Packing coloring of generalized Sierpiński graphs

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The packing chromatic number \(\chi_\rho(G)\) of a graph \(G\) is the smallest integer \(c\) such that the vertex set \(V(G)\) can be partitioned into sets \(X_1, \ldots, X_c\), with the condition that vertices in \(X_i\) have pairwise distance greater than \(i\). In this paper, we consider the packing chromatic number of several families of Sierpiński–type graphs. We establish the packing chromatic numbers of generalized Sierpiński graphs \(S_n^G\) where \(G\) is a path or a cycle (with exception of the cycle of length five) as well as for two families where \(G\) is a connected graph of order four. Furthermore, we prove that the packing chromatic number in the family of Sierpiński–triangle graphs \(ST_n^4\) is bounded from above by 20.

Keywords: coloring, packing coloring, generalized Sierpiński graph

1 Introduction

A \(c\)-coloring of a graph \(G\) is a function \(f\) from \(V(G)\) onto a set \(C = \{1, 2, \ldots, c\}\) (with no additional constraints). The elements of \(C\) are called colors, while the set of vertices with the image (color) \(i\) is denoted by \(X_i\). Let \(u, v\) be vertices of a graph \(G\). The distance between \(u\) and \(v\) in \(G\), denoted by \(d_G(u, v)\), equals the length of a shortest \(u, v\)-path (i.e. a path between \(u\) and \(v\)) in \(G\).

Let \(f\) be a \(c\)-coloring of a graph \(G\) with the corresponding sequence of color classes \(X_1, \ldots, X_c\). If each color class \(X_i\) is a set of vertices with the property that any distinct pair \(u, v \in X_i\) satisfies \(d_G(u, v) > i\), then \(X_i\) is said to be an \(i\)-packing, while the sequence \(X_1, \ldots, X_c\) is called a packing \(c\)-coloring. The smallest integer \(c\) for which there exists a packing \(c\)-coloring of \(G\) is called the packing chromatic number of \(G\) and it is denoted by \(\chi_\rho(G)\), see [Brešar et al. (2007); Goddard et al. (2008)].

If \(n\) is a natural number, let \([n]\) denote the set \(\{0, 1, \ldots, n-1\}\).

Let \(G\) be an undirected graph with vertex set \([k]\). The generalized Sierpiński graph \(S_n^G\) of \(G\) of dimension \(n\) is the graph with vertex set \([k]^n\), while vertices \(u, v \in V(S_n^G)\) are adjacent if and only if there exists \(i \in \{1, 2, 3, \ldots, n\}\) such that:

(i) \(u_j = v_j\) if \(j < i\),
(ii) \(u_i \neq v_i\) and \(u_iv_i \in E(G)\).

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(iii) \( u_j = v_i \) and \( v_j = u_i \) if \( j > i \).

We can also say that if \( uv \) is an edge of \( S^n_G \), there is an edge \( xy \) of \( G \) such that the labels of \( u \) and \( v \) are:
\[
u = wxyy \ldots y, \quad v = wyxx \ldots x,\]
where \( x, y \in [k] \) and \( w \in [k]^\ell, \quad 0 \leq \ell \leq n - 1 \).

The generalized Sierpiński graph \( S^n_G \) can also be constructed recursively from \( k \) copies of \( S^{n-1}_G \) as follows:
- for each \( j \in [k] \) add the label \( j \) in front of the labels of all vertices in \( S^{n-1}_G \) and denote the obtained graph by \( jS^n_G \),
- for any edge \( xy \) of \( G \), add an edge between the vertices \( xyy \ldots y \) and \( yyx \ldots x \) in \( S^n_G \).

Generalized Sierpiński graphs \( S^n_G \) are a natural generalization of Sierpiński graphs with base \( p \), \( S^n_p \), which are generated from complete graphs. In other words, the generalized Sierpiński graph \( S^n_G \) coincides with \( S^n_p \). Partially motivated by the fact that Sierpiński graphs belong to a family of subcubic graphs, Brešar et al. [2016] determined bounds on the packing chromatic number of Sierpiński graphs with base 3. The exact chromatic number on this class of graph was recently determined (see Shao and Vesel).

The packing colorings of generalized Sierpiński graphs as well as of Sierpiński triangle graphs have been studied by Brešar and Ferme [2018] who determined the packing chromatic numbers of \( S^n_G \) for all connected graphs \( G \) on 4 vertices with the exception of two families (generated from \( K_4 - e \) and the paw graph), while for the packing chromatic number of Sierpiński triangle graphs an upper bound 31 was established.

This paper is organized as follows. In the next section, we describe basic definitions as well as the result which provides the upper bounds on the packing chromatic number of generalized Sierpiński graphs and Sierpiński triangle graphs. In Section 3, we report on the packing chromatic number of generalized Sierpiński graphs \( S^n_G \) where \( G \) is a connected graph of order four. With the presented results, we solve the problem of determining the packing chromatic number for this family of graphs. In Section 4, we consider the packing chromatic number of generalized Sierpiński graphs \( S^n_G \), where \( G \) is a path or a cycle. These numbers are determined for all paths and cycles with the exception of a cycle of length five where the exact numbers are found till the dimension six while for other dimensions an upper bound is provided. The paper is concluded with results on the packing chromatic number in the family of Sierpiński–triangle graphs \( ST^n_3 \). In particular, we show that this number is bounded from above by 20 and therefore substantially improve previous results.

