

On the insertion of n -powers

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received 16th Nov. 2017, revised 16th Jan. 2019, accepted 16th Jan. 2019.

In algebraic terms, the insertion of n -powers in words may be modelled at the language level by considering the pseudovariety of ordered monoids defined by the inequality $1 \leq x^n$. We compare this pseudovariety with several other natural pseudovarieties of ordered monoids and of monoids associated with the Burnside pseudovariety of groups defined by the identity $x^n = 1$. In particular, we are interested in determining the pseudovariety of monoids that it generates, which can be viewed as the problem of determining the Boolean closure of the class of regular languages closed under n -power insertions. We exhibit a simple upper bound and show that it satisfies all pseudoidentities which are provable from $1 \leq x^n$ in which both sides are regular elements with respect to the upper bound.

Keywords: Regular language, polynomial closure, pseudovariety, finite ordered monoid, pseudoidentity, Burnside pseudovariety

1 Introduction

Pseudovarieties of ordered monoids have been introduced in the theory of finite semigroups as a tool that, via a generalization of Eilenberg's correspondence and the syntactic monoid, provides a classifier for classes of regular languages, cf. Pin (1995b, 1997). More generally than varieties of languages, they classify the so-called positive varieties of languages. As for the original version of Eilenberg's correspondence for pseudovarieties of monoids, the extended version prompted additional interest in studying pseudovarieties of ordered monoids, particularly in the context of concatenation hierarchies of regular languages, which provided the initial motivation for introducing them.

Even before pseudovarieties of ordered monoids were considered, ordered monoids had already been shown to play a role in the theory of finite semigroups. A notable instance is a direct algebraic proof of the fact that every finite \mathcal{J} -trivial monoid is a quotient of some finite ordered monoid satisfying the inequality $1 \leq x$, see Straubing and Thérien (1988), a fact that turns out to be equivalent to Simon's characterization of piecewise testable languages as those whose syntactic monoid is a finite \mathcal{J} -trivial monoid, see Simon (1975), one of the classical results that led to the formulation of Eilenberg's correspondence, cf. Eilenberg (1976). Note that the language counterpart of the pseudovariety of ordered monoids defined by the

*Partial support by CMUP (UID/MAT/00144/2019) which is funded by FCT (Portugal) with national (MCTES) and European structural funds (FEDER) under the partnership agreement PT2020.

†Supported by Grant 15-02862S of the Grant Agency of the Czech Republic.

inequality $1 \leq x$ is the class of regular languages that are closed under inserting arbitrary words in each of their elements.

Another important instance of an inequality of the form $1 \leq x^\alpha$ is the weakest such inequality, namely $1 \leq x^\omega$. The pseudovariety of monoids generated by the class of ordered monoids it defines was the object of deep research in the 1980's which led to many alternative descriptions, from block groups to power groups, as well as the language counterpart given by the Boolean-polynomial closure of the class of all group languages. A discussion of such results, whose key ingredient is due to Ash (1991), may be found in Henckell et al. (1991); Pin (1995a). Most descriptions of that pseudovariety involve some construction on the pseudovariety \mathbf{G} of all finite groups, such as power groups (PG), the semidirect product $\mathbf{J} * \mathbf{G}$, the Mal'cev product $\mathbf{J} \overline{m} \mathbf{G}$, and block groups (BG). The relationships between such constructions starting from an arbitrary pseudovariety of groups instead of the pseudovariety \mathbf{G} have been extensively studied by Auinger and Steinberg, see Steinberg (2000, 2001b,a); Auinger and Steinberg (2003, 2004, 2005b,a). In particular, the situation is radically different from the well-behaved case of \mathbf{G} for the Burnside pseudovariety defined by the identity $x^n = 1$ for $n \geq 2$.

The aim of our paper is to investigate the pseudovariety of ordered monoids $\llbracket 1 \leq x^n \rrbracket$ defined by the inequality $1 \leq x^n$, which is the algebraic counterpart of the positive variety of languages closed under the insertion of n -powers. We are also interested in the Boolean closure of that positive variety, for which decidability of membership remains an open problem. It corresponds to the pseudovariety of monoids generated by $\llbracket 1 \leq x^n \rrbracket$, which may be viewed as an extension of the case $1 \leq x$ and a restriction of the case $1 \leq x^\omega$ by bounding the exponent. We compare these pseudovarieties with the classical constructions on the corresponding Burnside pseudovariety, defined by $x^n = 1$, and with the best upper bound we have been able to find. This is the pseudovariety $(\mathbf{BG})_n$ of block groups defined by the pseudoidentity $(xy^\omega z)^{\omega+1} = (xy^n z)^{\omega+1}$. We also propose an ordered version of the pseudoidentity proof scheme introduced by the authors, see Almeida and Klíma (2018). Finally, we show that if a pseudoidentity over $(\mathbf{BG})_n$ whose sides are regular pseudowords may be proved from $1 \leq x^n$, then it is trivial, which gives some evidence towards our upper bound being optimal.

2 Background

We assume the reader is familiar with the basics of finite semigroup theory, particularly with pseudovarieties, pseudoidentities, and relatively free profinite monoids. For details, see Pin (1986); Almeida (1995, 2005); Rhodes and Steinberg (2009); Almeida and Costa. In particular, recall that a *profinite monoid* is a compact zero-dimensional monoid. For a pseudovariety \mathbf{V} of monoids, the pro- \mathbf{V} monoid freely generated by a set A is denoted $\overline{\Omega}_A \mathbf{V}$. By a *V-pseudoidentity* we mean a formal equality $u = v$ with $u, v \in \overline{\Omega}_A \mathbf{V}$ for some finite set A . For a set Σ of \mathbf{V} -pseudoidentities, the class of all monoids from \mathbf{V} satisfying all pseudoidentities from Σ is denoted $\llbracket \Sigma \rrbracket$. Most often, we consider \mathbf{M} -pseudoidentities, where \mathbf{M} is the pseudovariety of all finite monoids. Elements of $\overline{\Omega}_A \mathbf{M}$ are sometimes called *pseudowords*.

For an element s of a profinite monoid, s^ω denotes the unique idempotent in the closed subsemigroup $\overline{\langle s \rangle}$ generated by s , while $s^{\omega-1}$ denotes the inverse of $s^{\omega+1} = s^\omega s$ in the unique maximal subgroup of $\overline{\langle s \rangle}$. For a nonzero integer k , $s^{\omega+k}$ stands for $(s^{|k|})^{\omega+\varepsilon}$, where ε is the sign of k .

By an *ordered monoid* we mean a monoid with a partial order that is compatible with the monoid operation, cf. Pin (1997). A pseudovariety of ordered monoids is a nonempty class of such structures that is closed under taking images under order-preserving homomorphisms, subsemigroups under the induced order, and finite direct products. The theory of pseudovarieties of ordered monoids is a natural extension

of the unordered case. For a pseudovariety \mathbf{V} of ordered monoids, forgetting the order of its elements, we may consider the pseudovariety of monoids $\langle \mathbf{V} \rangle$ it generates. On the other hand, an unordered monoid may be viewed as an ordered one under the trivial partial order, given by equality in the monoid. For a pseudovariety of monoids \mathbf{V} , the pseudovariety of ordered monoids \mathbf{V}' that the members of \mathbf{V} generate when ordered trivially consists precisely of all monoids in \mathbf{V} under all possible compatible orders. Thus, it is natural to identify \mathbf{V} and \mathbf{V}' and we do so it freely throughout this paper.

By a (*pseudo*)inequality we mean a formal inequality $u \leq v$ with $u, v \in \overline{\Omega}_A M$ for some finite set A . The class of all finite ordered monoids satisfying a given set Σ of inequalities is also denoted $[[\Sigma]]$.

