Packing coloring of generalized Sierpiński graphs

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received 1st Oct. 2018, revised 22nd Jan. 2019, accepted 24th Jan. 2019.

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer c such that the vertex set $V(G)$ can be partitioned into sets $X_1, ..., X_c$, with the condition that vertices in X_i have pairwise distance greater than i. In this paper, we consider the packing chromatic number of several families of Sierpiński–type graphs. We establish the packing chromatic numbers of generalized Sierpiński graphs S^n_G where G is a path or a cycle (with exception of the cycle of length five) as well as for two families where G is a connected graph of order four. Furthermore, we prove that the packing chromatic number in the family of Sierpiński–triangle graphs ST^n_4 is bounded from above by 20.

Keywords: coloring, packing coloring, generalized Sierpiński graph

1 Introduction

A c*-coloring* of a graph G is a function f from $V(G)$ onto a set $C = \{1, 2, \dots, c\}$ (with no additional constraints). The elements of C are called *colors*, while the set of vertices with the image (color) i is denoted by X_i. Let u, v be vertices of a graph G. The *distance* between u and v in G, denoted by $d_G(u, v)$, equals the length of a shortest u, v-path (i.e. a path between u and v) in G.

Let f be a c-coloring of a graph G with the corresponding sequence of color classes $X_1, ..., X_c$. If each color class X_i is a set of vertices with the property that any distinct pair $u, v \in X_i$ satisfies $d_G(u, v) > i$, then X_i is said to be an i*-packing*, while the sequence $X_1, ..., X_c$ is called a *packing* c*-coloring*. The smallest integer c for which there exists a packing c-coloring of G is called the *packing chromatic number of* G and it is denoted by $\chi_\rho(G)$, see Brešar et al. (2007); Goddard et al. (2008).

If n is a natural number, let $[n]$ denote the set $\{0, 1, \dots, n-1\}$,

Let G be an undirected graph with vertex set $[k]$. The *generalized Sierpiński graph* S^n_G *of* G *of dimension* n is the graph with vertex set $[k]^n$, while vertices $u, v \in V(S^n_G)$ are adjacent if and only if there exists $i \in \{1, 2, 3, \dots, n\}$ such that:

(i) $u_j = v_j$ if $j < i$,

(ii) $u_i \neq v_i$ and $u_iv_i \in E(G)$,

*Supported by the Slovenian Research Agency under the grant J2-7357.

†Supported by the Slovenian Research Agency under the grants P1-0297, J1-7110 and J1-9109.

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

We can also say that if uv is an edge of S_G^n, there is an edge xy of G such that the labels of u and v are: $u = wxyy\dots y$, $v = wyxx\dots x$, where $x, y \in [k]$ and $w \in [k]^\ell$, $0 \le \ell \le n-1$.

The generalized Sierpiński graph S_G^n can also be constructed recursively from k copies of S_G^{n-1} as follows:

- for each $j \in [k]$ add the label j in front of the labels of all vertices in S_G^{n-1} and denote the obtained graph by jS_G^n,

- for any edge xy of G, add an edge between the vertices $xyy\dots y$ and $yxx\dots x$ in S_G^n.

Generalized Sierpiński graphs S_G^n are a natural generalization of *Sierpiński graphs with base* p, S_p^n, which are generated from complete graphs. In other words, the generalized Sierpiński graph $S_{K_p}^n$ coincides with S_p^n. Partially motivated by the fact that Sierpiński graphs belong to a family of subcubic graphs, Brešar et al. (2016) determined bounds on the packing chromatic number of Sierpiński graphs with base 3. The exact chromatic number on this class of graph was recently determined (see Shao and Vesel).

The packing colorings of generalized Sierpiński graphs as well as of Sierpiński triangle graphs have been studied by Brešar and Ferme (2018) who determined the packing chromatic numbers of S_G^n for all connected graphs G on 4 vertices with the exception of two families (generated from $K_4 - e$ and the paw graph), while for the packing chromatic number of Sierpiński triangle graphs an upper bound 31 was established.

This paper is organized as follows. In the next section, we describe basic definitions as well as the result which provides the upper bounds on the packing chromatic number of generalized Sierpiński graphs and Sierpiński triangle graphs. In Section 3, we report on the packing chromatic number of generalized Sierpiński graphs S_G^n where G is a connected graph of order four. With the presented results, we solve the problem of determining the packing chromatic number for this family of graphs. In Section 4, we consider the packing chromatic number of generalized Sierpiński graphs S_G^n, where G is a path or a cycle. These numbers are determined for all paths and cycles with the exception of a cycle of length five where the exact numbers are found till the dimension six while for other dimensions an upper bound is provided. The paper is concluded with results on the packing chromatic number in the family of Sierpiński–triangle graphs ST_3^n. In particular, we show that this number is bounded from above by 20 and therefore substantially improve previous results.

2 Preliminaries

Let $j \in V(G)$. If $u \in V(S_G^{n-1})$, then the corresponding vertex in S_G^n (a "copy" of u in jS_G^{n-1}) is of the form ju. Vertices of S_G^n of the form i^n, $i \in [k]$, are called the *extreme vertices*. Note that only the extreme vertices of any S_G^{n-1} can be end-vertices of edges between distinct copies of S_G^{n-1} in S_G^n. It follows that for $u, v \in V(S_G^{n-1})$ and $ju, jv \in V(S_G^n)$ we have $d_{S_G^{n-1}}(u,v) = d_{S_G^n}(ju, jv)$ (see also Klavžar et al. (2013)).

If $ij \in E(G)$, let ${}^{ij}S_G^\ell$ be the graph obtained from S_G^ℓ by adding the edge between the extreme vertices i^ℓ and j^ℓ. We say that f is an *extendable packing* c*-coloring of* S_G^ℓ if f is a packing c-coloring of ${}^{ij}S_G^\ell$ for every $ij \in E(G)$.

If H_1 and H_2 are subgraphs of a graph G, let $d_G(H_1, H_2)$ denote the distance between H_1 and H_2, i.e. the minimal distance between a vertex of H_1 and a vertex of H_2 in G.

Proposition 1 *Let G be an undirected graph with vertex set $[k]$, $i, j \in [k]$, and $n > \ell$. If a generalized Sierpiński graph S_G^ℓ admits an extendable packing c-coloring such that $d_{S_G^{\ell+1}}(iS_G^\ell, jS_G^\ell) > c$ for every $ij \notin E(G)$, $i \neq j$, then $\chi_\rho(S_G^n) \leq c$.*

Proof: Let f be an extendable packing c-coloring of S_G^ℓ, and let f' be a c-coloring of $S_G^{\ell+1}$ such that f' restricted to jS_G^ℓ equals f for every $j \in E(G)$.

We first show that f' is a packing c-coloring of $S_G^{\ell+1}$. Let $u = iu', v = jv' \in V(S_G^{\ell+1})$ where $u', v' \in V(S_G^\ell)$ and $i, j \in V(G)$. We have to show that for $f(u) = f(v) = t$, $t \leq c$, we have $d_{S_G^{\ell+1}}(u, v) > t$.