2 Preliminaries

Let \( j \in V(G) \). If \( u \in V(S^{n-1}_G) \), then the corresponding vertex in \( S^n_G \) (a “copy” of \( u \) in \( jS^{n-1}_G \)) is of the form \( ju \). Vertices of \( S^n_G \) of the form \( i^n, \quad i \in [k] \), are called the extreme vertices. Note that only the extreme vertices of any \( S^{n-1}_G \) can be end-vertices of edges between distinct copies of \( S^{n-1}_G \) in \( S^n_G \). It follows that for \( u, v \in V(S^{n-1}_G) \) and \( ju, jv \in V(S^n_G) \) we have \( d_{S^{n-1}_G}(u, v) = d_{S^n_G}(ju, jv) \) (see also Klavžar et al. [2013]).

If \( ij \in E(G) \), let \( ij S^n_G \) be the graph obtained from \( S^n_G \) by adding the edge between the extreme vertices \( i^j \) and \( j^i \). We say that \( f \) is an extendable packing \( c \)-coloring of \( S^n_G \) if \( f \) is a packing \( c \)-coloring of \( ij S^n_G \) for every \( ij \in E(G) \).

If \( H_1 \) and \( H_2 \) are subgraphs of a graph \( G \), let \( d_G(H_1, H_2) \) denote the distance between \( H_1 \) and \( H_2 \), i.e. the minimal distance between a vertex of \( H_1 \) and a vertex of \( H_2 \) in \( G \).
Proposition 1 Let $G$ be an undirected graph with vertex set $[k]$, $i, j \in [k]$, and $n > t$. If a generalized Sierpiński graph $S_G^t$ admits an extendable packing $c$-coloring such that $d_{S_G^t}(ij, jS_G^t) > c$ for every $ij \notin E(G)$, $i \neq j$, then $\chi_p(S_G^0) \leq c$.

Proof: Let $f$ be an extendable packing $c$-coloring of $S_G^t$, and let $f'$ be a $c$-coloring of $S_G^{t+1}$ such that $f'$ restricted to $jS_G^t$ equals $f$ for every $j \in E(G)$.

We first show that $f'$ is a packing $c$-coloring of $S_G^{t+1}$. Let $u = iu', v = jv' \in V(S_G^{t+1})$ where $u', v' \in V(S_G^t)$ and $i, j \in V(G)$. We have to show that for $f(u) = f(v) = t$, $t \leq c$, we have $d_{S_G^{t+1}}(u, v) > t$.

If $i = j$, then $iu'$ and $jv'$ belongs to the same copy of $S_G^t$ and since $f$ is a packing $c$-coloring of $S_G^t$, the claim readily follows. If $i \neq j$ and $ij \notin E(G)$, then by $d_{S_G^{t+1}}(iu, jS_G^t) > c$, we have that $d_{S_G^{t+1}}(iu', jv') > c$. Finally, if $ij \in E(G)$, we claim that $d_{S_G^{t+1}}(iu', jv') \geq d_{S_G^{t+1}}(iu, jv') > t$. Note that $ij'$ and $ji'$ are the only vertices of $iS_G^t$ and $jS_G^t$, respectively, that are connected with an edge. It follows that

$$d_{S_G^{t+1}}(iu', jv') = d_{S_G^{t+1}}(i j', jv') + d_{S_G^{t+1}}(ij', jv').$$

Since $j'$ and $i'$ are adjacent in $ijS_G^t$, we have

$$d_{ijS_G^t}(u', v') \leq d_{S_G^t}(u', j') + 1 + d_{S_G^t}(i', v').$$

Equalities $d_{S_G^{t+1}}(iu', ij') = d_{S_G^t}(u', j')$ and $d_{S_G^{t+1}}(ij', jv') = d_{S_G^t}(i', v')$ now yield the assertion. It follows that $f'$ is a packing $c$-coloring of $S_G^{t+1}$.

We next show that $f'$ is a packing $c$-coloring of $ijS_G^{t+1}$. Let $i, j \in V(G)$, $u', v' \in V(S_G^t)$ and $u = iu', v = jv' \in V(S_G^{t+1})$. We show that for $f(u) = f(v) = t$, $t \leq c$, we have $d_{ijS_G^{t+1}}(iu', jv') \geq d_{S_G^{t+1}}(iu, jv') > t$. By the same argument as above, it suffices to show the claim only for $ij \in E(G)$. Note that $ijS_G^{t+1}$ is the graph obtained from $S_G^{t+1}$ by adding the edge between $i^{t+1}$ and $j^{t+1}$. If a shortest path between $iu'$ and $jv'$ does not contain the edge between $i^{t+1}$ and $j^{t+1}$, then $d_{ijS_G^{t+1}}(iu', jv') = d_{S_G^{t+1}}(iu', jv')$ and by the discussion above the claim follows. Otherwise, we have

$$d_{ijS_G^{t+1}}(iu', jv') = d_{S_G^{t+1}}(iu', i^{t+1}) + 1 + d_{S_G^{t+1}}(i^{t+1}, jv').$$