For a pseudovariety \mathbf{V} of ordered monoids, there is also a pro- \mathbf{V} monoid freely generated by a set A , denoted $\overline{\Omega}_A \mathbf{V}$ which, as a topological monoid, coincides with $\overline{\Omega}_A \langle \mathbf{V} \rangle$. It may be viewed as the quotient of $\overline{\Omega}_A M$ by the (compatible closed) quasiorder \leq defined by $u \leq v$ when \mathbf{V} satisfies the inequality $u \leq v$.

Mutatis mutandis, instead of monoids one may consider semigroups. For a pseudovariety \mathbf{V} of monoids, we usually also denote by \mathbf{V} the pseudovariety of semigroups it generates. Occasionally, we refer to pseudovarieties of ordered semigroups.

There are several pseudovarieties that play an important role in this paper. Among them are the pseudovariety \mathbf{J} of all finite \mathcal{J} -trivial monoids, the pseudovariety \mathbf{A} of all finite aperiodic monoids, and the pseudovariety \mathbf{Sl} of all finite semilattices. Some operators on pseudovarieties are also relevant. For a pseudovariety \mathbf{V} of semigroups, \mathbf{EV} denotes the pseudovariety of all finite monoids whose idempotents generate a semigroup from \mathbf{V} , \mathbf{DV} denotes the pseudovariety of all finite semigroups whose regular \mathcal{D} -classes are subsemigroups from \mathbf{V} , and \mathbf{PV} denotes the pseudovariety generated by all power semigroups of the semigroups from \mathbf{V} . For a pseudovariety of groups \mathbf{H} , \mathbf{BH} and $\overline{\mathbf{H}}$ denote the pseudovarieties of all finite monoids whose blocks are groups from \mathbf{H} and whose subgroups belong to \mathbf{H} , respectively. When \mathbf{V} is a pseudovariety of ordered semigroups and \mathbf{W} is a pseudovariety of monoids, the Mal'cev product $\mathbf{V} \overline{\circ} \mathbf{W}$ consists of all finite ordered monoids for which there is a relational morphism into a monoid from \mathbf{W} such that the preimage of each idempotent is a member of \mathbf{V} .

Given a language L over a finite alphabet A , meaning a subset of the free monoid A^* , the associated *syntactic quasiorder* is the quasiorder on A^* defined by $u \leq v$ if, for all $x, y \in A^*$, $xuy \in L$ implies $xvy \in L$.⁽ⁱ⁾ The *syntactic ordered monoid* of L , denoted $\text{Synt}(L)$, is the quotient ordered monoid by the quasiorder \leq , meaning the quotient of A^* by the congruence $\leq \cap \geq$, endowed with the partial order induced by \leq . By the *syntactic monoid* of L we mean the same monoid $\text{Synt}(L)$ but with no reference to the order.

3 Preliminary results

Consider the pseudovarieties $\mathbf{J}^+ = [[1 \leq x]]$ and $\mathbf{Ll}^+ = [[x^\omega \leq x^\omega y x^\omega]]$, respectively of ordered monoids and of ordered semigroups. By (Pin and Weil, 1997, Theorem 5.9), for a pseudovariety of monoids \mathbf{V} , the polynomial closure⁽ⁱⁱ⁾ of \mathbf{V} is the pseudovariety of ordered monoids $\text{Pol } \mathbf{V} = \mathbf{Ll}^+ \overline{\circ} \mathbf{V}$. As was proved by Pin and Weil (1996), $\mathbf{Ll}^+ \overline{\circ} \mathbf{V}$ is defined by the inequalities of the form $u^\omega \leq u^\omega v u^\omega$ such that the pseudoidentities $u = v = v^2$ hold in \mathbf{V} . In particular, in case \mathbf{V} is a pseudovariety of groups, one may

⁽ⁱ⁾ In the literature, one often finds the syntactic quasiorder defined to be the reverse quasiorder (see Almeida et al. (2015) for historical details).

⁽ⁱⁱ⁾ meaning the pseudovariety of ordered monoids $\text{Pol } \mathbf{V}$ corresponding to the positive variety of languages generated by the class of languages which, for a finite alphabet A , consists of the products of the form $L_0 a_1 L_1 \cdots a_n L_n$, where the $a_i \in A$ and the L_i are \mathbf{V} -languages.

take $u = 1$, so that the defining inequalities for $\text{Pol } \mathbf{V}$ are reduced to $1 \leq v$ whenever \mathbf{V} satisfies $v = 1$. This observation proves the following statement.

Lemma 3.1 ((Steinberg, 2000, Corollary 3.1)). *If \mathbf{H} is a pseudovariety of groups, then $\text{Pol } \mathbf{H} = \mathbf{J}^+ \widehat{\circlearrowleft} \mathbf{H}$.* \square

The following result allows us to separate two pseudovarieties of interest.

Lemma 3.2. *For $n \geq 2$, $\llbracket 1 \leq x^n \rrbracket$ is not contained in $\mathbf{J}^+ \widehat{\circlearrowleft} \llbracket x^n = 1 \rrbracket$.*

Proof: Consider first the case where $n \geq 3$. In the Burnside pseudovariety $\llbracket x^n = 1 \rrbracket$, we have $(y^{n-1}x)^{n-1}(xy)^{n-1}x^2 = x^{n-1}yy^{n-1}x^{n-1}x^2 = 1$. Hence, the inequality $1 \leq (y^{n-1}x)^{n-1}(xy)^{n-1}x^2$ holds in the Mal'cev product $\mathbf{J}^+ \widehat{\circlearrowleft} \llbracket x^n = 1 \rrbracket$. Let L be the language over the alphabet $A = \{x, y, t\}$ given by

$$L = \{pu^nq \in A^* : u \in A^+, p, q \in A^*\} \cup \{w \in A^* : |w| \geq (n+1)^2\}.$$

Then, L is a cofinite language, whence it is regular. Since u^n appears in L in every context, the syntactic ordered monoid $\text{Synt}(L)$ satisfies the inequality $1 \leq x^n$. Note also that $t \cdot 1 \cdot t^{n-1}$ belongs to L but, since $n \geq 3$, $t \cdot (y^{n-1}x)^{n-1}(xy)^{n-1}x^2 \cdot t^{n-1}$ does not as it is a word of length $(n-1)(n+3) + 3 = n^2 + 2n$ which does not contain any factor of the form u^n with $u \neq 1$. Hence, $\text{Synt}(L)$ fails the inequality $1 \leq (y^{n-1}x)^{n-1}(xy)^{n-1}x^2$.

In case $n = 2$, we consider instead the inequality $1 \leq xyzxzy$. Let $A = \{x, y, z, t\}$ and consider the language

$$L = \{pu^2q \in A^* : u \in A^+, p, q \in A^*\} \cup \{w \in A^* : |w| \geq 9\}.$$

The argument proceeds as in the previous case, where the essential ingredient that needs to be noted is that the word $txyzxzyt$ has no square factor. \square

Corollary 3.3. *For $n \geq 2$, the pseudovariety $\llbracket 1 \leq x^n \rrbracket$ is not of the form $\text{Pol } \mathbf{V}$ for any pseudovariety of monoids \mathbf{V} .*

Proof: Let \mathbf{V} be a pseudovariety of monoids and suppose that $\llbracket 1 \leq x^n \rrbracket = \text{Pol } \mathbf{V}$. Since $\mathbf{V} \subseteq \text{Pol } \mathbf{V}$, it follows that \mathbf{V} satisfies the inequality $1 \leq x^n$, whence also the identity $x^n = 1$ so that, in particular, \mathbf{V} must be a pseudovariety of groups. By Lemma 3.1, we deduce that $\text{Pol } \mathbf{V} = \mathbf{J}^+ \widehat{\circlearrowleft} \mathbf{V}$. By Lemma 3.2, $\mathbf{J}^+ \widehat{\circlearrowleft} \mathbf{V}$ satisfies a pseudoidentity that fails in $\llbracket 1 \leq x^n \rrbracket$, which entails $\llbracket 1 \leq x^n \rrbracket \not\subseteq \text{Pol } \mathbf{V}$, in contradiction with the initial assumption. \square

In contrast with Corollary 3.3, for the pseudovariety \mathbf{G} of all finite groups, (Pin and Weil, 1997, Theorem 5.9 and 2.7) yield the equalities $\text{Pol } \mathbf{G} = \mathbf{Ll}^+ \widehat{\circlearrowleft} \mathbf{G} = \llbracket 1 \leq x^\omega \rrbracket$. It is important to notice that the known proof of the latter equality depends on a deep theorem of Ash (1991). On the other hand, for $n = 1$, for the trivial pseudovariety $\mathbf{l} = \llbracket x = 1 \rrbracket$, we have $\text{Pol } \mathbf{l} = \mathbf{J}^+ \widehat{\circlearrowleft} \mathbf{l} = \mathbf{J}^+ = \llbracket 1 \leq x \rrbracket$.

