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Nordhaus-Gaddum Type Results for Total Domination

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A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. In this paper we study Nordhaus-Gaddum-type results for total domination. We examine the sum and product of $\gamma_t(G_1)$ and $\gamma_t(G_2)$ where $G_1 \oplus G_2 = K(s,s)$, and γ_t is the total domination number. We show that the maximum value of the sum of the total domination numbers of G_1 and G_2 is $2s+4$, with equality if and only if $G_1 = sK_2$ or $G_2 = sK_2$, while the maximum value of the product of the total domination numbers of G_1 and G_2 is $\max\{8s, \lfloor (s+6)^2/4 \rfloor\}$.

Keywords: Nordhaus-Gaddum; Total domination; Relative complement

1 Introduction

In this paper, we continue the study of total domination in graphs which was introduced by Cockayne, Dawes, and Hedetniemi Cockayne et al. (1980). A *total dominating set*, abbreviated TDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S = V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater Haynes et al. (1998a,b). A recent survey of total domination in graphs can be found in Henning (2009).

In 1956 the original paper Nordhaus and Gaddum (1956) by Nordhaus and Gaddum appeared. In it they gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters; see, for example, Chartrand and Mitchem (1971); Füredi et al. (2005). Nordhaus-Gaddum inequalities involving domination parameters in graphs have been studied in several papers; see, for example, Arumugan and Thuraiswamy (1996); Cockayne et al. (1980); Erfang et al. (2004); Favaron et al. (2010); Goddard and Henning (2003); Goddard et al.

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(1992); Harary and Haynes (1996); Hattingh et al. (2008); Jaeger and Payan (1972); Joseph and Arumugam (1995); Payan and Xuong (1982) and elsewhere. In particular, we remark that in the introductory paper on total domination, Cockayne, Dawes, and Hedetniemi Cockayne et al. (1980) proved a Nordhaus-Gaddum bound for the sum of the total domination numbers of a graph and its complement, while the authors Henning et al. (2011) gave a bound for their product.

Theorem 1 (Cockayne et al. (1980); Henning et al. (2011)) *Let G be a graph of order n such that neither G nor $\overline{G}$ contains isolated vertices. Then the following holds.*
(a) $\gamma_t(G) + \gamma_t(\overline{G}) \leq n + 2$.
(b) $\gamma_t(G)\gamma_t(\overline{G}) \leq 2n$.
In both inequalities, equality holds if and only if G or $\overline{G}$ consists of disjoint copies of K_2.

For an overview of Nordhaus-Gaddum inequalities for domination-related parameters we refer the reader to Chapter 10 in the domination book by Haynes, Hedetniemi, and Slater Haynes et al. (1998a).

Plesník k (1978) was the first to extend Nordhaus and Gaddum's results to the case where the complete graph is factored into several factors. Goddard, Henning and Swart Goddard et al. (1992) continued this approach and considered the domination number and $G_1 \oplus G_2 \oplus G_3 = K_n$. They also looked at another variation on Nordhaus-Gaddum type results in which they extended the concept by considering $G_1 \oplus G_2 = K(s,s)$ rather than $G_1 \oplus G_2 = K_n$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.)

In this paper, we focus our attention on Nordhaus-Gaddum type results for the total domination number. In particular, we establish upper bounds on the sums and products of the total domination numbers of G_1 and G_2, where $G_1 \oplus G_2 = K(s,s)$, and neither G_1 nor G_2 contains isolated vertices.

1.1 Notation

For notation and graph theory terminology we in general follow Haynes et al. (1998a). Specifically, let $G = (V,E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v\in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. For subsets $S, T \subseteq V$, the set S *totally dominates* the set T if $T \subseteq N(S)$, while S *dominates* T if $T \subseteq N[S]$. In particular, if S dominates V, then S is called a *dominating set* in G. The minimum degree among the vertices of G is denoted by $\delta(G)$. A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n, while a complete graph on n vertices is denoted by K_n. If G is a disjoint union of k copies of a graph F, we write $G = kF$. For disjoint subsets X and Y of V, we let $G[X,Y]$ denote the set of all edges of G between X and Y.