Since $j'$ and $i'$ are adjacent in $ijS_G^t$, we have

$$d_{ijS_G^t}(u', v') \leq d_{S_G^t}(u', j') + 1 + d_{S_G^t}(i', v').$$

Equalities $d_{S_G^{t+1}}(iu', i^{t+1}) = d_{S_G^t}(u', i')$ and $d_{S_G^{t+1}}(j^{t+1}, jv') = d_{S_G^t}(j', v')$ now yield the assertion. Since $f'$ is a packing $c$-coloring of $ijS_G^{t+1}$ for every $ij \in E(G)$, we showed that $f'$ is an extendable packing $c$-coloring of $S_G^{t+1}$. In other words, we showed that a packing $c$-coloring of $S_G^t$ can be obtained by using $f$ for every copy of $S_G^t$ in $S_G^0$. This assertion completes the proof.

The questions of packing coloring for various finite and infinite graphs have been reduced to SAT problems already in Shao and Vesel (2015), Martin et al. (2017) and Koro and Vesel (2018). In this paper, we applied this approach for searching packing $c$-colorings and extendable packing $c$-colorings of generalized Sierpiński graphs as well as Sierpiński triangle graphs. In particular, we applied propositional
formulas presented in [Korže and Vesel] (2018) which transform an instance of a packing coloring problem into a propositional satisfiability test (SAT). We used the SAT-solvers Glucose Syrup 4.1 (see [Audemard and Simon]) and Cryptominisat 5 (see [Soós]) in order to find the solutions of the derived propositional formulas.

Some of the constructed colorings that provide improved upper bounds are not present in the paper. Interested readers are invited to visit the website [https://omr.fnm.si/wp-content/uploads/2017/06/SierpinskiP.pdf](https://omr.fnm.si/wp-content/uploads/2017/06/SierpinskiP.pdf) where all of the obtained colorings are given.

3 Generalized Sierpiński graphs with base graphs on 4 vertices

There are altogether six generalized Sierpiński graphs with base graphs on 4 vertices. As mentioned previously, the packing chromatic numbers of $S_n^G$ for all connected graphs $G$ on 4 vertices have been determined in [Brešar and Ferme] (2018) with the exception of $K_4 - e$ (the graph obtained by removing an edge from $K_4$) and the **paw graph** (the graph obtained by joining one vertex of $K_3$ to a $K_1$).

Fig. 1: A packing 9-coloring of $S_{K_4-e}^3$
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In this section, we establish the packing chromatic numbers for these two families of graphs.

**Theorem 1** If $S^n_{K_4-e}$ is the generalized Sierpiński graph of $K_4 - e$ of dimension $n$, then

$$
\chi_p(S^n_{K_4-e}) = \begin{cases} 
3, & n = 1 \\
6, & n = 2 \\
8, & n = 3 \\
9, & n = 4 \\
10, & n \geq 5 
\end{cases}
$$

**Fig. 2:** An extendable packing 10-coloring of $S^n_{K_4-e}$
Fig. 3: A packing 5-coloring of $S_{\text{paw}}^2$ (left) and packing 7-coloring of $S_{\text{paw}}^3$ (right)

**Proof:** For $n \leq 3$, the result is presented in Bresar and Ferme (2018). Since we showed with SAT solver that neither a packing 8-coloring of $S_{K_4-e}^4$ nor a packing 9-coloring of $S_{K_4-e}^5$ can be obtained, we have the lower bound for $n = 4$ as well as for $n \geq 5$.

The upper bound for $n = 4$ was obtained by a packing 9-coloring of $S_{K_4-e}^4$ presented in Fig. 1 while for $n \geq 5$ we obtained an extendable packing 10-coloring of $S_{K_4-e}^5$ presented in Fig. 2.

**Theorem 2** If $S_{\text{paw}}^n$ is the generalized Sierpiński graph of the paw of dimension $n$, then

$$\chi_{\rho}(S_{\text{paw}}^n) = \begin{cases} 
3, & n = 1 \\
5, & n = 2 \\
7, & n = 3 \\
8, & n \geq 4 
\end{cases}$$

**Proof:** For $n = 1$, this is the result presented in Bresar and Ferme (2018), as well as the lower bounds for $n = 2$ and $n = 3$.

Since we showed with our SAT solver that a packing 7-coloring of $S_{\text{paw}}^n$ cannot be obtained, we have the lower bound for $n \geq 4$.

The upper bounds were obtained by the following constructions: a packing 5-coloring of $S_{\text{paw}}^2$, and a packing 7-coloring of $S_{\text{paw}}^3$ presented in Fig. 3 while for $n \geq 4$ we obtained an extendable packing 8-coloring of $S_{\text{paw}}^5$ presented in Fig. 4.

4 Generalized Sierpiński graphs with base graphs paths and cycles

**Proposition 2** Let $n$ and $k$ be integers with $k \geq 4$ and $n \geq 3$. If $S_{P_k}^n$ is the generalized Sierpiński graph of $P_k$ of dimension $n$, then
(i) \( \chi_p(S^2_{P_k}) \geq 4 \).
(ii) \( \chi_p(S^3_{P_k}) \geq 5 \).