Next, we recall some related results.

Theorem 3.4 (Higgins and Margolis (2000)). *The pseudovariety \mathbf{G} is the only pseudovariety of groups \mathbf{H} such that $\mathbf{A} \cap \text{ESl}$ is contained in $\text{DA} \widehat{\circlearrowleft} \mathbf{H}$.*

The following is an immediate application of the preceding theorem which has already been observed in (Steinberg, 2001a, Proposition 13).

Corollary 3.5. *For a pseudovariety of groups \mathbf{H} , the equality $\mathbf{J} \widehat{\circlearrowleft} \mathbf{H} = \text{BH}$ holds if and only if $\mathbf{H} = \mathbf{G}$.*

Proof: The equalities $J^{\circledast m} G = BG = EJ$ are well-known (Margolis and Pin, 1985, Theorem 4.7), see also (Pin, 1995a, Proposition 7.4). For the converse, it suffices to observe that $BH = EJ \cap \bar{H} \supseteq ESI \cap A$ while $J^{\circledast m} H \subseteq DA^{\circledast m} H$ and apply Theorem 3.4 to conclude that, if $BH \subseteq J^{\circledast m} H$, then $H = G$. \square

The following theorem summarizes several results of Steinberg (2000) related with our present purposes. In it, a pseudovariety of groups H is said to be *arborescent* if it satisfies the equality $(H \cap \text{Ab}) * H = H$, where Ab is the pseudovariety of all finite Abelian groups; the adjective arborescent comes from tree-like homological properties of the Cayley graphs of the free pro- H groups, see Almeida and Weil (1994) and also (Ribes, 2017, Theorem 2.5.3).

Theorem 3.6. *Let H be a pseudovariety of groups.*

1. $\langle J^+ * V \rangle = J * V$ for every pseudovariety of monoids V (Steinberg, 2000, Proposition 2.2).
2. $J^+ \circledast m H \subseteq J * H$ (Steinberg, 2000, Proposition 5.2).
3. $\text{PH} \subseteq J * H = \langle \text{Pol} H \rangle \subseteq J^{\circledast m} H$ (Steinberg, 2000, Theorem 5.2 and Corollary 5.2).
4. In case H is arborescent, $\text{PH} = J * H = \langle \text{Pol} H \rangle = J^{\circledast m} H$ (Steinberg, 2000, Theorem 5.9).

A related problem for which only partial answers seem to be known is the following.

Problem 3.7. *When does the equality $\langle J^+ \circledast m V \rangle = J^{\circledast m} V$ hold for a given pseudovariety V of monoids?*

Note that, from Lemma 3.1 and Theorem 3.6 it follows that, if H is a pseudovariety of groups, then

$$\text{Pol} H = J^+ \circledast m H \subseteq J * H = \langle \text{Pol} H \rangle \subseteq J^{\circledast m} H.$$

In particular, we get the following result.

Corollary 3.8. *If H is a pseudovariety of groups, then $\langle J^+ \circledast m H \rangle = J * H$.* \square

Remark 3.9. Note that the inclusion $J * H \subseteq J^{\circledast m} H$ may be strict. A characterization of the pseudovarieties of groups for which equality holds is given in (Auinger and Steinberg, 2004, Theorem 8.3). These are the so-called arborescent pseudovarieties of groups, defined in terms of certain geometrical properties of the Cayley graphs of the corresponding relatively free profinite groups. That $J * H \subsetneq J^{\circledast m} H$ for all nontrivial pseudovarieties satisfying some identity of the form $x^n = 1$ had previously been shown in (Steinberg, 2001b, Theorem 7.32).

In general, we may use the basis theorems for the Mal'cev, see Pin and Weil (1996), and semidirect products, see Almeida and Weil (1998) and also (Rhodes and Steinberg, 2009, Section 3.7), to obtain bases of pseudoidentities:

- using Knast (1983) and Almeida and Weil (1998), $J * H$ is defined by the pseudoidentities of the form

$$(xy)^\omega xt(zt)^\omega = (xy)^\omega (zt)^\omega$$

for all pseudowords x, y, z, t such that the pseudoidentities $x = z$ and $xy = zt = 1$ hold in H ;

- $J^{\circledast m} H$ is defined by the pseudoidentities of the forms

$$u^{\omega+1} = u^\omega \text{ and } (uv)^\omega = (vu)^\omega$$

for all pseudowords u, v such that the pseudoidentities $u^2 = u = v$ hold in H .

In the particular case where $H = \llbracket x^n = 1 \rrbracket$, since this pseudovariety is locally finite by Zelmanov's solution of the restricted Burnside problem, see Zelmanov (1991), we may take above x, y, z, t, u, v to be words. More precisely the H-free groups are finite and computable (see the discussion on (Rhodes and Steinberg, 2009, Page 369)). Hence, given a finite monoid, to test membership in the pseudovarieties $J * \llbracket x^n = 1 \rrbracket$ and $J \widehat{\circ} \llbracket x^n = 1 \rrbracket$, one only needs to check a computable finite set of effectively verifiable pseudoidentities in the respective bases above. Indeed, given a finite monoid M , with a generating subset A , one only needs to consider the above pseudoidentities in which the words x, y, z, t, u, v have length at most $|M| \times \overline{\Omega}_A \llbracket x^n = 1 \rrbracket$. Hence, both those product pseudovarieties have decidable membership problem, which adds to the interest in comparing them with the main object of our investigation, namely the pseudovariety $\langle \llbracket 1 \leq x^n \rrbracket \rangle$.

4 The pseudovariety $(BG)_n$

We introduce three alternative bases of pseudoidentities for a pseudovariety naturally associated with BG and a positive integer n . Recall that BG may be defined by the pseudoidentity $(x^\omega y)^\omega = (y x^\omega)^\omega$.

Lemma 4.1. *The pseudovarieties*

$$\begin{aligned} U_n &= \llbracket x^{\omega+n} = x^\omega, (xy^n)^\omega = (y^n x)^\omega \rrbracket, \\ V_n &= \llbracket x^{\omega+n} = x^\omega, (xy^\omega)^\omega = (y^n x)^\omega \rrbracket, \\ W_n &= \llbracket (xy^\omega z)^{\omega+1} = (xy^n z)^{\omega+1} \rrbracket \cap \text{BG} \end{aligned}$$

coincide.

Proof: ($W_n \subseteq U_n$) Taking $x = z = 1$ in $(xy^\omega z)^{\omega+1} = (xy^n z)^{\omega+1}$, we obtain $y^\omega = y^{\omega+n}$. If, instead, we take $x = 1$, we get $(y^\omega z)^{\omega+1} = (y^n z)^{\omega+1}$ which, by raising both sides to the power ω , yields $(y^\omega z)^\omega = (y^n z)^\omega$. Similarly, we also have $(zy^\omega)^\omega = (zy^n)^\omega$ in W_n . Since W_n is contained in BG, it also satisfies $(y^\omega z)^\omega = (zy^\omega)^\omega$. Hence, W_n satisfies $(zy^n)^\omega = (y^n z)^\omega$.