2 Total Domination Number and Relative Complement

If G is a subgraph of H, then the graph $H - E(G)$ is called the *complement of G relative to H*. In Goddard et al. (1992), the authors determined the graphs H with respect to which complements are always unique in the following sense: if G_1 and G_2 are isomorphic subgraphs of H, then their complements $H - E(G_1)$ and $H - E(G_2)$ are isomorphic.

Theorem 2 (Goddard et al. (1992)) *Let H be a graph without isolated vertices with respect to which complements are always unique. Then H is one of the following: (a)* $rK(1, s)$*, (b)* rK_3*, (c)* K_s*, (d)* C_5*, or (e)* $K(s, s)$*, for some integers* r *and/or* s*.*

The above results suggest that the complete bipartite graph $K(s, s)$ is an obvious alternate to K_n in Nordhaus-Gaddum results. In Goddard et al. (1992), the sums and products of $\psi(G_1)$ and $\psi(G_2)$ are examined where $G_1 \oplus G_2 = K(s, s)$, and ψ is the independence, domination, or independent domination number, inter alia. In this section we take ψ to be the total domination number.

2.1 Notation

For the rest of the section we shall assume that $K(s, s)$ has partite sets $\mathcal{L}$ and $\mathcal{R}$ (standing for "left" and "right"), and that $G_1 \oplus G_2 = K(s, s)$ where neither G_1 nor G_2 have an isolated vertex. For $G \subset K(s, s)$, let $\delta^L(G)$ denote the minimum degree of a vertex of $\mathcal{L}$. Define $\delta^R(G)$, $\Delta^L(G)$ and $\Delta^R(G)$ similarly. Further we shall abbreviate parameters by writing γ_t^i for $\gamma_t(G_i)$, ψ_i for $\psi(G_i)$ for parameters $\psi \neq \gamma_t$, $N_i(v)$ for $N_{G_i}(v)$, E_i for $E(G_i)$, V_i for $V(G_i)$, and m_i for the size of G_i, for $i \in \{1, 2\}$.

Recall that for subsets $S, T \subseteq V$, the set S totally dominates the set T if $T \subseteq N(S)$. Let $G \subset K(s, s)$. A *left total dominating set*, abbreviated left TDS, of G is a set of vertices that totally dominates $\mathcal{L}$; that is, a left TDS is a set S of vertices of G such that $\mathcal{L} \subseteq N(S)$. Necessarily a left TDS of G belongs to $\mathcal{R}$. The minimum cardinality of a left TDS in G is called the *left domination number*, denoted $\ell(G)$, of G. Similarly, a *right total dominating set*, abbreviated right TDS, of G is a set of vertices that totally dominates $\mathcal{R}$ and the minimum cardinality of a right TDS in G is the *right domination number*, denoted $r(G)$, of G. We note that a right TDS of G belongs to $\mathcal{L}$.

2.2 Preliminary Results

Cockayne et al. Cockayne et al. (1980) obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 3 (Cockayne et al. (1980)) *If G is a connected graph of order* $n \geq 3$*, then* $\gamma_t(G) \leq 2n/3$*.*

We shall need the following observation.

Observation 4 *Let* $G_1 \oplus G_2 = K(s, s)$*, and let* $i \in \{1, 2\}$*. Then the following hold.*
(a) $\delta_i^L + \Delta_{3-i}^L = \delta_i^R + \Delta_{3-i}^R = s$.
(b) $\delta_i^L + \delta_{3-i}^R \leq s$.

Proof. Part (a) is immediate. Counting edges in G_1 and G_2, we have $m_i \geq s\delta_i^L$ and $m_{3-i} \geq s\delta_{3-i}^R$, and so $s^2 = m_i + m_{3-i} \geq s(\delta_i^L + \delta_{3-i}^R)$, or, equivalently, $\delta_i^L + \delta_{3-i}^R \leq s$. $\Box$

The following lemma will prove to be useful.