If $i = j$, then iu' and jv' belongs to the same copy of S_G^ℓ and since f is a packing c-coloring of S_G^ℓ, the claim readily follows. If $i \neq j$ and $ij \notin E(G)$, then by $d_{S_G^{\ell+1}}(iS_G^\ell, jS_G^\ell) > c$, we have that $d_{S_G^{\ell+1}}(iu', jv') > c$. Finally, if $ij \in E(G)$, we claim that $d_{S_G^{\ell+1}}(iu', jv') \geq d_{{}^{ij}S_G^\ell}(u', v') > t$. Note that ij^ℓ and ji^ℓ are the only vertices of iS_G^ℓ and jS_G^ℓ, respectively, that are connected with an edge. It follows that

$$d_{S_G^{\ell+1}}(iu', jv') = d_{S_G^{\ell+1}}(iu', ij^\ell) + 1 + d_{S_G^{\ell+1}}(ji^\ell, jv').$$

Since j^ℓ and i^ℓ are adjacent in ${}^{ij}S_G^\ell$, we have

$$d_{{}^{ij}S_G^\ell}(u', v') \leq d_{S_G^\ell}(u', j^\ell) + 1 + d_{S_G^\ell}(i^\ell, v').$$

Equalities $d_{S_G^{\ell+1}}(iu', ij^\ell) = d_{S_G^\ell}(u', j^\ell)$ and $d_{S_G^{\ell+1}}(ji^\ell, jv') = d_{S_G^\ell}(i^\ell, v')$ now yield the assertion. It follows that f' is a packing c-coloring of $S_G^{\ell+1}$.

We next show that f' is a packing c-coloring of ${}^{ij}S_G^{\ell+1}$. Let $i, j \in V(G)$, $u', v' \in V(S_G^\ell)$ and $u = iu', v = jv' \in V(S_G^{\ell+1})$. We show that for $f(u) = f(v) = t$, $t \leq c$, we have $d_{{}^{ij}S_G^{\ell+1}}(iu', jv') \geq d_{{}^{ij}S_G^\ell}(u', v') > t$. By the same argument as above, it suffices to show the claim only for $ij \in E(G)$. Note that ${}^{ij}S_G^{\ell+1}$ is the graph obtained from $S_G^{\ell+1}$ by adding the edge between $i^{\ell+1}$ and $j^{\ell+1}$. If a shortest path between iu' and jv' does not contain the edge between $i^{\ell+1}$ and $j^{\ell+1}$, then $d_{{}^{ij}S_G^{\ell+1}}(iu', jv') = d_{S_G^{\ell+1}}(iu', jv')$ and by the discussion above the claim follows. Otherwise, we have

$$d_{{}^{ij}S_G^{\ell+1}}(iu', jv') = d_{S_G^{\ell+1}}(iu', i^{\ell+1}) + 1 + d_{S_G^{\ell+1}}(j^{\ell+1}, jv').$$

Since j^ℓ and i^ℓ are adjacent in ${}^{ij}S_G^\ell$, we have

$$d_{{}^{ij}S_G^\ell}(u', v') \leq d_{S_G^\ell}(u', j^\ell) + 1 + d_{S_G^\ell}(i^\ell, v').$$

Equalities $d_{S_G^{\ell+1}}(iu', i^{\ell+1}) = d_{S_G^\ell}(u', i^\ell)$ and $d_{S_G^{\ell+1}}(j^{\ell+1}, jv') = d_{S_G^\ell}(j^\ell, v')$ now yield the assertion.

Since f' is a packing c-coloring of ${}^{ij}S_G^{\ell+1}$ for every $ij \in E(G)$, we showed that f' is an extendable packing c-coloring of $S_G^{\ell+1}$. In other words, we showed that a packing c-coloring of S_G^n can be obtained by using f for every copy of S_G^ℓ in S_G^n. This assertion completes the proof. □

The questions of packing coloring for various finite and infinite graphs have been reduced to SAT problems already in Shao and Vesel (2015), Martin et al. (2017) and Korže and Vesel (2018). In this paper, we applied this approach for searching packing c-colorings and extendable packing c-colorings of generalized Sierpiński graphs as well as Sierpiński triangle graphs. In particular, we applied propositional formulas presented in Korže and Vesel (2018) which transform an instance of a packing coloring problem into a propositional satisfiability test (SAT). We used the SAT-solvers Glucose Syrup 4.1 (see Audemard and Simon) and Cryptominisat 5 (see Soos) in order to find the solutions of the derived propositional formulas.

Some of the constructed colorings that provide improved upper bounds are not present in the paper. Interested readers are invited to visit the website `https://omr.fnm.um.si/wp-content/uploads/2017/06/SierpinskiP.pdf` where all of the obtained colorings are given.

3 Generalized Sierpiński graphs with base graphs on 4 vertices

There are altogether six generalized Sierpiński graphs with base graphs on 4 vertices. As mentioned previously, the packing chromatic numbers of S_G^n for all connected graphs G on 4 vertices have been determined in Brešar and Ferme (2018) with the exception of $K_4 - e$ (the graph obtained by removing an edge from K_4) and the *paw graph* (the graph obtained by joining one vertex of K_3 to a K_1).

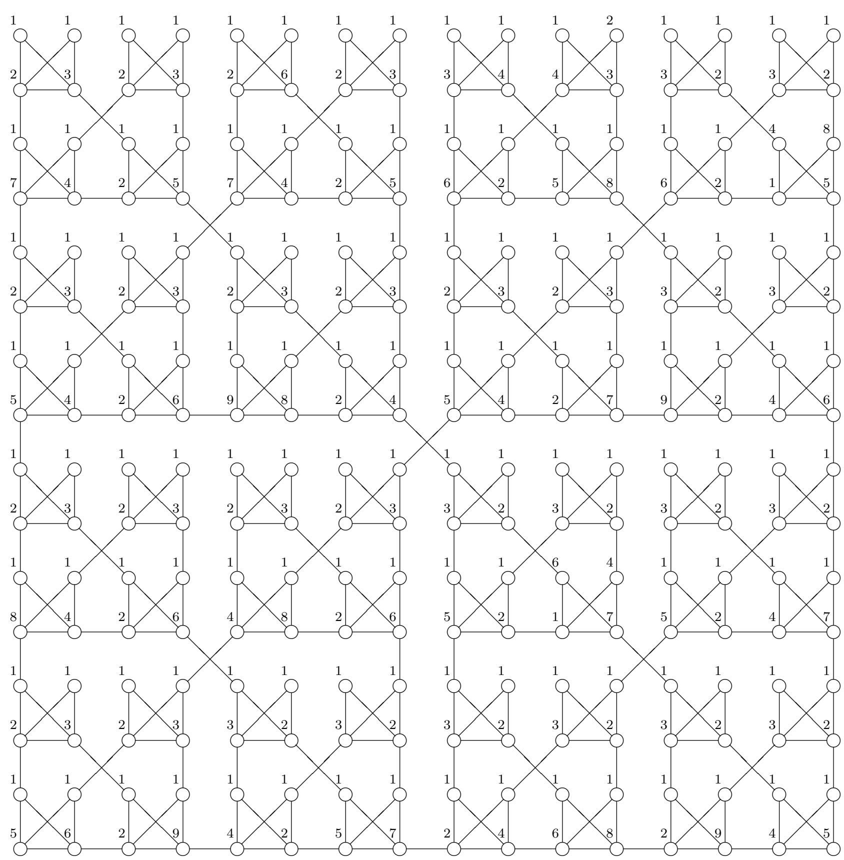

Fig. 1: A packing 9-coloring of $S^4_{K_4-e}$

In this section, we establish the packing chromatic numbers for these two families of graphs.