($U_n \subseteq V_n$) The proof consists in showing that several pseudoidentities hold in U_n . Throughout we write equality in the sense of pseudoidentities valid in U_n . Substituting y^ω for y in the pseudoidentity $(xy^n)^\omega = (y^n x)^\omega$, we obtain $(xy^\omega)^\omega = (y^\omega x)^\omega$. Associativity gives

$$(y^n x)^\omega = y^n (xy^n)^\omega (xy^n)^{\omega-1} x = y^n \cdot (y^n x)^\omega \cdot (xy^n)^{\omega-1} x,$$

which, by iteration, yields $(y^n x)^\omega = y^{nk} (y^n x)^\omega ((xy^n)^{\omega-1} x)^k$, so that, taking limits, we get $(y^n x)^\omega = y^\omega (y^n x)^\omega ((xy^n)^{\omega-1} x)^\omega$, whence $(y^n x)^\omega = y^\omega (y^n x)^\omega$ since y^ω is an idempotent. Similarly, we obtain

$$(y^n x)^\omega = y^\omega (y^n x)^\omega = y^\omega (xy^n)^\omega = y^\omega x \cdot (y^n x)^\omega \cdot y^n (xy^n)^{\omega-1},$$

whence $(y^n x)^\omega = (y^\omega x)^k (y^n x)^\omega (y^n (xy^n)^{\omega-1})^k$ and $(y^n x)^\omega = (y^\omega x)^\omega (y^n x)^\omega$. On the other hand, we have

$$(y^\omega x)^\omega = y^n (y^\omega x)^\omega = y^n (xy^\omega)^\omega = y^n x \cdot (y^\omega x)^\omega \cdot y^\omega (xy^\omega)^{\omega-1}$$

so that $(y^\omega x)^\omega = (y^n x)^\omega (y^\omega x)^\omega$ follows as above. Hence, the idempotents $(y^\omega x)^\omega$ and $(y^n x)^\omega$ are \mathcal{R} -equivalent. By symmetry, they are also \mathcal{L} -equivalent, whence they are equal. In particular, we get $(xy^\omega)^\omega = (y^n x)^\omega$.

($V_n \subseteq W_n$) Substituting y^ω for y in the pseudoidentity $(xy^\omega)^\omega = (y^n x)^\omega$, we obtain $(xy^\omega)^\omega = (y^\omega x)^\omega$, which is a defining pseudoidentity for BG. It remains to show that V_n satisfies the pseudoidentity $(xy^\omega z)^{\omega+1} = (xy^n z)^{\omega+1}$. Indeed, it satisfies the following pseudoidentities:

$$\begin{aligned} (xy^n z)^{\omega+1} &= x(y^n z x)^\omega y^n z = x(z x y^\omega)^\omega y^n z \\ &= x(z x y^\omega)^\omega z = x(z x y^\omega)^\omega y^\omega z \\ &= x(y^\omega z x)^\omega y^\omega z = (xy^\omega z)^{\omega+1}. \end{aligned}$$

This completes the proof of the lemma. \square

From hereon, we denote by $(BG)_n$ the pseudovariety of Lemma 4.1.

We now return to the pseudovariety of ordered monoids $\llbracket 1 \leq x^n \rrbracket$.

Proposition 4.2. *The pseudovariety $\llbracket 1 \leq x^n \rrbracket$ is contained in $(BG)_n$.*

Proof: The pseudovariety $\llbracket 1 \leq x^n \rrbracket$ is contained in $\llbracket 1 \leq x^\omega \rrbracket = \text{Pol } G \subseteq \text{BG}$. Moreover, it satisfies the following inequalities:

$$\begin{aligned} xy^\omega z &= x(y^n)^\omega z \\ &\leq x(y^n (zx)^n)^{\omega-1} y^n z = x(y^n z (xz)^{n-1} x)^{\omega-1} y^n z \\ &\leq x(y^n z (xy^n z)^{n-1} x)^{\omega-1} y^n z = (xy^n z)^{n(\omega-1)+1} = (xy^n z)^{\omega-n+1} \\ &\leq (xy^n z)^{\omega+1}, \end{aligned}$$

whence $xy^\omega z \leq (xy^n z)^{\omega+1}$. Raising both sides of the preceding inequality to the power $\omega + 1$, we obtain $(xy^\omega z)^{\omega+1} \leq (xy^n z)^{\omega+1}$. For the reverse inequality, just note that $\llbracket 1 \leq x^n \rrbracket$ satisfies $y^n \leq y^{mn}$ for every positive integer m and, therefore, also $y^n \leq y^\omega$. \square

The following lemma gathers some elementary properties of the pseudovariety BG.

Lemma 4.3. *The pseudovariety BG satisfies the following pseudoidentities:*

- $(xy^{\omega+1})^\omega = y^{\omega-1} (y^{\omega+1} x)^\omega y^{\omega+1}$;
- $(xy^\omega z)^\omega (xz)^{\omega+1} = (xy^\omega z)^{\omega+1} = (xz)^{\omega+1} (xy^\omega z)^\omega$;
- $(xy^\omega z)^\omega (xz)^\omega = (xy^\omega z)^\omega = (xz)^\omega (xy^\omega z)^\omega$;
- $(xy^\omega z)^\omega (xt^\omega z)^\omega = (xy^\omega z)^{\omega+1} (xt^\omega z)^{\omega-1}$.

Proof: The following pseudoidentities hold in BG:

$$\begin{aligned}
(xy^{\omega+1})^\omega &= (xy y^\omega)^\omega = (y^\omega xy)^\omega y^\omega = y^{\omega-1}(y^{\omega+1}x)^\omega y^{\omega+1}; \\
(xy^\omega z)^\omega &= x(y^\omega z x)^\omega y^\omega z = x(zxy^\omega)^\omega y^\omega z = x(zxy^\omega)^\omega z \\
&= x(y^\omega z x)^\omega z = (xy^\omega z)^\omega xz \\
\therefore (xy^\omega z)^{\omega+k} &= (xy^\omega z)^\omega (xz)^k \text{ for every } k \geq 1 \\
\therefore (xy^\omega z)^{\omega+1} &= (xy^\omega z)^\omega (xz)^{\omega+1} \text{ and } (xy^\omega z)^\omega = (xy^\omega z)^\omega (xz)^\omega \\
\therefore (xy^\omega z)^{\omega+k} &= (xz)^k (xy^\omega z)^\omega \text{ by symmetry} \\
\therefore (xy^\omega z)^{\omega+1} (xt^\omega z)^{\omega-1} &= (xy^\omega z)^\omega (xz) \cdot (xz)^{\omega-1} (xt^\omega z)^\omega \\
&= (xy^\omega z)^\omega (xz)^\omega (xt^\omega z)^\omega = (xy^\omega z)^\omega (xt^\omega z)^\omega,
\end{aligned}$$

which completes the proof. \square

Corollary 4.4. *The pseudovariety $(\text{BG})_n$ satisfies the pseudoidentities*

$$\begin{aligned}
(xy^n z)^{\omega+1} &= (xz)^{\omega+1} (xy^n z)^\omega = (xy^n z)^\omega (xz)^{\omega+1} \\
(xy^n z)^\omega &= (xz)^\omega (xy^n z)^\omega = (xy^n z)^\omega (xz)^\omega. \quad \square
\end{aligned}$$

5 Comparing several pseudovarieties

We start with a simple observation regarding the pseudovariety defined by the Mal'cev product $J \circledast \llbracket x^n = 1 \rrbracket$.

Lemma 5.1. $J \circledast \llbracket x^n = 1 \rrbracket \subseteq (\text{BG})_n$.