**Proof:** (i) Note that \( A = \{11, 12, 13, 14, 20, 21, 22, 23\} \) is a subset of \( V(S^2_{P_k}) \). It can be seen that the graph induced by \( A \) is isomorphic to the graph \( H' \) depicted in the left-hand side of Fig. Since it shown in [Brešar and Ferme 2018, Lemma 4] that \( \chi_p(H') \geq 4 \), this case is settled.

(ii) The set \( B = \{111, 112, 113, 121, 120, 122, 123, 132, 131, 133, 134, 211, 210, 201, 200, 202, 203, 212, 213, 221, 222\} \) is a subset of \( V(S^3_{P_k}) \). Moreover, the graph induced by \( B \) is isomorphic to the graph \( H \) depicted in the right-hand side of Fig. Since it shown in [Brešar and Ferme 2018, Lemma 4] that \( \chi_p(H) \geq 5 \), the assertion follows. \( \square \)

**Proposition 3** Let \( k \geq 3 \). If \( S^2_{P_k} \) is the generalized Sierpiński graph of \( P_k \) of dimension 2, then
Fig. 5: Graph $H'$ (left) and $H$ (right)

$$
\chi_\rho(S^2_{P_k}) = \begin{cases} 
3, & k = 3 \\
4, & k \geq 4 
\end{cases}
$$

**Proof:** Since $S^2_{P_3}$ induces a path with seven vertices, we have $\chi_\rho(S^2_{P_3}) \geq 3$. Let $f$ be a function from $V(S^2_{P_k})$ to the set of integers, defined by $f(00) = f(02) = f(10) = f(12) = f(20) = f(22) = 1$, $f(01) = f(21) = 2$, and $f(11) = 3$. Obviously, $f$ is a packing 3-coloring of $S^2_{P_3}$ and this case is settled.

For $k \geq 4$, the lower bound is given by Proposition 2. In order to prove the upper bound, note that by the definition of the generalized Sierpiński graph, $S^2_{P_k}$ is composed of $k$ copies of $P_k$ such that an edge joins a vertex $x$ in $iP_k$ with a vertex $y$ in $jP_k$ if and only if $i = j \pm 1$. The edge $xy$ is called a cross edge. Note that $x$ and $y$ are of the form $ij$ and $ji$, respectively.

Let $f$ be a 4-coloring of $S^2_{P_k}$ defined as follows:

$$
f(ij) = \begin{cases} 
1, & j \equiv 0 \pmod{2} \\
2, & j \equiv 1 \pmod{4} \text{ and } i \neq j \\
3, & j \equiv 3 \pmod{4} \text{ and } i \neq j \\
4, & j \equiv 1 \pmod{2} \text{ and } i = j 
\end{cases}
$$

If $i$ is even, then $f$ restricted to $iP_k$ is the sequence $1, 2, 1, 3, 1, 2, 1, 3, \ldots$, while for $i$ odd, we obtain the same sequence with the exception that it admits exactly one vertex with color 4 at the position $i$. It is straightforward to see, that both sequences imply a packing 4-coloring of $P_k$. In order to show that $f$ is a packing 4-coloring of $S^2_{P_k}$, we have to consider vertices which are “close” to a cross edge $xy$. If $i \in [k - 1]$, we may assume without loss of generality that $x = ij = i(i + 1)$ and $y = ji = (i + 1)i$. Let $j \geq 1$, $j \leq k - 2$ and $j \leq k - 3$ for $x^- := i(j - 1)$, $x^- := i(j - 2)$, $x^+ := i(j + 1)$ and $x^{++} := i(j + 2)$, respectively. Analogously, we define $y^-, y^+, y^{++}$ and $y^{+++}$. Since either $f(x) = 1$ or $f(y) = 1$, for all vertices $u$ and $v$, $u \neq v$, with $f(u) = f(v) = 1$ we have $d(u, v) \geq 2$. Moreover, by definition of $f$, for $f(u) = f(v) = 4$ we have $d(u, v) \geq 6$. It follows that we have to study only vertices with color 2 or 3.

By definition of $f$, for $x = ij$ and $y = ji$ we have

$$
f(x) = \begin{cases} 
1, & j \equiv 0 \pmod{2} \\
2, & j \equiv 1 \pmod{4} \\
3, & j \equiv 3 \pmod{4} 
\end{cases}
$$
and

\[ f(y) = \begin{cases} 
1, & j \equiv 1 \pmod{2} \\
2, & j \equiv 2 \pmod{4} \\
3, & j \equiv 0 \pmod{4} 
\end{cases} \]

Thus, we consider the following cases.

(i) If \( f(x) = 1 \), then either \( f(y) = 2 \) or \( f(y) = 3 \). We have that \( f(x^-) = 4 \) and \( f(x^+) = f(x^+) = 1 \). If \( f(y) = 2 \) (resp. \( f(y) = 3 \)), then \( f(x^-) = 3 \) (resp. \( f(x^+) = 2 \)). The vertices closest to \( x^+ \) in \((i + 1)P_k\) with color 3 (resp. 2) are \( y^- \) and \( y^+ \). Since both are at distance four from \( x^+ \), this case is settled.