Theorem 1 *If $S^n_{K_4-e}$ is the generalized Sierpiński graph of K_4-e of dimension n, then*

$$\chi_\rho(S^n_{K_4-e}) = \begin{cases} 3, & n=1 \\ 6, & n=2 \\ 8, & n=3 \\ 9, & n=4 \\ 10, & n\geq 5 \end{cases}$$

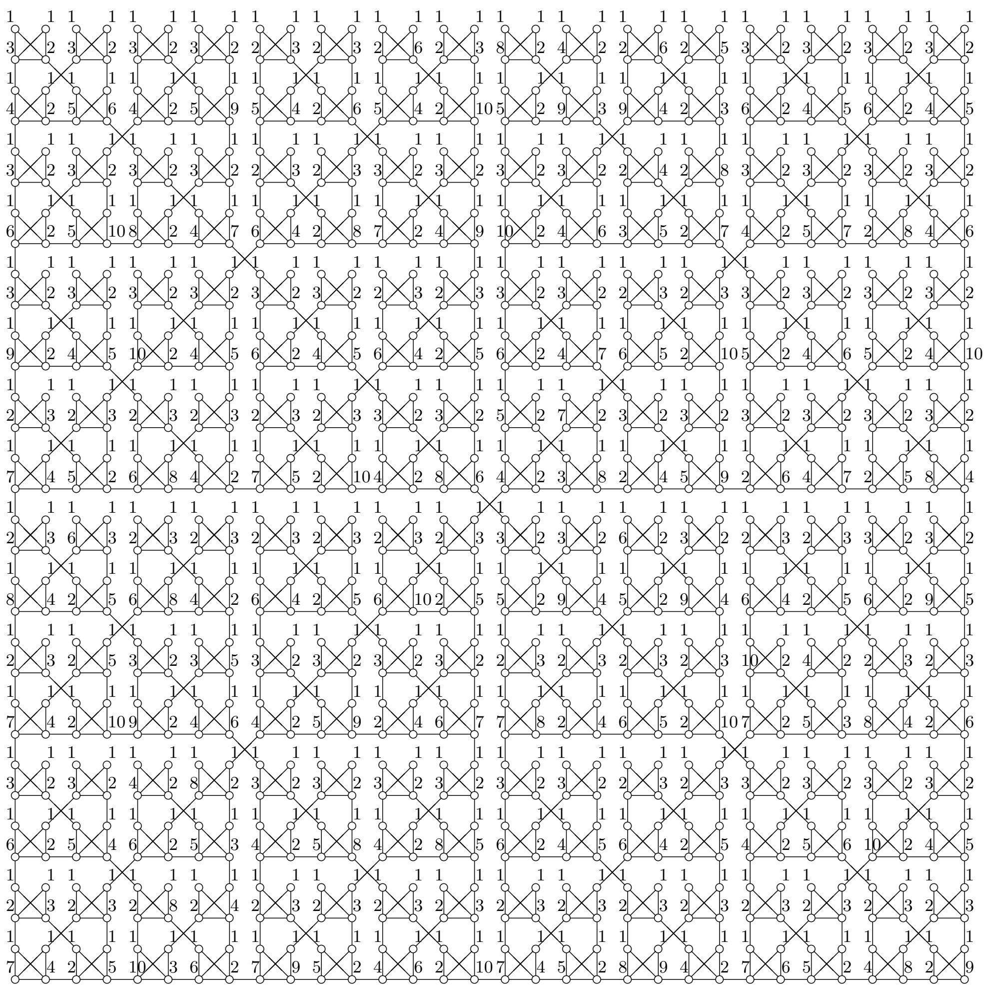

Fig. 2: An extendable packing 10-coloring of $S^5_{K_4-e}$

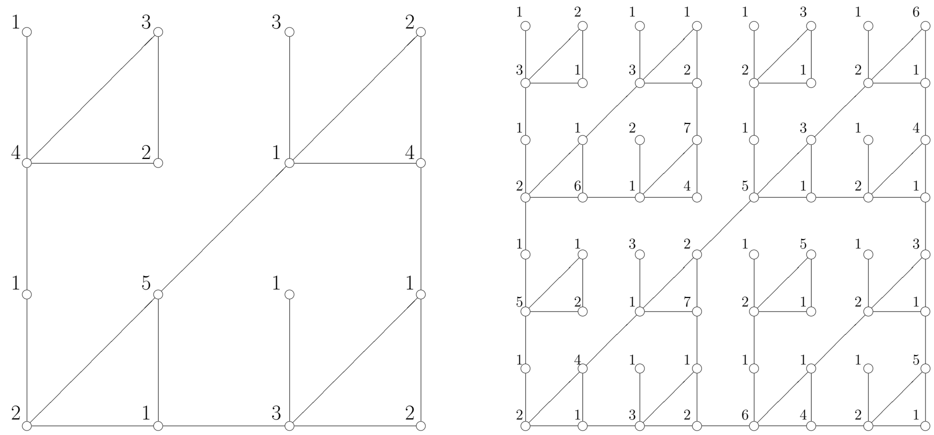

Fig. 3: A packing 5-coloring of S^2_{paw} (left) and packing 7-coloring of S^3_{paw} (right)

Proof: For $n \leq 3$, the result is presented in Brešar and Ferme (2018). Since we showed with SAT solver that neither a packing 8-coloring of $S^4_{K_4-e}$ nor a packing 9-coloring of $S^5_{K_4-e}$ can be obtained, we have the lower bound for $n = 4$ as well as for $n \geq 5$.

The upper bound for $n = 4$ was obtained by a packing 9-coloring of $S^4_{K_4-e}$ presented in Fig. 1, while for $n \geq 5$ we obtained an extendable packing 10-coloring of $S^5_{K_4-e}$ presented in Fig. 2. □

Theorem 2 *If S^n_{paw} is the generalized Sierpiński graph of the paw of dimension n, then*

$$\chi_\rho(S^n_{paw}) = \begin{cases} 3, & n = 1 \\ 5, & n = 2 \\ 7, & n = 3 \\ 8, & n \geq 4 \end{cases}$$

Proof: For $n = 1$, this is the result presented in Brešar and Ferme (2018), as well as the lower bounds for $n = 2$ and $n = 3$.

Since we showed with our SAT solver that a packing 7-coloring of S^4_{paw} cannot be obtained, we have the lower bound for $n \geq 4$.

The upper bounds were obtained by the following constructions: a packing 5-coloring of S^2_{paw} and a packing 7-coloring of S^3_{paw} presented in Fig. 3, while for $n \geq 4$ we obtained an extendable packing 8-coloring of S^5_{paw} presented in Fig. 4.

□

4 Generalized Sierpiński graphs with base graphs paths and cycles

Proposition 2 *Let n and k be integers with $k \geq 4$ and $n \geq 3$. If $S^n_{P_k}$ is the generalized Sierpiński graph of P_k of dimension n, then*