Proof: The pseudoidentities $x^n = 1 = y(x^n y)^{\omega-1}$ are valid in $\llbracket x^n = 1 \rrbracket$. Hence the Mal'cev product $J \circledast \llbracket x^n = 1 \rrbracket$ satisfies the pseudoidentities

$$\begin{aligned}
x^{\omega+n} &= (x^n)^{\omega+1} = (x^n)^\omega = x^\omega \\
(x^n y)^\omega &= (x^n \cdot y(x^n y)^{\omega-1})^\omega = (y(x^n y)^{\omega-1} \cdot x^n)^\omega = (yx^n)^\omega,
\end{aligned}$$

which proves the lemma. \square

Note that the preceding proof may be adapted to show that the pseudovariety $J \circledast \llbracket x^n = 1 \rrbracket$ satisfies the pseudoidentity $(ux)^\omega = (xu)^\omega$ whenever the pseudoidentity $u = 1$ holds in $\llbracket x^n = 1 \rrbracket$. Such a pseudoidentity $(ux)^\omega = (xu)^\omega$ may however fail in $(\text{BG})_n$. For example, in case $n = 2$, we may take $u = yztytz$ and the resulting pseudoidentity fails in the syntactic monoid of the language $(abcdadc)^*$ over the alphabet $\{a, b, c, d\}$, which lies in $\text{BG} \cap \llbracket x^3 = x^2 \rrbracket$, as may be easily checked with the aid of computer calculations, and, therefore also in $(\text{BG})_2$. In particular, $J \circledast \llbracket x^2 = 1 \rrbracket$ does not contain $(\text{BG})_2$. Below, we prove the stronger statement that $\llbracket 1 \leq x^n \rrbracket \not\subseteq A \circledast \llbracket x^n = 1 \rrbracket$ (Corollary 5.7).

For a language L , denote by $F(L)$ the set of all factors of words from L . Note that $F(L)$ is regular if so is L . Given natural numbers k and ℓ , for shortness we denote by $w^{k+\ell*}$ the set of all powers of the word w whose exponent is of the form $k + \ell n$ for some non-negative integer n .

Proposition 5.2. Consider the following language over the alphabet $A = \{a, b, c\}$:

$$L_2 = \left(A^* \setminus F((abcacb)^*) \right) \cup (abcacb)^{1+2*}.$$

Then the syntactic ordered monoid of L_2 belongs to $\llbracket 1 \leq x^2 \rrbracket$ and fails the pseudoidentity $(xyzxzy)^{\omega+1} = (xyzxzy)^\omega$.

To prove the first part of Proposition 5.2, we establish the following two lemmas.

Lemma 5.3. Suppose that x, u, y are words such that xu^2y belongs to the language $(abcacb)^*$. Then $|u|$ is a multiple of 6 and $xy \in (abcacb)^*$.

Proof: We first note that the case where $u = 1$ is obvious while it is impossible that u has length 1 since no square of a letter belongs to $F((abcacb)^*)$. Hence, we may consider the prefix of length 2 of u , which we denote v . Then v must be one of the words ab, bc, ca, ac, cb, ba . Let p, q be words such that pvq belongs to $(abcacb)^*$. Note that, whatever the value of v , its first letter can only appear in one position within $abcacb$, so that $|p|$ is completely determined modulo 6 by the value of v . For instance, if $v = ab$, then we must have $|p| \equiv 0 \pmod{6}$.

Let $u = vw$. By the above, since $xvwvwy$ is a power of $abcacb$, we conclude that $|x| \equiv |xvw| \pmod{6}$, which yields $|u| = |vw| \equiv 0 \pmod{6}$. Thus, whichever position in $abcacb$ the factor u^2 starts in the power xu^2y of $abcacb$, the factor y starts exactly in the same position. Hence, xy belongs to $(abcacb)^*$. \square

Lemma 5.4. The syntactic ordered monoid of L_2 satisfies the inequality $1 \leq x^2$.

Proof: We must show that, if p, q are words such that $pq \in L_2$, then $pu^2q \in L_2$ for every word $u \in A^*$.

Suppose first that pq belongs to the language $A^* \setminus F((abcacb)^*)$. We claim that pu^2q belongs to the same language, whence to L_2 . For this purpose, we argue by contradiction, assuming pu^2q is a factor of some power of $abcacb$, that is, there exist words x, y such that xpu^2qy belongs to $(abcacb)^*$. By Lemma 5.3, it follows that $xpqy$ also belongs to $(abcacb)^*$, which contradicts the assumption that pq does not belong to $F((abcacb)^*)$.

Hence, we may assume that pq belongs to $(abcacb)^{1+2*}$, so that there is an integer k such that $pq = (abcacb)^{1+2k}$. If pu^2q is not in $F((abcacb)^*)$, then it belongs to L_2 and we are done. Thus, we assume that pu^2q belongs to $F((abcacb)^*)$ and we choose words x, y such that $xpu^2qy = (abcacb)^\ell$ for some integer ℓ . By Lemma 5.3, there is some integer ℓ' such that $|u| = 6\ell'$ and $xpqy = (abcacb)^{\ell-2\ell'}$. Since $pq = (abcacb)^{1+2k}$, and there are no nontrivial overlaps between the word $abcacb$ with itself, there exist integers r, s such that $x = (abcacb)^r$ and $y = (abcacb)^s$. This yields the equality $\ell - 2\ell' = r + s + 1 + 2k$, that is, $\ell - (r + s) = 1 + 2k + 2\ell'$, which shows that $pu^2q = (abcacb)^{1+2k+2\ell'}$ is a word in $(abcacb)^{1+2*}$, whence also in L_2 . \square

Proof of Proposition 5.2: In view of Lemma 5.4, to complete the proof of Proposition 5.2, it remains to show that the syntactic monoid of L_2 fails the pseudoidentity $(xyzxzy)^{\omega+1} = (xyzxzy)^\omega$. Indeed, substituting the syntactic classes of a, b, c respectively for the variables x, y, z , we obtain for $(xyzxzy)^{\omega+1}$ the syntactic class of a word of the form $(abcacb)^k$, where k is odd, which belongs to L_2 , whereas for ℓ even, $(abcacb)^\ell$ does not belong to L_2 ; hence, the value we obtain for $(xyzxzy)^\omega$ cannot be the same as for $(xyzxzy)^{\omega+1}$. \square

For $n \geq 3$, the argument is similar, but we need to work with a more complicated word.

Proposition 5.5. *Let $n \geq 3$, $A = \{a, b\}$, $w = (b^{n-1}a)^{n-1}(ab)^{n-1}a^2$, and consider the following language:*

$$L_n = (A^* \setminus F(w^*)) \cup w^{1+n*}.$$

Then the syntactic ordered monoid of L_n belongs to $\llbracket 1 \leq x^n \rrbracket$ and fails the pseudoidentity

$$((y^{n-1}x)^{n-1}(xy)^{n-1}x^2)^{\omega+1} = ((y^{n-1}x)^{n-1}(xy)^{n-1}x^2)^\omega.$$

The proof of Proposition 5.5 proceeds along the same lines of the above proof for Proposition 5.2. The only point where there is an essential difference is in the analogue of Lemma 5.3, and that is the only detail which we present here. The role of the number 6 is now played by $n^2 + n = |(b^{n-1}a)^{n-1}(ab)^{n-1}a^2|$.

Lemma 5.6. *Let $n \geq 3$, $w = (b^{n-1}a)^{n-1}(ab)^{n-1}a^2$, and suppose that x, u, y are words such that xu^ny belongs to w^* . Then $n^2 + n$ divides $|u|$ and xy also belongs to w^* .*

Proof: In case $u = 1$, the result is immediate. Suppose a^2b^2 is not a factor of u^n . Then, u^n must be a factor of $a(b^{n-1}a)^{n-1}(ab)^{n-1}a^2b$, which is easily seen to be impossible for a nonempty word u . Hence, a^2b^2 must be a factor of u^n and, therefore, also of u^2 .