(ii) If \( f(y) = 1 \), then either \( f(x) = 3 \) or \( f(x) = 2 \). We have that \( f(x^+) = f(x^-) = 1 \). If \( f(x) = 3 \) (resp. \( f(x) = 2 \)), then \( f(x^-) = f(x^+) = 2 \) (resp. \( f(x^-) = f(x^+) = 3 \)). Since \( d(x^-, y) = d(x^+, y) = 3 \), we have to consider only vertices close to \( x \). Note that \( f(y^-) = 4 \), thus, vertices in \((i + 1)P_k\) of color 3 (resp. 2) are clearly at distance at least four from \( y \) and the assertion follows. \( \square \)

**Theorem 3** Let \( k \) and \( n \) be integers. If \( S^n_{P_k} \) is the generalized Sierpiński graph of \( P_k \) of dimension \( n \), then

\[
\chi_\rho(S^n_{P_k}) = \begin{cases} 
3, & k = 3 \text{ and } n \geq 2 \\
4, & k \geq 4 \text{ and } n = 2 \\
5, & k \geq 4 \text{ and } n \geq 3 
\end{cases}
\]

**Proof:** For \( n = 2 \) this is Proposition 3. While for \( n \geq 3 \) the lower bound is given by Proposition 2. If \( k = 3 \) and \( n \geq 3 \) note that the packing 3-coloring of \( S^n_{P_2} \) presented in the proof of Proposition 3 is an extendable packing 3-coloring of this graph.

Let \( n \geq 3, s, i, j \in [k] \) and \( w \in [k]^{n-3} \). We define a 5-labeling \( f \) of \( S^n_{P_k} \) as follows:

\[
f(wsij) = \begin{cases} 
1, & j \equiv 0 \pmod{2} \\
2, & j \equiv 1 \pmod{4} \text{ and } i \neq j \\
3, & j \equiv 3 \pmod{4} \text{ and } i \neq j \\
4, & j \equiv 1 \pmod{2} \text{ and } i = j \text{ and } s \equiv 0 \pmod{2} \\
5, & j \equiv 1 \pmod{2} \text{ and } i = j \text{ and } s \equiv 1 \pmod{2} 
\end{cases}
\]

From the proof of Proposition 3, it follows that a restriction of \( f \) to a copy of \( S^n_{P_k} \) in \( S^n_{P_k} \) is either a packing 4-coloring or a packing 5-coloring of \( S^n_{P_k} \).

In order to see that \( f \) is a packing 5-coloring of \( S^n_{P_k} \), note that an edge of \( S^n_{P_k} \) that does not belong to a copy of \( S^n_{P_k} \) connects vertices \( u \) and \( v \) of the form \( u = wi^{n-\ell-1}, v = wij^{n-\ell-1}, i, j \in [k], 0 \leq \ell \leq n - 3, w \in [k]^{\ell} \text{ and } |i - j| = 1. \)

Assume that \( f \) admits vertices \( \alpha \in V(zjS^n_{P_k}) \) and \( \beta \in V(wiS^n_{P_k}) \) such that \( f(\alpha) = f(\beta) = \xi \) and \( d(\alpha, \beta) \leq \xi \). We may assume without loss of generality that \( i \equiv 0 \pmod{2} \). Thus, by definition of \( f \), we have \( f(u) = 1 \) and \( f(v) \in \{4, 5\} \). Note that \( v \) is of degree three, say \( N(v) = \{u, u', u''\} \). Moreover, all vertices in \( N(v) \) have color 1. It follows that \( \xi \neq 1 \). If \( f(v) = 4 \) (resp. \( f(v) = 5 \)), then the copy of \( S^n_{P_k} \) that contains \( u \) does not admit a vertex with color 4 (resp. 5). It follows that \( \xi \notin \{4, 5\} \). Finally, since the vertices of \( N(u) \setminus \{v\} \) are at distance four from the vertices of \( (N(u') \cup N(u'')) \setminus \{v\} \), we have
Fig. 6: An extendable packing 5-coloring of $S_{C_{10}}^2$.

$\xi \not\in \{2, 3\}$ and we obtain a contradiction. Since we showed that $f$ is a packing 5-coloring of $S_n^p$, the proof is complete.

Let the $n \geq 2$, $i, j \in [k]$ and $w \in [k]^{n-2}$. Note that $S_n^p$ is composed of $k^{n-2}$ copies of $S_{C_k}^2$. We will say that vertices of the form $wij$ belong to a copy of $S_{C_k}^2$ denoted as $wS_{C_k}^2$. Moreover, $wS_{C_k}^2$ is composed of $k$ copies of $C_k$ and a copy that admits the vertex $wij$ will be denoted as $iC_k$.

In the proofs of below propositions, we define a packing coloring of $wS_{C_k}^2$ for every $w \in [k]^{n-2}$ such that the coloring of every $iC_k$ is given either explicitly or as a sequence of colors for vertices $wi, wi(i + 1), wi(i + 2), \ldots, wi(i + k - 1)$ (addition modulo $k$). Let also $wxyz$ stand for the sequence of colors $wxyz$ which can be repeated as needed in order to color all vertices of $iC_k$.