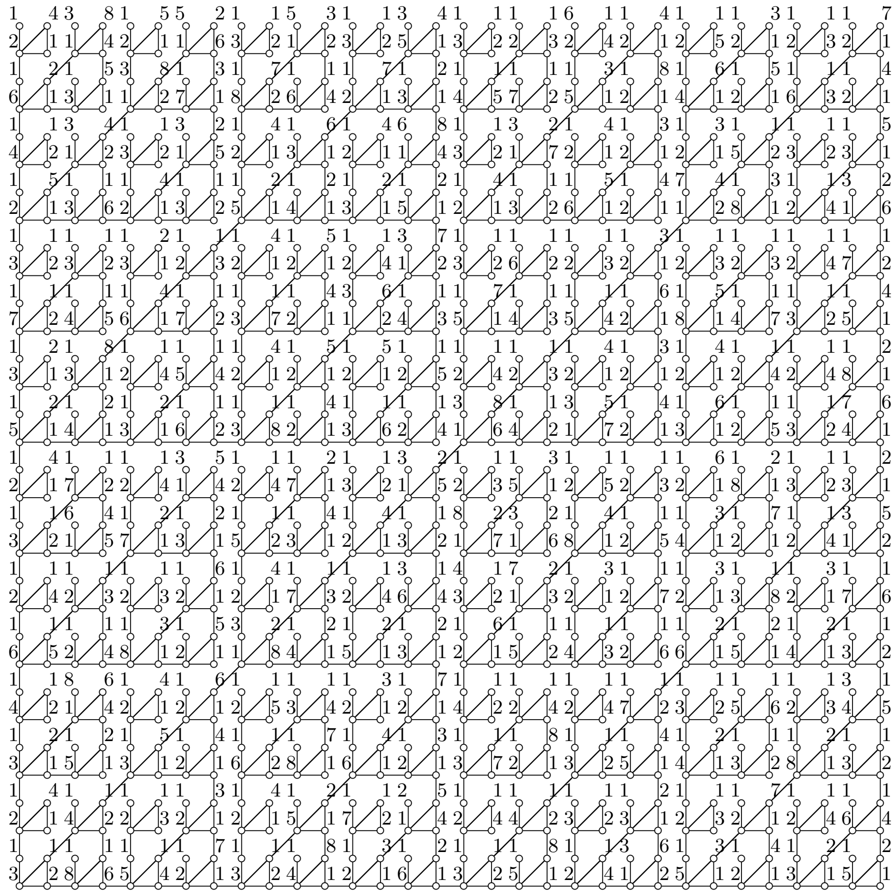

Fig. 4: An extendable packing 8-coloring of S^5_{paw}

(i) $\chi_\rho(S^2_{P_k}) \geq 4$,
(ii) $\chi_\rho(S^n_{P_k}) \geq 5$.

Proof: (i) Note that $A = \{11, 12, 13, 14, 20, 21, 22, 23\}$ is a subset of $V(S^2_{P_k})$. It can be seen that the graph induced by A is isomorphic to the graph H' depicted in the left-hand side of Fig 5. Since it shown in (Brešar and Ferme, 2018, Lemma 4) that $\chi_\rho(H') \geq 4$, this case is settled.

(ii) The set $B = \{111, 112, 113, 121, 120, 122, 123, 132, 131, 133, 134, 211, 210, 201, 200, 202, 203, 212, 213, 221, 222\}$ is a subset of $V(S^3_{P_k})$. Moreover, the graph induced by B is isomorphic to the graph H depicted in the right-hand side of Fig 5. Since it shown in (Brešar and Ferme, 2018, Lemma 4) that $\chi_\rho(H) \geq 5$, the assertion follows. $\square$

Proposition 3 *Let* $k \geq 3$. *If* $S^2_{P_k}$ *is the generalized Sierpiński graph of* P_k *of dimension* 2, *then*

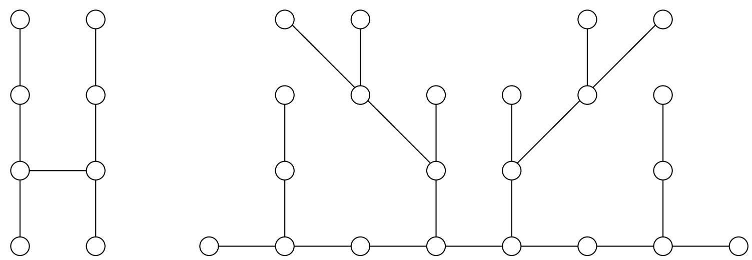

Fig. 5: Graph H' (left) and H (right)

$$\chi_\rho(S^2_{P_k}) = \begin{cases} 3, & k = 3 \\ 4, & k \geq 4 \end{cases}$$

Proof: Since $S^2_{P_3}$ induces a path with seven vertices, we have $\chi_\rho(S^2_{P_k}) \geq 3$. Let f be a function from $V(S^2_{P_3})$ to the set of integers, defined by $f(00) = f(02) = f(10) = f(12) = f(20) = f(22) = 1$, $f(01) = f(21) = 2$ and $f(11) = 3$. Obviously, f is a packing 3-coloring of $S^2_{P_3}$ and this case is settled.

For $k \geq 4$, the lower bound is given by Proposition 2. In order to prove the upper bound, note that by the definition of the generalized Sierpiński graph, $S^2_{P_k}$ is composed of k copies of P_k such that an edge joins a vertex x in iP_k with a vertex y in jP_k if and only if $i = j \pm 1$. The edge xy is called a *cross edge*. Note that x and y are of the form ij and ji, respectively.

Let f be a 4-coloring of $S^2_{P_k}$ defined as follows:

$$f(ij) = \begin{cases} 1, & j \equiv 0 \pmod 2 \\ 2, & j \equiv 1 \pmod 4 \text{ and } i \neq j \\ 3, & j \equiv 3 \pmod 4 \text{ and } i \neq j \\ 4, & j \equiv 1 \pmod 2 \text{ and } i = j \end{cases}$$

If i is even, then f restricted to iP_k is the sequence $1, 2, 1, 3, 1, 2, 1, 3, \ldots$, while for i odd, we obtain the same sequence with the exception that it admits exactly one vertex with color 4 at the position i. It is straightforward to see, that both sequences imply a packing 4-coloring of P_k. In order to show that f is a packing 4-coloring of $S^2_{P_k}$, we have to consider vertices which are "close" to a cross edge xy. If $i \in [k-1]$, we may assume without loss of generality that $x = ij = i(i+1)$ and $y = ji = (i+1)i$. Let $j \geq 1$, $j \geq 2$, $j \leq k-2$ and $j \leq k-3$ for $x^- := i(j-1)$, $x^{--} := i(j-2)$, $x^+ := i(j+1)$ and $x^{++} := i(j+2)$, respectively. Analogously, we define y^{--}, y^-, y^+ and y^{++}. Since either $f(x) = 1$ or $f(y) = 1$, for all vertices u and v, $u \neq v$, with $f(u) = f(v) = 1$ we have $d(u,v) \geq 2$. Moreover, by definition of f, for $f(u) = f(v) = 4$ we have $d(u,v) \geq 6$. It follows that we have to study only vertices with color 2 or 3.

By definition of f, for $x = ij$ and $y = ji$ we have

$$f(x) = \begin{cases} 1, & j \equiv 0 \pmod 2 \\ 2, & j \equiv 1 \pmod 4 \\ 3, & j \equiv 3 \pmod 4 \end{cases}$$

and

$$f(y)=\begin{cases}1, & j\equiv 1 \pmod 2\\ 2, & j\equiv 2 \pmod 4\\ 3, & j\equiv 0 \pmod 4\end{cases}$$

Thus, we consider the following cases.

(i) If $f(x)=1$, then either $f(y)=2$ or $f(y)=3$. We have that $f(x^-)=4$ and $f(x^{--})=f(x^{++})=1$. If $f(y)=2$ (resp. $f(y)=3$), then $f(x^+)=3$ (resp. $f(x^+)=2$). The vertices closest to x^+ in $(i+1)P_k$ with color 3 (resp. 2) are y^{--} and y^{++}. Since both are at distance four from x^+, this case is settled.

(ii) If $f(y)=1$, then either $f(x)=3$ or $f(x)=2$. We have that $f(x^+)=f(x^-)=1$. If $f(x)=3$ (resp. $f(x)=2$), then $f(x^{--})=f(x^{++})=2$ (resp. $f(x^{--})=f(x^{++})=3$). Since $d(x^{--},y)=d(x^{++},y)=3$, we have to consider only vertices close to x. Note that $f(y^-)=4$, thus, vertices in $(i+1)P_k$ of color 3 (resp. 2) are clearly at distance at least four from y and the assertion follows. □

Theorem 3 *Let k and n be integers. If $S^n_{P_k}$ is the generalized Sierpiński graph of P_k of dimension n, then*

$$\chi_\rho(S^n_{P_k})=\begin{cases}3, & k=3 \text{ and } n\ge 2\\ 4, & k\ge 4 \text{ and } n=2\\ 5, & k\ge 4 \text{ and } n\ge 3\end{cases}$$

Proof: For $n=2$ this is Proposition 3, while for $n\ge 3$ the lower bound is given by Proposition 2. If $k=3$ and $n\ge 3$ note that the packing 3-coloring of $S^2_{P_3}$ presented in the proof of Proposition 3 is an extendable packing 3-coloring of this graph.