Lemma 5 *Let* $G_1 \oplus G_2 = K(s, s)$*, where neither* G_1 *nor* G_2 *have an isolated vertex, and let* $i \in \{1, 2\}$*. Then the following hold.*
(a) $2 \leq \ell_i, r_i \leq s$.
(b) $\gamma_t^i = \ell_i + r_i$.
(c) $\ell_i \leq 1 + \delta_{3-i}^R$ *and* $r_i \leq 1 + \delta_{3-i}^L$.
(d) $(\ell_i - 2)(r_{3-i} - 1) \leq \delta_{3-i}^R - 1$ *and* $(r_i - 2)(\ell_{3-i} - 1) \leq \delta_{3-i}^L - 1$.
(e) $\gamma_t^i \leq 2 + 2\min\{\Delta_{3-i}^R, \Delta_{3-i}^L\}$.

Proof. (a) Part (a) follows from the observation that $1 \le \delta_i \le \Delta_i \le s-1$.
(b) Part (b) follows from the observation that every TDS of G_1 can be partitioned into two sets, one contained in $\mathcal{L}$ that totally dominates $\mathcal{R}$ and the other contained in $\mathcal{R}$ that totally dominates $\mathcal{L}$.
(c) Let $v \in \mathcal{R}$ be a vertex of degree Δ_1^R in G_1. We now consider the set $N_2(v)$ and note that $|N_2(v)| = \delta_2^R$. For each vertex $u \in N_2(v)$, select an arbitrary neighbor u' in G_1. Let $U = \cup\{u'\}$, where the union is taken over all vertices $u \in N_2(v)$. Then, $U \cup \{v\}$ is a left TDS in G_1, and so $\ell_1 \le |U|+1 \le |N_2(v)|+1 = 1+\delta_2^R$. The other results follow by symmetry.
(d) Let $v \in \mathcal{R}$ have degree Δ_1^R in G_1 and let $X = \mathcal{L} \setminus N_1(v)$. We now partition X into subsets $X_1, \dots, X_k$ of size at most $r_2 - 1$ such that k is as small as possible. For each $i = 1, \dots, k$, the set X_i does not dominate $\mathcal{R}$ in G_2, and so there exists a $y_i \in \mathcal{R} \setminus N_2(X_i)$ which therefore totally dominates X_i in G_1. Hence, $\{v, y_1, \dots, y_k\}$ totally dominates $\mathcal{L}$ in G_1. Since $k = \lceil |X|/(r_2-1) \rceil$ and $|X| = s - \Delta_1^R = \delta_2^R$, we deduce that $\ell_1 \le 1 + \lceil \delta_2^R/(r_2-1) \rceil \le 2 + (\delta_2^R - 1)/(r_2-1)$, or, equivalently, $(\ell_1 - 2)(r_2 - 1) \le \delta_2^R - 1$. The other results follow by symmetry.
(e) Let $v \in \mathcal{R}$ have degree Δ_1^R in G_1. Let $X = N_1(v)$, and let $u \in \mathcal{L} \setminus X$. Further let $Y = N_1(u)$ and $Z = \mathcal{R} \setminus (Y \cup \{v\})$. If some vertex $y \in Y$ dominates X in G_1, then the degree of y in G_1 exceeds that of v, a contradiction. Hence every $y \in Y$ is adjacent to at least one vertex of X in G_2, and so X totally dominates Y in G_2. For each vertex $x \in X$, select an arbitrary neighbor x' in G_2. Let $X' = \cup_{x \in X}\{x'\}$ and note that X' totally dominates X in G_2 and $|X'| \le |X|$. Moreover, $\{u, v\}$ totally dominates $V \setminus (X \cup Y)$ in G_2. Thus, $X \cup X' \cup \{u, v\}$ is a TDS in G_2, and so $\gamma_t^2 \le 2 + 2|X| = 2 + 2\Delta_1^R$. Similarly by choosing $v \in \mathcal{L}$ to have degree Δ_1^L in G_1, we have that $\gamma_t^2 \le 2 + 2\Delta_1^L$. Thus, $\gamma_t^2 \le 2 + 2\min\{\Delta_1^R, \Delta_1^L\}$. Similarly, $\gamma_t^1 \le 2 + 2\min\{\Delta_2^R, \Delta_2^L\}$. □