Now, there is only one position where the factor a^2b^2 appears in w^2 , namely as $(b^{n-1}a)^{n-1}(ab)^{n-1} \cdot a^2b^2 \cdot b^{n-2}a(b^{n-1}a)^{n-2}(ab)^{n-1}$. In w^k , two such consecutive positions are at distance $n^2 + n$. Hence, whenever pa^2b^2q is a power of w , the value of $|p|$ modulo $n^2 + n$ is constant.

Let $u^2 = pa^2b^2q$, where a^2b^2 is not a factor of p . By assumption, we know that $xu^ny \in w^*$. Since $xp \cdot a^2b^2 \cdot qu^{n-2}y = xup \cdot a^2b^2 \cdot qu^{n-3}y$ is a power of w , we conclude from the preceding paragraph that $|xp| \equiv |xup| \pmod{n^2 + n}$, which yields $|u| \equiv 0 \pmod{n^2 + n}$. Hence, in the factorization of xu^ny as a power of w , the position in w where the factor x ends must be followed, in a later occurrence of w , precisely by the position where the factor y starts. Thus, the factor u^n may be removed to show that xy is also a power of w . \square

Combining Propositions 5.2 and 5.5, we obtain the following result.

Corollary 5.7. *For $n \geq 2$, the pseudovariety of ordered monoids $\llbracket 1 \leq x^n \rrbracket$ is not contained in $A \textcircled{m} \llbracket x^n = 1 \rrbracket$. In particular, $\llbracket 1 \leq x^n \rrbracket$ is contained in neither $J \textcircled{m} \llbracket x^n = 1 \rrbracket$ nor $J * \llbracket x^n = 1 \rrbracket$.*

Proof: We observe that the pseudoidentities $(xyzxzy)^{\omega+1} = (xyzxzy)^\omega$ and

$$((y^{n-1}x)^{n-1}(xy)^{n-1}x^2)^{\omega+1} = ((y^{n-1}x)^{n-1}(xy)^{n-1}x^2)^\omega$$

hold in the pseudovariety $A \textcircled{m} \llbracket x^n = 1 \rrbracket$ respectively in case $n = 2$ and $n > 2$. Hence, it suffices to apply, respectively, Propositions 5.2 and 5.5. \square

Another pseudovariety of interest is $(EJ)_n$, which is defined to be the class of all finite monoids M such that the submonoid generated by $\{s^n : s \in M\}$ is \mathcal{J} -trivial. This is clearly contained in EJ, where only the submonoid generated by the idempotents is required to be \mathcal{J} -trivial, and satisfies the pseudoidentity $x^{\omega+n} = x^\omega$ so that $(EJ)_n \subseteq B \llbracket x^n = 1 \rrbracket$.

The diagram in Figure 1 summarizes the known inclusions between various pseudovarieties that we have considered so far. We next justify the strict inclusions depicted in the diagram in case $n \geq 2$.

- (5) This follows from Lemma 3.2. Alternatively, it also follows from (4) since, for the two other sides of the diamond involving the inclusions (4) and (5), one goes up by taking the pseudovariety of monoids generated by a pseudovariety of ordered monoids.
- (6) The equality $\langle \llbracket 1 \leq x^n \rrbracket \rangle = \llbracket 1 \leq x^n \rrbracket$ means that, for every $(M, \leq) \in \llbracket 1 \leq x^n \rrbracket$, we have $(M, =) \in \llbracket 1 \leq x^n \rrbracket$, so that $\llbracket 1 \leq x^n \rrbracket = \llbracket x^n = 1 \rrbracket$, which contradicts (5).
- (7) The argument is similar to that given for (6) and is omitted.

For (4), we may also prove the following stronger result.

Proposition 5.8. *Whenever $n \geq 2$, there is no pseudovariety V such that $\langle \llbracket 1 \leq x^n \rrbracket \rangle = J * V$.*

Proof: Assume to the contrary that there is such a pseudovariety V . In particular, we have $J * V \subseteq \text{BG}$. If $\text{Sl} \subseteq V$, then it follows that $\text{Sl} * \text{Sl} \subseteq \text{BG}$. However, it is easy to see that the monoid consisting of the two-element left-zero semigroup with an identity adjoined satisfies the identities defining $\text{Sl} * \text{Sl}$ (Almeida, 1995, Exercise 10.3.7), while it is not in BG . Hence, Sl is not contained in the pseudovariety of monoids V , which implies that V is a pseudovariety of groups. On the other hand, we know that $(J * V) \cap G = V \cap G$ (see (Almeida, 1995, Proposition 10.1.7)). We conclude that

$$V = V \cap G = (J * V) \cap G = \langle \llbracket 1 \leq x^n \rrbracket \rangle \cap G = \llbracket x^n = 1 \rrbracket,$$

and so $\llbracket 1 \leq x^n \rrbracket \subseteq J * V = J * \llbracket x^n = 1 \rrbracket$, which contradicts Corollary 5.7. \square

We can also prove the following result for the Mal'cev product.

Proposition 5.9. *If there is some pseudovariety of monoids V such that $\langle \llbracket 1 \leq x^n \rrbracket \rangle = J \circledast V$, then $J \circledast \llbracket x^n = 1 \rrbracket \subseteq \langle \llbracket 1 \leq x^n \rrbracket \rangle$.*

Proof: Suppose that the equality $\langle \llbracket 1 \leq x^n \rrbracket \rangle = J \circledast V$ holds. It follows that

$$\llbracket x^n = 1 \rrbracket = G \cap \langle \llbracket 1 \leq x^n \rrbracket \rangle = (J \circledast V) \cap G = V \cap G, \quad (1)$$

where only the last equality remains to be justified. The inclusion \supseteq is a consequence of $V \subseteq J \circledast V$. For the reverse inclusion, suppose that G is a group from $J \circledast V$. Then, there is a relational morphism $\mu : G \rightarrow V$ onto some $V \in V$ such that, for every idempotent e from V , $\mu^{-1}(e)$ is a semigroup from J . Since G is a group and $J \cap G$ is the trivial pseudovariety, consisting only of singleton monoids, we deduce that $\mu^{-1}(e) = \{1\}$ for every idempotent e from V . Consider now the canonical factorization of μ : μ may be viewed as a submonoid of $G \times V$; we denote φ and ψ respectively the restrictions to μ of the first and second component projections of the product $G \times V$; then, as a relation, $\mu = \varphi^{-1}\psi$. Since φ is onto, there is a subgroup H of μ such that $\varphi(H) = G$ (see, for instance, (Rhodes and Steinberg, 2009, Proposition 4.1.44)). Consider the subgroup $K = \psi(H)$ of V . Note that

$$\psi^{-1}(1_K) = \{(g, 1_K) \in \mu : g \in G\} = \mu^{-1}(1_K) \times \{1_K\} = \{(1, 1_K)\}.$$

Thus, the group kernel of the homomorphism ψ is trivial, so that H and K are isomorphic, whence G belongs to V since so does K .

From (1), we know that $\llbracket x^n = 1 \rrbracket$ is contained in V , which finally entails the required inclusion $J \circledast \llbracket x^n = 1 \rrbracket \subseteq \langle \llbracket 1 \leq x^n \rrbracket \rangle$. \square

Although we are interested mainly in the comparison of several pseudovarieties and the computation of $\langle \llbracket 1 \leq x^n \rrbracket \rangle$ in case n is an integer with $n \geq 2$, the cases $n = 1$ and $n = \omega$ are also of great interest. In fact, they have deserved considerable attention in the literature.

The case $n = 1$ is that of the pseudovariety $J^+ = \llbracket 1 \leq x \rrbracket$. It has been shown to be equivalent to a celebrated theorem of Simon (1975) that $\langle J^+ \rangle = J$. A direct algebraic proof of this fact has been given by Straubing and Thérien (1988). In this case, we also have $B\llbracket x = 1 \rrbracket \neq (EJ)_1 = (BG)_1 = J = J * \llbracket x = 1 \rrbracket$.