Let $n\ge 3$, $s,i,j\in[k]$ and $w\in[k]^{n-3}$. We define a 5-labeling f of $S^n_{P_k}$ as follows:

$$f(wsij)=\begin{cases}1, & j\equiv 0 \pmod 2\\ 2, & j\equiv 1 \pmod 4 \text{ and } i\ne j\\ 3, & j\equiv 3 \pmod 4 \text{ and } i\ne j\\ 4, & j\equiv 1 \pmod 2 \text{ and } i=j \text{ and } s\equiv 0 \pmod 2\\ 5, & j\equiv 1 \pmod 2 \text{ and } i=j \text{ and } s\equiv 1 \pmod 2\end{cases}$$

From the proof of Proposition 3 it follows that a restriction of f to a copy of $S^2_{P_k}$ in $S^n_{P_k}$ is either a packing 4-coloring or a packing 5-coloring of $S^2_{P_k}$.

In order to see that f is a packing 5-coloring of $S^n_{P_k}$, note that an edge of $S^n_{P_k}$ that does not belong to a copy of $S^2_{P_k}$ connects vertices u and v of the form $u=wij^{n-\ell-1}$, $v=wji^{n-\ell-1}$, $i,j\in[k]$, $0\le\ell\le n-3$, $w\in[k]^\ell$ and $|i-j|=1$.

Assume that f admits vertices $\alpha\in V(zjS^2_{P_k})$ and $\beta\in V(wiS^2_{P_k})$ such that $f(\alpha)=f(\beta)=\xi$ and $d(\alpha,\beta)\le\xi$. We may assume without loss of generality that $i\equiv 0 \pmod 2$. Thus, by definition of f, we have $f(u)=1$ and $f(v)\in\{4,5\}$. Note that v is of degree three, say $N(v)=\{u,u',u''\}$. Moreover, all vertices in $N(v)$ have color 1. It follows that $\xi\ne 1$. If $f(v)=4$ (resp. $f(v)=5$), then the copy of $S^2_{P_k}$ that contains u does not admit a vertex with color 4 (resp. 5). It follows that $\xi\notin\{4,5\}$. Finally, since the vertices of $N(u)\setminus\{v\}$ are at distance four from the vertices of $(N(u')\cup N(u''))\setminus\{v\}$, we have

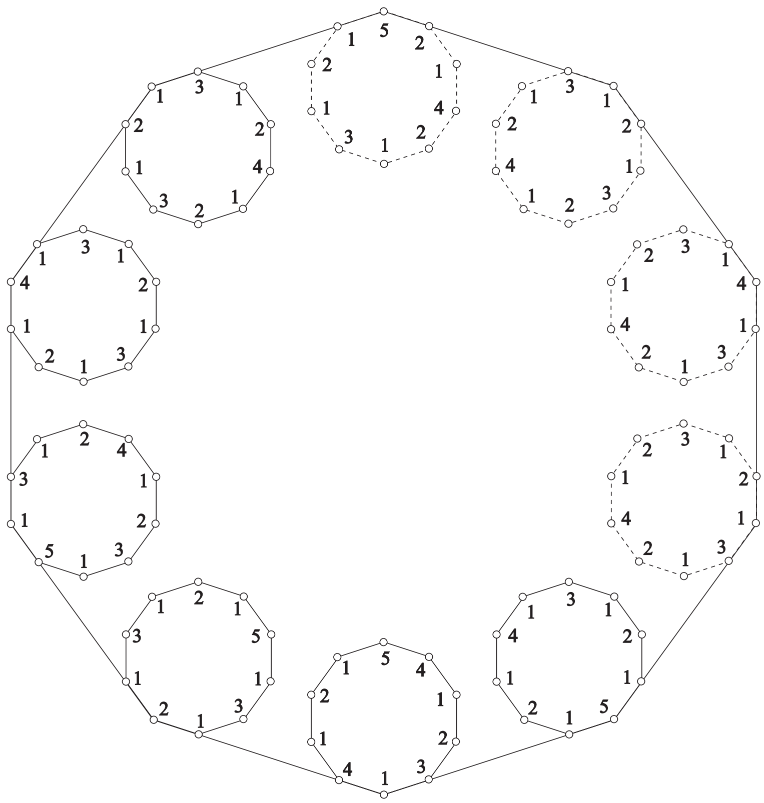

Fig. 6: An extendable packing 5-coloring of $S^2_{C_{10}}$

$\xi \notin \{2, 3\}$ and we obtain a contradiction. Since we showed that f is a packing 5-coloring of $S^n_{P_k}$, the proof is complete. □

Let the $n \geq 2$, $i, j \in [k]$ and $w \in [k]^{n-2}$. Note that $S^n_{C_k}$ is composed of k^{n-2} copies of $S^2_{C_k}$. We will say that vertices of the form wij belong to a copy of $S^2_{C_k}$ denoted as $wS^2_{C_k}$. Moreover, $wS^2_{C_k}$ is composed of k copies of C_k and a copy that admits the vertex wij will be denoted as iC_k.

In the proofs of below propositions, we define a packing coloring of $wS^2_{C_k}$ for every $w \in [k]^{n-2}$ such that the coloring of every iC_k is given either explicitly or as a sequence of colors for vertices $wii, wi(i+1), wi(i+2), \ldots, wi(i+k-1)$ (addition modulo k). Let also $\overline{wxyz}$ stand for the sequence of colors $wxyz$ which can be repeated as needed in order to color all vertices of iC_k.

Proposition 4 *Let $k \geq 10$ and k is even or $k \equiv 0 \ (mod\ 4)$. If $S^2_{C_k}$ is the generalized Sierpiński graph of C_k of dimension 2, then $\chi_\rho(S^2_{C_k}) = 4$*

Proof: Note that $S^2_{C_k}$ is composed of k copies of C_k such that for $i, j \in [k]$ an edge joins a vertex x in iC_k with a vertex y in jC_k if and only if $i = j + 1$ (addition modulo k). Moreover, x and y are of the form ij and ji, respectively.