2.3 The Sum

In this section, we look at $G_1 \oplus G_2 = K(s,s)$ and the upper bound on $\gamma_t(G_1) + \gamma_t(G_2)$, where neither G_1 nor G_2 have an isolated vertex. We shall establish:

Theorem 6 *If $G_1 \oplus G_2 = K(s,s)$ where neither G_1 nor G_2 has an isolated vertex, then $\gamma_t(G_1) + \gamma_t(G_2) \le 2s+4$, with equality if and only if $G_1 = sK_2$ or $G_2 = sK_2$.*

Proof. That the upper bound of $2s+4$ on the sum $\gamma_t^1 + \gamma_t^2$ is sharp, may be seen by taking $G_1 = sK_2$, in which case $\gamma_t^1 = 2s$ and $\gamma_t^2 = 4$, or, by symmetry, taking $G_2 = sK_2$.

We now prove the upper bound holds. Renaming sets, if necessary, we may assume without loss of generality that $r_1 = \min\{\ell_1, r_1, \ell_2, r_2\}$. By Lemma 5(a), $r_1 \ge 2$. If $r_1 = 2$, then we have that

$$\begin{array}{rcll} \gamma_t^1 + \gamma_t^2 & = & r_1 + \ell_1 + \gamma_t^2 & \text{(by Lemma 5(b))} \\ & \le & 2 + (1+\delta_2^R) + (2 + 2\Delta_1^R) & \text{(by Lemma 5(c) and 5(e))} \\ & = & 5 + s + \Delta_1^R & \text{(by Observation 4)} \\ & \le & 2s+4 & \text{(since } \Delta_1^R \le s-1). \end{array}$$

Suppose, further, that $\gamma_t^1 + \gamma_t^2 = 2s+4$ (and still $r_1 = 2$). Then equality occurs throughout the above inequality chain. In particular, this requires that $\gamma_t^2 = 2 + 2\Delta_1^R$ and that $\Delta_1^R = s-1$. But then $\gamma_t^2 = 2s$. This is only possible if $G_2 = sK_2$.

Hence we may assume that $r_1 \ge 3$, for otherwise the desired result follows. By Lemma 5(c), $r_1 \le 1 + \delta_2^L$, implying that $\delta_2^L \ge 2$. Similarly, since $\ell_1 \ge r_1 \ge 3$, we note that $\delta_2^R \ge 2$. Since $r_2 \ge r_1 \ge 3$, Lemma 5(d) implies that $\ell_1 \leq 2 + (\delta_2^R - 1)/(r_2 - 1) \leq 2 + (\delta_2^R - 1)/2$. Similarly, $r_1 \leq 2 + (\delta_2^L - 1)/(\ell_2 - 1) \leq 2 + (\delta_2^L - 1)/2$. By Lemma 5(b), $\gamma_t^1 = \ell_1 + r_1$, and so

$$\gamma_t^1 \leq 3 + (\delta_2^R + \delta_2^L)/2. \tag{1}$$

By Lemma 5(e), $\gamma_t^2 \leq 2 + 2\min\{\Delta_1^R, \Delta_1^L\} \leq 2 + \Delta_1^R + \Delta_1^L$. By Observation 4, we note that $\Delta_1^R = s - \delta_2^R$ and $\Delta_1^L = s - \delta_2^L$. Hence,

$$\gamma_t^2 \leq 2 + 2s - \delta_2^R - \delta_2^L. \tag{2}$$

Adding Equation (1) and Equation (2), we have that $\gamma_t^1 + \gamma_t^2 \leq 2s + 5 - (\delta_2^R + \delta_2^L)/2$. However as observed earlier, $r_1 \geq 3$ implies that $\delta_2^L \geq 2$ and $\delta_2^R \geq 2$. Hence, $\gamma_t^1 + \gamma_t^2 \leq 2s + 5 - 2 < 2s + 4$. $\square$