In case $n = \omega$, we know that $J * G = BG$ (see Pin (1995a)) and $J^+ \overline{m} G = \llbracket 1 \leq x^\omega \rrbracket$ (Pin and Weil, 1997, Theorem 2.7). There are proofs of these facts in the literature that depend on a deep result of Ash (1991). In the case of the first equality, an alternative “constructive” proof can be found in Auinger and Steinberg (2005b). Hence, by Corollary 3.8, the equality $\langle \llbracket 1 \leq x^\omega \rrbracket \rangle = BG$ holds.

6 Algebraically provable inequalities

An inequality $u' \leq v'$ is said to be a *direct consequence* of the inequality $u \leq v$ if $u, v \in \overline{\Omega}_A M$, $u', v' \in \overline{\Omega}_B M$, and there is a continuous homomorphism $\varphi : \overline{\Omega}_A M \rightarrow \overline{\Omega}_B M$ such that $u' = \varphi(u)$ and $v' = \varphi(v)$.

By an *algebraic proof* of an inequality $u \leq v$ from a set Σ of inequalities we mean a pair of finite sequences of pseudowords $(x_i y_i z_i)_{i=1, \dots, m}$ and $(t_i)_{i=1, \dots, m}$ such that $u = x_1 y_1 z_1$, $v = x_m t_m z_m$, each inequality $y_i \leq t_i$ is a direct consequence of some inequality from Σ , and $x_i t_i z_i = x_{i+1} y_{i+1} z_{i+1}$ ($i = 1, \dots, m-1$). In case there exists such a sequence, we also say that the inequality $u \leq v$ is *algebraically provable* from Σ . A pseudoidentity $u = v$ is *algebraically provable* from Σ if both inequalities $u \leq v$ and $v \leq u$ have that property.

Note that, in particular, $u \leq v$ is algebraically provable from $1 \leq x^n$ if and only if v may be obtained from u by a finite sequence of insertions of factors of the form w^n .

Since our aim is to show that $\langle \llbracket 1 \leq x^n \rrbracket \rangle = (BG)_n$ and we already know that the inclusion from left to right holds, by Reiterman’s theorem, see Reiterman (1982), this amounts to showing that every pseudoidentity valid in the pseudovariety $\langle \llbracket 1 \leq x^n \rrbracket \rangle$ is also valid in $(BG)_n$. The following proposition gives a key connection between inequalities provable from $1 \leq x^n$ and pseudidentities valid in $(BG)_n$.

Proposition 6.1. *The pseudovariety $(BG)_n$ satisfies the following pseudidentities:*

- (a) $v^{\omega+1} = u^{\omega+1} v^\omega = v^\omega u^{\omega+1}$ whenever the inequality $u \leq v$ is algebraically provable from $1 \leq x^n$;
- (b) $v^\omega = u^\omega v^\omega = v^\omega u^\omega$ whenever the inequality $u \leq v$ is algebraically provable from $1 \leq x^n$;
- (c) $u^{\omega+1} = v^{\omega+1}$ whenever the pseudoidentity $u = v$ is algebraically provable from $1 \leq x^n$.

Proof: (a) Consider an algebraic proof of $u \leq v$ from $1 \leq x^n$, given by a pair of sequences of pseudowords $(x_i z_i)_{i=1, \dots, m}$ and $(y_i)_{i=1, \dots, m}$. Note that, for each i , since $1 \leq y_i$ is a direct consequence of $1 \leq x^n$, there is t_i such that $y_i = t_i^n$. Let $u_i = x_i z_i$ for $i = 1, \dots, m$ and $u_{m+1} = x_m t_m^n z_m = v$. We prove, by induction on i , that

$$(BG)_n \text{ satisfies } u_i^{\omega+1} = u^{\omega+1} u_i^\omega. \quad (2)$$

For $i = m+1$, this gives one of the pseudidentities in (a), the other one being obtained dually.

Since $u_1 = u$, (2) is immediate for $i = 1$. Suppose that (2) holds for a certain $i \leq m$. Then, in view of Corollary 4.4 and the induction hypothesis, $(\text{BG})_n$ satisfies the following pseudoidentities:

$$\begin{aligned} u_{i+1}^{\omega+1} &= (x_i t_i^n z_i)^{\omega+1} = (x_i z_i)^{\omega+1} (x_i t_i^n z_i)^\omega = u_i^{\omega+1} u_{i+1}^\omega \\ &= u^{\omega+1} u_i^\omega u_{i+1}^\omega = u^{\omega+1} (x_i z_i)^\omega (x_i t_i^n z_i)^\omega \\ &= u^{\omega+1} (x_i t_i^n z_i)^\omega = u^{\omega+1} u_{i+1}^\omega, \end{aligned}$$

which completes the induction step for the proof of (2).

(b) This can be established by a slight modification of the proof of (a), namely by replacing (2) by $(\text{BG})_n$ satisfies $u_i^\omega = u^\omega u_i^\omega$.

(c) From (a) and (b), it follows that $(\text{BG})_n$ satisfies the pseudoidentities

$$v^{\omega+1} = u^{\omega+1} v^\omega = u^{\omega+1} u^\omega v^\omega = u^{\omega+1}.$$

□

7 More general proofs

Let Σ be a set of inequalities $u \leq v$ with u and v pseudowords over some finite alphabet. We are interested in allowing more general proofs of the validity of inequalities in the pseudovariety $\llbracket \Sigma \rrbracket$ than those considered in Section 6. For simplicity, we fix the finite set A of variables on which we consider such provable inequalities. The definitions below extend to the case of inequalities those previously considered by the authors for pseudoidentities, see Almeida and Klíma (2018).

For each ordinal α , we define recursively a set Σ_α of inequalities over A as follows:

- Σ_0 consists of all diagonal pairs (w, w) , with $w \in \overline{\Omega}_A M$, together with all pairs of the form $(x\varphi(u)y, x\varphi(v)y)$ such that $u \leq v$ is an inequality from Σ , say with $u, v \in \overline{\Omega}_B M$, $\varphi : \overline{\Omega}_B M \rightarrow \overline{\Omega}_A M$ is a continuous homomorphism, and $x, y \in \overline{\Omega}_A M$;
- $\Sigma_{2\alpha+1}$ is the transitive closure of the binary relation $\Sigma_{2\alpha}$;
- $\Sigma_{2\alpha+2}$ is the topological closure of the relation $\Sigma_{2\alpha+1}$ in the space $\overline{\Omega}_A M \times \overline{\Omega}_A M$;
- if α is a limit ordinal, then $\Sigma_\alpha = \bigcup_{\beta < \alpha} \Sigma_\beta$.

Note that Σ_1 consists of the algebraically provable inequalities and that, if $\Sigma_{\alpha+2} = \Sigma_\alpha$, then Σ_α is both transitive and topologically closed, so that $\Sigma_\beta = \Sigma_\alpha$ for every ordinal β with $\beta \geq \alpha$. Since $\overline{\Omega}_A M$ is a metric space, such a condition must hold for α at most the least uncountable ordinal (see (Almeida and Klíma, 2018, Proposition 3.1) for a justification in the unordered case, which applies equally well to the ordered case). Hence, the union $\tilde{\Sigma} = \bigcup_\alpha \Sigma_\alpha$ defines a transitive closed binary relation on $\overline{\Omega}_A M$.

Consider a binary relation θ on $\overline{\Omega}_A M$. We say that θ is *stable* if $(u, v) \in \theta$ and $x, y \in \overline{\Omega}_A M$ implies $(xuy, xvy) \in \theta$. We also say that θ is *fully invariant* if, for every continuous endomorphism φ of $\overline{\Omega}_A M$ and $(u, v) \in \theta$, we have $(\varphi(u), \varphi(v)) \in \theta$.

The next result is the order analog of (Almeida and Klíma, 2018, Proposition 3.1), with a proof following the very same lines, which is therefore omitted.