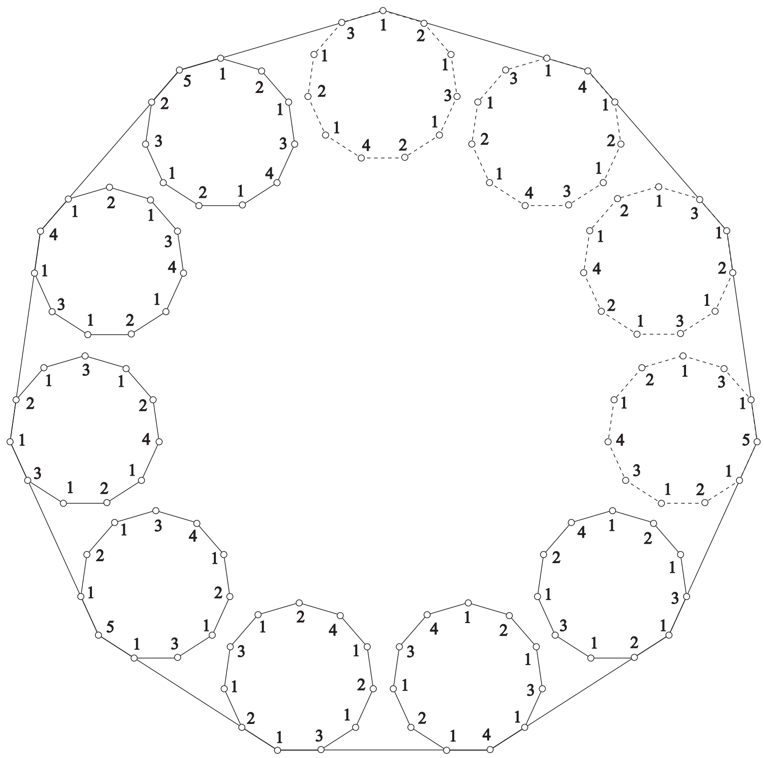

Fig. 7: An extendable packing 5-coloring of $S^2_{C_{11}}$

If $k \equiv 0 \pmod 4$, then f is a 4-coloring of $S^2_{C_k}$ defined by:

$$f(ij) = \begin{cases} 1, & j \equiv 0 \pmod 2 \\ 2, & j \equiv 1 \pmod 4 \text{ and } i \neq j \\ 3, & j \equiv 3 \pmod 4 \text{ and } i \neq j \\ 4, & j \equiv 1 \pmod 2 \text{ and } i = j \end{cases}$$

For $k \geq 10$ and $k \equiv 2 \pmod 4$, we define a 4-coloring of $S^2_{C_k}$ as follows. Let $i, j \in [k]$ and let ij be a vertex of $S^2_{C_k}$. We label iC_k such that we start at the vertex ii. If i is even (resp. i is odd), we use the sequence $121314\overline{1213}$ (resp. $4121314213\overline{1213}$).

We can see analogously as in the proof of Proposition 3 that above colorings are packing 4-coloring of $S^2_{C_k}$ for $k \equiv 0$ $(mod\ 4)$ and $k \equiv 2$ $(mod\ 4)$, respectively.

□

Proposition 5 *Let $n \geq 2$ and $k \notin \{5, 6, 7\}$. If $S^n_{C_k}$ is the generalized Sierpiński graph of C_k of dimension n, then $\chi_\rho(S^n_{C_k}) \leq 5$.*

Proof:

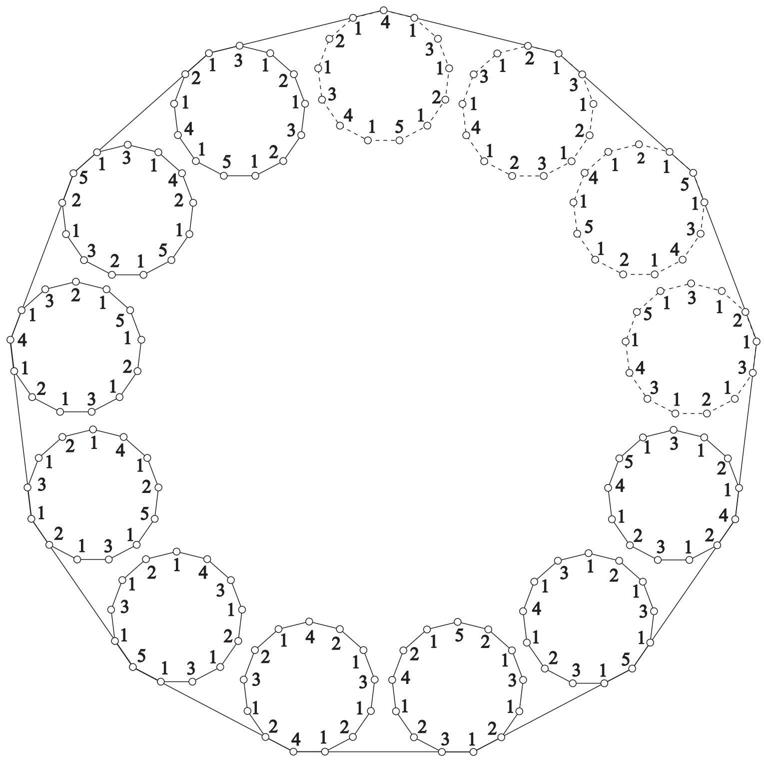

Fig. 8: An extendable packing 5-coloring of $S^2_{C_{13}}$

Let $k = 4t$, $s, i, j \in [k]$ and $w \in [k]^{n-3}$. We define a 5-coloring f of $S^n_{C_{4t}}$ as follows:

$$f(wsij) = \begin{cases} 1, & j \equiv 0 \pmod 2 \\ 2, & j \equiv 1 \pmod 4 \text{ and } i \neq j \\ 3, & j \equiv 3 \pmod 4 \text{ and } i \neq j \\ 4, & j \equiv 1 \pmod 2 \text{ and } i = j \text{ and } s \equiv 0 \pmod 2 \\ 5, & j \equiv 1 \pmod 2 \text{ and } i = j \text{ and } s \equiv 1 \pmod 2 \end{cases}$$

Analogously as in the proof of Theorem 3, we can show that f is a packing 5-coloring of $S^n_{C_{4t}}$.

Let $t \geq 2$, $k = 4t+2$, $i, j \in [k]$ and $w \in [k]^{n-2}$. We define a coloring of $wS^2_{C_{4t+2}}$ for every $w \in [k]^{n-2}$. More precisely, we color iC_{4t+2} of $wS^2_{C_{4t+2}}$ such that we start at the vertex ii and consecutively label its vertices with the sequences presented in Table 1.

In order to see that the coloring defined in Table 1 is a packing 5-coloring of $S^n_{C_{4t+2}}$, observe Fig. 6 which represent the application of this procedure to $S^2_{C_{10}}$. Since we can see that the obtained coloring is extendable, we showed that $\chi_\rho(S^n_{C_{10}}) \leq 5$. Note that the procedure for $t \geq 3$ and $n = 2$ repeats the coloring of the first four cycles (depicted by the dashed line in Fig. 6) in a way that the coloring remains extendable. Moreover, since the coloring of each C_{4t+2} is extended either with the pattern $\overline{1312}$ or $\overline{1213}$,

Tab. 1: A coloring of iC_{4t+2} in $S^n_{C_{4t+2}}$

$i < 4(t-1)$

$i \equiv 0 \pmod 4$	$i \equiv 1 \pmod 4$	$i \equiv 2 \pmod 4$	$i \equiv 3 \pmod 4$
$52142\overline{1312}1$	$\overline{1213}2142131$	$4\overline{1312}41231$	$\overline{1312}412312$

$i \geq 4(t-1)$

$i = 4(t-1)$	$i = 4(t-1)+1$	$i=4(t-1)+2$	$i=4(t-1)+3$	$i=4(t-1)+4$	$i=4(t-1)+5$
$51214\overline{1312}1$	$14121\overline{3121}541231$	$21\overline{3121}5131$	$5\overline{1312}412315$	$413\overline{1213}121$	$\overline{1312}412312$

the conditions of extendable coloring are still fulfilled. Thus, we obtain an extendable packing 5-coloring of $S^2_{C_{4t+2}}$ and the case is settled.

For $k = 4t+3$, we define a packing 5-coloring of $S^2_{C_k}$ as described in Table 2. Since we can obtain by this procedure an extendable packing 5-coloring of $S^2_{C_{11}}$ depicted in Fig. 7, we can show by using the same arguments as above that $\chi_\rho(S^n_{C_{4k+3}}) \leq 5$.