**Proposition 4** Let $k \geq 10$ and $k$ is even or $k \equiv 0 \pmod{4}$. If $S_{C_k}^2$ is the generalized Sierpinski graph of $C_k$ of dimension 2, then $\chi_p(S_{C_k}^2) = 4$.

**Proof:** Note that $S_{C_k}^2$ is composed of $k$ copies of $C_k$ such that for $i, j \in [k]$ an edge joins a vertex $x$ in $iC_k$ with a vertex $y$ in $jC_k$ if and only if $i = j + 1$ (addition modulo $k$). Moreover, $x$ and $y$ are of the form $ij$ and $ji$, respectively.
If $k \equiv 0 \pmod{4}$, then $f$ is a 4-coloring of $S^2_{C_k}$ defined by:

$$f(ij) = \begin{cases} 
1, & j \equiv 0 \pmod{2} \\
2, & j \equiv 1 \pmod{4} \text{ and } i \neq j \\
3, & j \equiv 3 \pmod{4} \text{ and } i \neq j \\
4, & j \equiv 1 \pmod{2} \text{ and } i = j 
\end{cases}$$

For $k \geq 10$ and $k \equiv 2 \pmod{4}$, we define a 4-coloring of $S^2_{C_k}$ as follows. Let $i, j \in [k]$ and let $ij$ be a vertex of $S^2_{C_k}$. We label $iC_k$ such that we start at the vertex $ii$. If $i$ is even (resp. $i$ is odd), we use the sequence $1213141213$ (resp. $41213142131213$).

We can see analogously as in the proof of Proposition 3 that above colorings are packing 4-coloring of $S^2_{C_k}$ for $k \equiv 0 \pmod{4}$ and $k \equiv 2 \pmod{4}$, respectively.

Proposition 5 Let $n \geq 2$ and $k \not\in \{5, 6, 7\}$. If $S^n_{C_k}$ is the generalized Sierpiński graph of $C_k$ of dimension $n$, then $\chi_p(S^n_{C_k}) \leq 5$.

Proof:
Let $k = 4t$, $s, i, j \in [k]$ and $w \in [k]^{n-3}$. We define a 5-coloring $f$ of $S^0_{C_{4t}}$ as follows:

$$f(wsij) = \begin{cases} 
1, & j \equiv 0 \pmod{2} \\
2, & j \equiv 1 \pmod{4} \text{ and } i \neq j \\
3, & j \equiv 3 \pmod{4} \text{ and } i \neq j \\
4, & j \equiv 1 \pmod{2} \text{ and } i = j \text{ and } s \equiv 0 \pmod{2} \\
5, & j \equiv 1 \pmod{2} \text{ and } i = j \text{ and } s \equiv 1 \pmod{2}
\end{cases}$$

Analogously as in the proof of Theorem 3, we can show that $f$ is a packing 5-coloring of $S^0_{C_{4t}}$.

Let $t \geq 2$, $k = 4t+2$, $i, j \in [k]$ and $w \in [k]^{n-2}$. We define a coloring of $wS^0_{C_{4t+2}}$ for every $w \in [k]^{n-2}$. More precisely, we color $iC_{4t+2}$ of $wS^2_{C_{4t+2}}$ such that we start at the vertex $ii$ and consecutively label its vertices with the sequences presented in Table 1.

In order to see that the coloring defined in Table 1 is a packing 5-coloring of $S^0_{C_{4t+2}}$, observe Fig. 6 which represent the application of this procedure to $S^0_{C_{10}}$. Since we can see that the obtained coloring is extendable, we showed that $\chi_p(S^0_{C_{10}}) \leq 5$. Note that the procedure for $t \geq 3$ and $n = 2$ repeats the coloring of the first four cycles (depicted by the dashed line in Fig. 6) in a way that the coloring remains extendable. Moreover, since the coloring of each $C_{4t+2}$ is extended either with the pattern 1312 or 1213,
Packing coloring of generalized Sierpiński graphs

Tab. 1: A coloring of $iC_{t+2}$ in $S_{C_{t+2}}^{n}$

<table>
<thead>
<tr>
<th>$i &lt; 4(t-1)$</th>
<th>$i \equiv 0 \pmod{4}$</th>
<th>$i \equiv 1 \pmod{4}$</th>
<th>$i \equiv 2 \pmod{4}$</th>
<th>$i \equiv 3 \pmod{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5214213121</td>
<td>12132142131</td>
<td>413241231</td>
<td>1312412312</td>
<td></td>
</tr>
</tbody>
</table>

$i \geq 4(t-1)$

<table>
<thead>
<tr>
<th>$i = 4(t-1)$</th>
<th>$i = 4(t-1) + 1$</th>
<th>$i = 4(t-1) + 2$</th>
<th>$i = 4(t-1) + 3$</th>
<th>$i = 4(t-1) + 4$</th>
<th>$i = 4(t-1) + 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5121413121</td>
<td>141123121541231</td>
<td>2131275131</td>
<td>51312412315</td>
<td>4131213121</td>
<td>1312412312</td>
</tr>
</tbody>
</table>

the conditions of extendable coloring are still fulfilled. Thus, we obtain an extendable packing 5-coloring of $S_{C_{t+2}}^{n}$ and the case is settled.