2.4 The Product

In this section, we look at $G_1 \oplus G_2 = K(s,s)$ and the upper bound on $\gamma_t(G_1)\gamma_t(G_2)$, where neither G_1 nor G_2 has an isolated vertex. We shall establish:

Theorem 7 *If $G_1 \oplus G_2 = K(s,s)$ where neither G_1 nor G_2 have an isolated vertex, then $\gamma_t(G_1)\gamma_t(G_2) \leq \max\{8s, \lfloor (s+6)^2/4 \rfloor\}$.*

Proof. That the upper bound of $8s$ on the product $\gamma_t^1\gamma_t^2$ is achievable, may be seen by taking $G_1 = sK_2$ or $G_2 = sK_2$. We remark that if $G_1 \cong B_{12}$, where B_{12} is the bipartite cubic graph of order $n = 12$ shown in Figure 1, then $G_2 \cong B_{12}$ and $\gamma_t^1\gamma_t^2 = 36 = \lfloor (s+6)^2/4 \rfloor$. This shows that the upper bound of $\lfloor (s+6)^2/4 \rfloor$ on the product $\gamma_t^1\gamma_t^2$ is achievable.

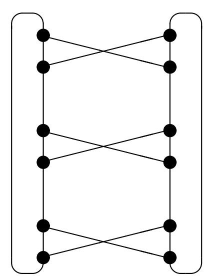

Fig. 1: The bipartite cubic graph B_{12}.

If $\gamma_t^1 = 4$, then since $\gamma_t^2 \leq 2s$, we have that $\gamma_t^1\gamma_t^2 \leq 8s$. Further if $\gamma_t^1\gamma_t^2 = 8s$, then $\gamma_t^2 = 2s$ which is only possible if $G_2 = sK_2$. Similarly, if $\gamma_t^2 = 4$, then $\gamma_t^1\gamma_t^2 \leq 8s$, with equality if and only if $G_1 = sK_2$. Hence we may assume that $\gamma_t^1 \geq 5$ and $\gamma_t^2 \geq 5$, for otherwise the desired result follows. In particular, this implies that both G_1 and G_2 are connected. Renaming sets, if necessary, we may assume without loss of generality that $r_1 = \min\{\ell_1, r_1, \ell_2, r_2\}$. By Lemma 5(a), $r_1 \geq 2$.

We proceed further with the following claim.

CLAIM A: If $r_1 \geq 3$, then $\gamma_t^1\gamma_t^2 \leq \lfloor (s+6)^2/4 \rfloor$.

PROOF. An identical argument as in the proof of Theorem 6, shows that Equation (1) holds; that is, $\gamma_t^1 \leq 3 + (\delta_2^R + \delta_2^L)/2$. By a symmetric argument, $\gamma_t^2 \leq 3 + (\delta_1^R + \delta_1^L)/2$. Adding these two inequalities yields $\gamma_t^1 + \gamma_t^2 \leq 6 + (\delta_1^L + \delta_2^L + \delta_1^R + \delta_2^R)/2$. Hence, by Observation 4(b), $\gamma_t^1 + \gamma_t^2 \leq s + 6$. Since the geometric mean is at most the arithmetic mean, we note that $\sqrt{\gamma_t^1\gamma_t^2} \le (\gamma_t^1+\gamma_t^2)/2$, and so $\gamma_t^1\gamma_t^2 \le \lfloor (s+6)^2/4 \rfloor$. This completes the proof of Claim A. $_{(\Box)}$

By Claim A we may assume that $r_1 = 2$. We now let $\ell_1 = k+1$ and continue with the following claim.
CLAIM B: We may assume that the following hold.
(a) $\ell_2 \le s-k$.
(b) $4 \le r_2 < s/k+1$.
(c) $4 \le k < s/3$.