Tab. 2: A coloring of iC_{4t+3} in $S^n_{C_{4t+3}}$

$i\equiv 0 \pmod 4$	$i\equiv 1 \pmod 4$	$i\equiv 2 \pmod 4, i\neq 4t+2$	$i\equiv 3 \pmod 4$	$i = 4t+2$
$1213124\overline{1213}1$	$412134\overline{1213}1$	$1213124\overline{1213}1$	$512134\overline{1213}1$	$512134\overline{1213}2$

For $k = 4t+1$, $t \geq 3$, we can construct a packing 5-coloring of $S^2_{C_k}$ from an extendable packing 5-coloring of $S^2_{C_{13}}$ depicted in Fig. 8. The construction for $t \geq 4$ repeats the first four cycles (depicted by the dashed line) of this graph as needed, while the coloring for each cycle is extended with the pattern $\overline{1213}$ which can be found in every C_{13} of Figure 8. Note that the conditions of extendable coloring are still fulfilled with this procedure. Since we also found an extendable packing 5-coloring of $S^2_{C_9}$, it follows that $\chi_\rho(S^n_{C_{4k+3}}) \leq 5$ for $t \geq 2$. This assertion concludes the proof. $\square$

We will need the following result provided by SAT solver.

Proposition 6 *Let $S^2_{P_k}$ is the generalized Sierpiński graph of P_k of dimension 2. If f (resp. f') is an arbitrary packing 4-coloring of $S^2_{P_6}$ (resp. $S^2_{P_8}$), then*
(i) $f(22) \notin \{2,3\}$ and
(ii) $f'(33)$ and $f'(44)$ are not both equal to 1.

Corollary 1 *Let k be an odd integer. If $S^2_{C_k}$ is the generalized Sierpiński graph of C_k of dimension 2, then $\chi_\rho(S^2_{C_k}) \geq 5$.*

Proof: Let $k \geq 9$, k is odd and $i \in [k]$. Suppose that f is a packing 4-coloring of $S^2_{C_k}$. Note that $S^2_{P_6}$ and $S^2_{P_8}$ are subgraphs of $S^2_{C_k}$. By Proposition 6, for every external vertex ii we have that $f(ii) \notin \{2,3\}$. Moreover, two consecutive external vertices ii and $(i+1)(i+1)$ (addition modulo k) cannot be both colored by 1. It follows that the external vertices of $S^2_{C_k}$ are alternatively colored by 1 and 4. Since k is odd, this is clearly impossible and we obtain a contradiction. It follows that $\chi_\rho(S^2_{C_k}) \geq 5$ for every odd $k \geq 9$.

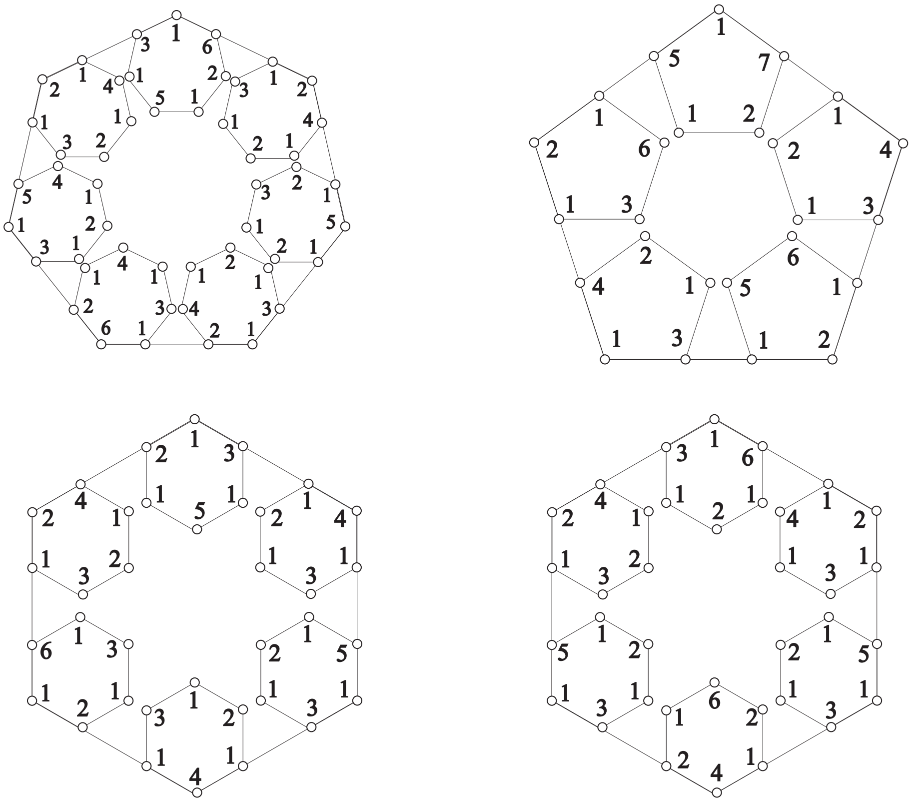

Fig. 9: An extendable packing 6-coloring of $S^2_{C_7}$, extendable packing 7-coloring of $S^2_{C_5}$, packing 5-coloring of $S^2_{C_6}$, extendable packing 6-coloring of $S^2_{C_6}$.

Since we establish by SAT solver that a packing 4-coloring of $S^2_{C_5}$ and $S^2_{C_7}$ also do not exist, the proof is complete. $\square$

Let $2\mathbb{N}$ and $2\mathbb{N}+1$ denote the set of even and odd natural numbers, respectively.

Theorem 4 *Let $n \geq 2$ and $k \geq 4$. If $S^n_{C_k}$ is the generalized Sierpiński graph of C_k of dimension n, then*

$$\chi_\rho(S^n_{C_k}) = \begin{cases} 4, & n = 2 \text{ and } k \in 2\mathbb{N} \setminus \{6\} \\ 5, & n \geq 3 \text{ and } k \notin \{5,6,7\} \text{ or } n = 2 \text{ and } k \in \{6\} \cup 2\mathbb{N}+1 \setminus \{5\} \\ 6, & n \geq 3 \text{ and } k \in \{6,7\} \text{ or } n \leq 6 \text{ and } k = 5 \end{cases}$$

Moreover, if $n \geq 7$, then $6 \leq \chi_\rho(S^n_{C_5}) \leq 7$.

Proof: First note that since $S^n_{P_k}$ is a subgraph of $S^n_{C_k}$, by Theorem 3 we have $\chi_\rho(S^2_{C_k}) \geq 4$ and $\chi_\rho(S^n_{C_k}) \geq 5$, $n \geq 3$. Moreover, if k is odd, we have $\chi_\rho(S^2_{C_k}) \geq 5$ by Corollary 1. Since we showed with SAT solver that a packing 5-colorings of $S^2_{C_5}$, $S^3_{C_6}$ and $S^3_{C_7}$, as well as a packing 4-coloring of $S^2_{C_6}$ do not exist, all lower bounds are settled.

In order to prove the upper bounds, we found a packing 6-coloring of $S^6_{C_5}$, a packing 5-coloring of $S^2_{C_6}$, an extendable packing 7-coloring of $S^2_{C_5}$, an extendable packing 6-coloring of $S^2_{C_6}$ and an extendable packing 6-coloring of $S^2_{C_7}$ (the last four colorings are depicted in Fig. 9). Since the other needed upper bounds follows from Propositions 4 and 5, the proof is complete.