Proposition 7.1. *The relation $\tilde{\Sigma}$ is a fully invariant closed stable quasiorder on $\overline{\Omega}_A M$. For every $(u, v) \in \tilde{\Sigma}$, the inequality $u \leq v$ is valid in $[[\Sigma]]$.*

The pairs from $\tilde{\Sigma}$, which are viewed as inequalities, are said to be *provable from Σ* . We also say that a pseudoidentity $u = v$ is *provable from Σ* if so are both inequalities $u \leq v$ and $v \leq u$.

An alternative way of looking at proofs, which is equivalent in the sense of capturing the same provable inequalities, is to consider a transfinite sequence of inequalities in which in each step we allow one of the inequalities of Σ_0 , we take $u \leq w$ if there are two previous steps of the form $u \leq v$ and $v \leq w$, or we take $u \leq v$ provided there is a sequence of earlier steps $(u_n \leq v_n)_n$ with $u = \lim u_n$ and $v = \lim v_n$. The last step in such a proof should be the inequality to be proved.

Several examples of such proofs, can be found in the proofs of Lemma 4.1 and Proposition 4.2. The following is the order analog of (Almeida and Klíma, 2018, Conjecture 3.2). Note that ample evidence for the unordered case is presented in Almeida and Klíma (2018).

Conjecture 7.2. *An inequality $u \leq v$ of elements from $\overline{\Omega}_A M$ is provable from Σ if and only if $[[\Sigma]]$ satisfies $u \leq v$.*

Taking into account the analogues of the results of (Almeida, 1995, Section 3.8) for inequalities, the conjecture is equivalent to showing that $\tilde{\Sigma}$ is a profinite relation in the sense of (Rhodes and Steinberg, 2009, Section 3.1), that is that $\tilde{\Sigma}$ is a closed stable quasiorder such that the quotient by the congruence obtained by taking the intersection with the dual of $\tilde{\Sigma}$ is a profinite monoid. There seems to be no obvious way of establishing such a property.

8 Pseudoidentities provable from $1 \leq x^n$

The aim of this section is to show that, at least under suitable hypotheses, pseudoidentities provable from $1 \leq x^n$ are valid in $(\text{BG})_n$. We start by extending Proposition 6.1.

Proposition 8.1. *The statement of Proposition 6.1 remains true if the adverb “algebraically” is removed.*

Proof: We only handle the analogue of part (a) as (b) is similar and the proof of (c) does not require any changes. So, let Σ consist of the single inequality $1 \leq x^n$ and consider a finite alphabet and the binary relations Σ_α over $\overline{\Omega}_A M$. We prove by transfinite induction on α that, whenever $(u, v) \in \Sigma_\alpha$, $(\text{BG})_n$ satisfies the pseudoidentity $u^{\omega+1}v^\omega = v^{\omega+1}$. The cases of $\alpha = 0$ and $\alpha = 2\beta + 1$, that is, respectively inequalities of the form $xz \leq xy^n z$ or that obtained from inequalities from $\Sigma_{2\beta}$ by transitivity, are already essentially handled in the proof of Proposition 6.1. For the case where $\alpha = 2\beta + 2$, we consider a sequence $(u_k, v_k)_k$ from $\Sigma_{2\beta+1}$ converging to the limit (u, v) . By the induction hypothesis, $(\text{BG})_n$ satisfies each of the pseudoidentities $u_k^{\omega+1}v_k^\omega = v_k^{\omega+1}$. Hence, taking limits on both sides, we conclude that $(\text{BG})_n$ also satisfies $u^{\omega+1}v^\omega = v^{\omega+1}$. Finally, in case α is a limit ordinal, the induction step is immediate since $\Sigma_\alpha = \bigcup_{\beta < \alpha} \Sigma_\beta$. \square

We say that a pseudoword $w \in \overline{\Omega}_A M$ has a certain property over $(\text{BG})_n$ if that property is verified by $\pi(w)$ where $\pi : \overline{\Omega}_A M \rightarrow \overline{\Omega}_A (\text{BG})_n$ is the unique continuous homomorphism sending each $a \in A$ to itself.

Theorem 8.2. *Let $u = v$ be a pseudoidentity provable from $1 \leq x^n$.*

(a) *If u and v are group elements over $(\text{BG})_n$, then $(\text{BG})_n$ satisfies the pseudoidentity $u = v$.*

(b) If there are pseudowords w and z such that the pseudoidentity $u = (wz)^\omega w$ is valid in $(\text{BG})_n$, then so are the pseudoidentities $u = v(zw)^\omega = (wz)^\omega v$.

(c) If u and v are regular over $(\text{BG})_n$, then $(\text{BG})_n$ satisfies the pseudoidentity $u = v$.

Proof: (a) By Proposition 8.1, from the hypothesis that $u = v$ is provable from $1 \leq x^n$, we deduce that $(\text{BG})_n$ satisfies $u^{\omega+1} = v^{\omega+1}$. But, since we are assuming that $(\text{BG})_n$ satisfies $u^{\omega+1} = u$ and $v^{\omega+1} = v$, it follows that it also satisfies $u = v$.

(b) From the assumption that $u = v$ is provable from $1 \leq x^n$ it follows that so is $uz = vz$. Part (a) yields that the pseudoidentities $(vz)^{\omega+1} = (uz)^{\omega+1} = (wz)^{\omega+1} = uz$ hold in $(\text{BG})_n$. Hence, so do $(uz)^\omega = (vz)^\omega = (wz)^\omega$ and the following pseudoidentities:

$$\begin{aligned} uz &= (uz)^{\omega+1} = (vz)^{\omega+1} = vz(uz)^\omega = vz(wz)^\omega \\ \therefore u &= uz \cdot (wz)^{\omega-1} w = vz(wz)^\omega \cdot (wz)^{\omega-1} w = v(zw)^\omega. \end{aligned}$$

The proof that $(\text{BG})_n$ satisfies the pseudoidentity $u = (wz)^\omega v$ is dual.

(c) Since u is regular over $(\text{BG})_n$, there are pseudowords w and z such that the pseudoidentity $u = (wz)^\omega w$ holds in $(\text{BG})_n$. From part (b), it follows that u is both \mathcal{R} and \mathcal{L} below v over $(\text{BG})_n$. By symmetry, we conclude that u and v lie in the same \mathcal{H} -class over $(\text{BG})_n$. Since u is \mathcal{R} -equivalent to $(wz)^\omega$ over $(\text{BG})_n$ and $u = (wz)^\omega v$ holds in $(\text{BG})_n$, so does $u = v$. \square

Theorem 8.2 may be viewed as a hint that the equality of pseudovarieties $\langle [1 \leq x^n] \rangle = (\text{BG})_n$ may hold. Should Conjecture 7.2 hold for the inequality $1 \leq x^n$, the evidence for the equality is even more compelling. At present, we must leave it as an open problem.

Another natural and weaker question is whether $J \overset{\text{m}}{\circlearrowleft} [x^n = 1]$ is contained in $\langle [1 \leq x^n] \rangle$. We have no further partial results in this direction than those that follow from Theorem 8.2.

Other questions worth investigating concerning the pseudovarieties in Figure 1 involve the corresponding relatively free profinite monoids. For instance, using the representation theorem for semidirect products (Almeida, 1995, Theorem 10.2.3), the fact that $\overline{\Omega}_A J$ is countable for every finite set A (Almeida, 1995, Proposition 8.2.1), and the local finiteness of the Burnside pseudovariety $[x^n = 1]$, see Zelmanov (1991), we deduce that $\overline{\Omega}_A (J * [x^n = 1])$ is also countable in case A is finite. We do not know if a similar property holds for any of the pseudovarieties $J \overset{\text{m}}{\circlearrowleft} [x^n = 1]$, $(\text{BG})_n$, or perhaps even $B[x^n = 1]$.

Acknowledgements

The authors would like to thank the referees for their suggestions and comments, which contributed to an improved presentation of our results.

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