For $k = 4t + 3$, we define a packing 5-coloring of $S_{C_{k}}^{n}$ as described in Table 2. Since we can obtain by this procedure an extendable packing 5-coloring of $S_{C_{4k+3}}^{n}$ depicted in Fig. 7, we can show by using the same arguments as above that $\chi_{p}(S_{C_{4k+3}}^{n}) \leq 5$.

Tab. 2: A coloring of $iC_{t+3}$ in $S_{C_{k}}^{n}$

<table>
<thead>
<tr>
<th>$i \equiv 0 \pmod{4}$</th>
<th>$i \equiv 1 \pmod{4}$</th>
<th>$i \equiv 2 \pmod{4}$, $i \neq 4t+2$</th>
<th>$i \equiv 3 \pmod{4}$</th>
<th>$i = 4t + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>121312412131</td>
<td>41231412131</td>
<td>121312412131</td>
<td>51213412131</td>
<td>51213412132</td>
</tr>
</tbody>
</table>

For $k = 4t + 1$, $t \geq 3$, we can construct a packing 5-coloring of $S_{C_{k}}^{2}$ from an extendable packing 5-coloring of $S_{C_{13}}^{2}$ depicted in Fig. 8. The construction for $t \geq 4$ repeats the first four cycles (depicted by the dashed line) of this graph as needed, while the coloring for each cycle is extended with the pattern $1213$ which can be found in every $C_{13}$ of Figure 8. Note that the conditions of extendable coloring are still fulfilled with this procedure. Since we also found an extendable packing 5-coloring of $S_{C_{k}}^{2}$, it follows that $\chi_{p}(S_{C_{4k+3}}^{n}) \leq 5$ for $t \geq 2$. This assertion concludes the proof.

We will need the following result provided by SAT solver.

**Proposition 6** Let $S_{P_{k}}^{2}$ be the generalized Sierpiński graph of $P_{k}$ of dimension 2. If $f$ (resp. $f'$) is an arbitrary packing 4-coloring of $S_{P_{k}}^{2}$ (resp. $S_{P_{k}}^{2}$), then

(i) $f(22) \not\in \{2, 3\}$ and

(ii) $f'(33)$ and $f'(44)$ are not both equal to 1.

**Corollary 1** Let $k$ be an odd integer. If $S_{C_{k}}^{2}$ is the generalized Sierpiński graph of $C_{k}$ of dimension 2, then $\chi_{p}(S_{C_{k}}^{2}) \geq 5$.

**Proof:** Let $k \geq 9$, $k$ is odd and $i \in [k]$. Suppose that $f$ is a packing 4-coloring of $S_{C_{k}}^{2}$. Note that $S_{C_{k}}^{2}$ and $S_{P_{k}}^{2}$ are subgraphs of $S_{C_{k}}^{2}$. By Proposition 6, for every external vertex $ii$ we have that $f(ii) \not\in \{2, 3\}$. Moreover, two consecutive external vertices $ii$ and $(i + 1)(i + 1)$ (addition modulo $k$) cannot be both colored by 1. It follows that the external vertices of $S_{C_{k}}^{2}$ are alternatively colored by 1 and 4. Since $k$ is odd, this is clearly impossible and we obtain a contradiction. It follows that $\chi_{p}(S_{C_{k}}^{2}) \geq 5$ for every odd $k \geq 9$. 

We will need the following result provided by SAT solver.

**Proposition 6** Let $S_{P_{k}}^{2}$ be the generalized Sierpiński graph of $P_{k}$ of dimension 2. If $f$ (resp. $f'$) is an arbitrary packing 4-coloring of $S_{P_{k}}^{2}$ (resp. $S_{P_{k}}^{2}$), then

(i) $f(22) \not\in \{2, 3\}$ and

(ii) $f'(33)$ and $f'(44)$ are not both equal to 1.

**Corollary 1** Let $k$ be an odd integer. If $S_{C_{k}}^{2}$ is the generalized Sierpiński graph of $C_{k}$ of dimension 2, then $\chi_{p}(S_{C_{k}}^{2}) \geq 5$.
Fig. 9: An extendable packing 6-coloring of $S_2^5$, extendable packing 7-coloring of $S_2^5$, packing 5-coloring of $S_2^6$, extendable packing 6-coloring of $S_2^6$.

Since we establish by SAT solver that a packing 4-coloring of $S_2^5$ and $S_2^7$ also do not exist, the proof is complete. \[\square\]

Let $2\mathbb{N}$ and $2\mathbb{N} + 1$ denote the set of even and odd natural numbers, respectively.