□

5 Sierpiński triangle graphs

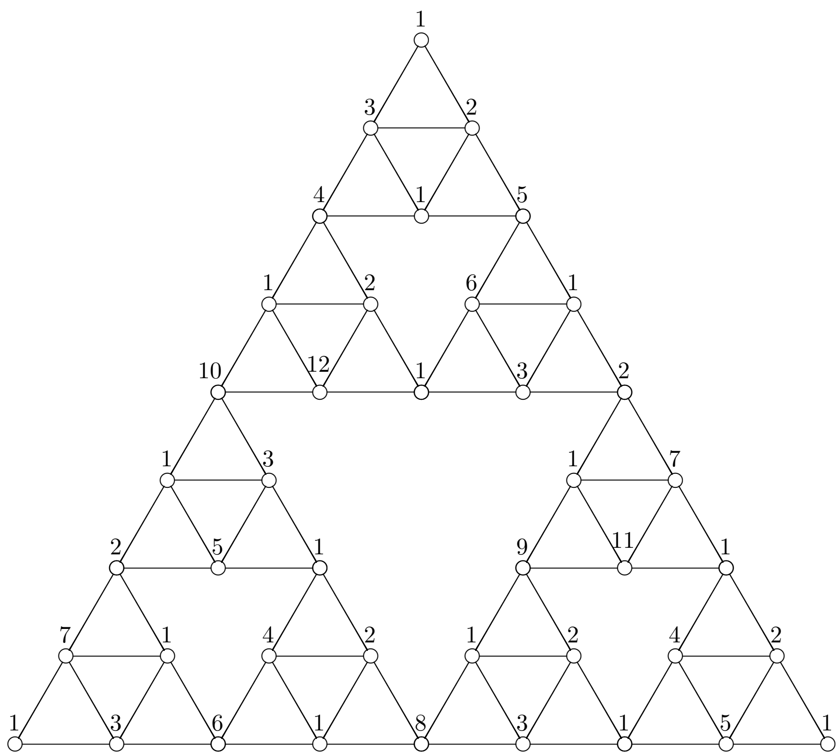

Fig. 10: A packing 12-coloring of ST^3_3

The *base-3 Sierpiński graphs* S^n are defined such that we start with $S^0 = K_1$. For $n \geq 1$, the vertex set of S^n is $[3]^n$ and the edge set is defined recursively as

$E(S^n) = \{\{is, it\} : i \in [3], \{s,t\} \in E(S^{n-1})\} \cup \{\{ij^{n-1}, ji^{n-1}\} | i, j \in [3], i \neq j\}$.

As mentioned in the introduction, for $n \geq 1$, the base-3 Sierpiński graphs S^n are generalized Sierpiński graphs where $G = K_3$. Obviously, for $n \geq 1$, S^n can be constructed from three copies of S^{n-1}.

Let n be a nonnegative integer. The class of the *Sierpiński triangle graphs* ST_3^n is obtained from S^{n+1} by contracting all non–clique edges.

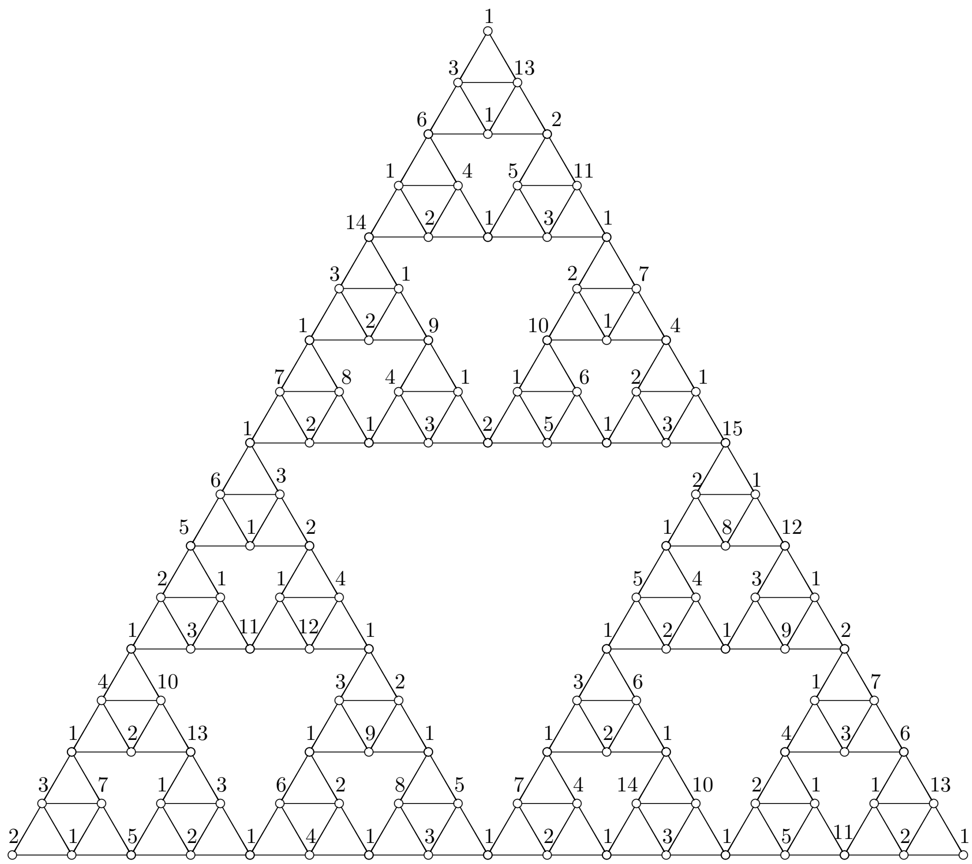

Fig. 11: A packing 15-coloring of ST_3^4

There are various other definitions of Sierpiński triangle graphs, which are based on the fact that their drawings in the plane represent approximations of the Sierpiński triangle fractal (see Hinz et al. (2017)). More intuitively, Sierpiński triangle graphs can be constructed by iteration. We start with a complete graph on 3 vertices, i.e. ST_3^0 is the triangle K_3. Assume now that ST_3^n is already constructed. ST_3^{n+1} is composed of three copies of ST_3^n, in a way that can be seen in Fig. 10, where ST_3^3 is composed of three copies of ST_3^2. Note that an extreme vertex of a copy of ST_3^2 is identified with an extreme vertex of another copy (this procedure is done for exactly two extreme vertices of each copy).

Brešar and Ferme (2018) established the packing chromatic number of Sierpiński triangle graphs ST_3^n for $n \leq 2$ and showed that the packing chromatic number for this class of graphs can be bounded above by 31. This bound is improved in the next theorem.

Theorem 5 *Let ST_3^n denote the Sierpiński triangle graph of dimension n.*

$$(i)\ \chi_\rho(ST_3^n) = \begin{cases} 3, & n=0 \\ 4, & n=1 \\ 8, & n=2 \\ 12 & n=3. \end{cases}$$

$$(ii)\ 12 \leq \chi_\rho(ST_3^4) \leq 15.$$

$$(iii)\ 12 \leq \chi_\rho(ST_3^5) \leq 19.$$

Moreover, if $n \geq 6$, then $12 \leq \chi_\rho(ST_3^n) \leq 20$

Proof: For $n \leq 2$, this is the result presented in Brešar and Ferme (2018). Since we showed with SAT solver that a packing 11-coloring of ST_3^3 cannot be obtained, we have $\chi_\rho(ST_3^n) \geq 12$ for every $n \geq 3$.

The upper bounds were obtained by the following constructions: a packing 12-coloring of ST_3^3 presented in Fig. 10, a packing 15-coloring of ST_3^4 presented in Fig. 11, a packing 19-coloring of ST_3^5, while for $n \geq 6$ we obtained an extendable packing 20-coloring of ST_3^6. The last two constructions can be obtained from the authors or in the webpage `https://omr.fnm.um.si/wp-content/uploads/2017/06/SierpinskiP.pdf`.

□

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