**Theorem 4** Let $n \geq 2$ and $k \geq 4$. If $S_{C_k}^n$ is the generalized Sierpinski graph of $C_k$ of dimension $n$, then

$$\chi_{\rho}(S_{C_k}^n) = \begin{cases} 
4, & n = 2 \text{ and } k \in 2\mathbb{N} \setminus \{6\} \\
5, & n \geq 3 \text{ and } k \not\in \{5, 6, 7\} \text{ or } n = 2 \text{ and } k \in \{6\} \cup 2\mathbb{N} + 1 \setminus \{5\} \\
6, & n \geq 3 \text{ and } k \in \{6, 7\} \text{ or } n \leq 6 \text{ and } k = 5
\end{cases}$$

Moreover, if $n \geq 7$, then $6 \leq \chi_{\rho}(S_{C_k}^n) \leq 7$.

**Proof:** First note that since $S_{P_k}^n$ is a subgraph of $S_{C_k}^n$, by Theorem 3 we have $\chi_{\rho}(S_{P_k}^n) \geq 4$ and $\chi_{\rho}(S_{P_k}^n) \geq 5$, $n \geq 3$. Moreover, if $k$ is odd, we have $\chi_{\rho}(S_{C_k}^n) \geq 5$ by Corollary 1. Since we showed
with SAT solver that a packing 5-colorings of $S_{C_5}^2$, $S_{C_6}^3$ and $S_{C_7}^3$, as well as a packing 4-coloring of $S_{C_6}^2$ do not exist, all lower bounds are settled.

In order to prove the upper bounds, we found a packing 6-coloring of $S_{C_6}^6$, a packing 5-coloring of $S_{C_6}^2$, an extendable packing 7-coloring of $S_{C_5}^5$, an extendable packing 6-coloring of $S_{C_6}^2$ and an extendable packing 6-coloring of $S_{C_7}^2$. (the last four colorings are depicted in Fig. 9). Since the other needed upper bounds follows from Propositions 4 and 5, the proof is complete.

\[ \square \]

5 Sierpiński triangle graphs

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig10}
\caption{A packing 12-coloring of $ST_3^3$}
\end{figure}

The base-3 Sierpiński graphs $S^n$ are defined such that we start with $S^0 = K_1$. For $n \geq 1$, the vertex set of $S^n$ is $[3]^n$ and the edge set is defined recursively as

$E(S^n) = \{\{is, it\} : i \in [3], \{s, t\} \in E(S^{n-1})\} \cup \{\{ij^{n-1}, ji^{n-1}\} | i, j \in [3], i \neq j\}.$

As mentioned in the introduction, for $n \geq 1$, the base-3 Sierpiński graphs $S^n$ are generalized Sierpiński graphs where $\tilde{G} = K_3$. Obviously, for $n \geq 1$, $S^n$ can be constructed from three copies of $S^{n-1}$.
Let $n$ be a nonnegative integer. The class of the Sierpiński triangle graphs $ST_3^n$ is obtained from $S^{n+1}$ by contracting all non–clique edges.

There are various other definitions of Sierpiński triangle graphs, which are based on the fact that their drawings in the plane represent approximations of the Sierpiński triangle fractal (see Hinz et al. (2017)). More intuitively, Sierpiński triangle graphs can be constructed by iteration. We start with a complete graph on 3 vertices, i.e. $ST_3^0$ is the triangle $K_3$. Assume now that $ST_3^n$ is already constructed. $ST_3^{n+1}$ is composed of three copies of $ST_3^n$, in a way that can be seen in Fig. 10, where $ST_3^3$ is composed of three copies of $ST_3^2$. Note that an extreme vertex of a copy of $ST_3^2$ is identified with an extreme vertex of another copy (this procedure is done for exactly two extreme vertices of each copy).

Brešar and Ferme (2018) established the packing chromatic number of Sierpiński triangle graphs $ST_3^n$ for $n \leq 2$ and showed that the packing chromatic number for this class of graphs can be bounded above by 31. This bound is improved in the next theorem.
Theorem 5 Let $ST^n_3$ denote the Sierpiński triangle graph of dimension $n$.

(i) $\chi_\rho(ST^n_3) = \begin{cases} 
3, & n = 0 \\
4, & n = 1 \\
8, & n = 2 \\
12, & n = 3.
\end{cases}$

(ii) $12 \leq \chi_\rho(ST^4_3) \leq 15$.

(iii) $12 \leq \chi_\rho(ST^5_3) \leq 19$.

Moreover, if $n \geq 6$, then $12 \leq \chi_\rho(ST^n_3) \leq 20$

Proof: For $n \leq 2$, this is the result presented in [Brešar and Ferme (2018)]. Since we showed with SAT solver that a packing 11-coloring of $ST^3_3$ cannot be obtained, we have $\chi_\rho(ST^n_3) \geq 12$ for every $n \geq 3$.

The upper bounds were obtained by the following constructions: a packing 12-coloring of $ST^4_3$ presented in Fig. 10, a packing 15-coloring of $ST^4_3$ presented in Fig. 11, a packing 19-coloring of $ST^5_3$, while for $n \geq 6$ we obtained an extendable packing 20-coloring of $ST^n_3$. The last two constructions can be obtained from the authors or in the webpage [https://omr.fnm.um.si/wp-content/uploads/2017/06/SierpinskiP.pdf](https://omr.fnm.um.si/wp-content/uploads/2017/06/SierpinskiP.pdf).

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