PROOF. **(a)** Let X be a minimum left TDS of G_1 and note that $X \subseteq \mathcal{R}$ and $|X| = \ell_1$. Let $X = \{x_1,\dots,x_{\ell_1}\}$. By the minimality of X, every vertex $x \in X$ has at least one neighbor in G_1 which has no other neighbor in X except for x. For each $x_i \in X$, let y_i be such a neighbor, and so $N_1(y_i) \cap X = \{x_i\}$. Let $Y = \{y_1,\dots,y_{\ell_1}\}$ and note that $Y \subseteq \mathcal{L}$ and that $|X| = |Y| = \ell_1$. Furthermore, $Y \setminus \{y_i\} \subseteq N_2(x_i)$ for all $i \in \{1,\dots,\ell_1\}$. Let $Y' = \mathcal{L} \setminus Y = \{y_1',\dots,y_{s-\ell_1}'\}$.

Suppose there are no edges between X and Y' in G_2, that is $G_2[X,Y'] = \emptyset$. Since G_2 is connected, there must exist vertices $y_{j_1} \in Y$ and y_{j_2}' in Y' with a common neighbor, x' say, in G_2. Necessarily, $x' \in \mathcal{R} \setminus X$. Now, for each $i \in \{1,\dots,s-\ell_1\} \setminus \{j_2\}$, we let $x_i' \in N_2(y_i')$, and let $x_{j_2}' = x'$. Let

$$X' = \bigcup_{i=1}^{s-\ell_1} \{x_i'\}$$

and note that $|X'| \le s-\ell_1$. But now $Y \setminus \{y_{j_1}\} \subseteq N_2(x_{j_1})$ and $Y' \cup \{y_{j_1}\} \subseteq N_2(X')$, and so $X' \cup \{x_{j_1}\}$ is a left TDS of G_2. Hence, $\ell_2 \le |X' \cup \{x_{j_1}\}| \le s-\ell_1+1 = s-k$, as desired.

Hence we may assume that $G_2[X,Y'] \ne \emptyset$. Thus for some $k_1 \in \{1,\dots,\ell_1\}$ and some $k_2 \in \{1,\dots,s-\ell_1\}$, we have $x_{k_1}y_{k_2}' \in E_2$. For each $i \in \{1,\dots,s-\ell_1\} \setminus \{k_2\}$, we let $x_i'' \in N_2(y_i')$, and let $x_{k_2}'' = x_{k_1}$. Let

$$X'' = \bigcup_{i=1}^{s-\ell_1} \{x_i''\}$$

and note that $|X''| \le s-\ell_1$. Let $k_3 \in \{1,\dots,\ell_1\} \setminus \{k_1\}$. Then, $Y \setminus \{y_{k_3}\} \subseteq N_2(x_{k_3})$ and $Y' \cup \{y_{k_3}\} \subseteq N_2(X'')$, and so $X'' \cup \{x_{k_3}\}$ is a left TDS of G_2. Hence, $\ell_2 \le |X'' \cup \{x_{k_3}\}| \le s-\ell_1+1 = s-k$, as desired.

(b) Suppose $r_2 \le 3$. Recall that $\ell_1 = k+1$, $r_1 = 2$, and by part (a), $\ell_2 \le s-k$. Therefore, $\gamma_t^1\gamma_t^2 = (\ell_1+r_1)(\ell_2+r_2) \le (k+3)(s-k+3)$. Since the geometric mean is at most the arithmetic mean, we have that $\sqrt{(k+3)(s-k+3)} \le (s+6)/2$, and so $\gamma_t^1\gamma_t^2 \le \lfloor (s+6)^2/4 \rfloor$. Hence we may assume that $r_2 \ge 4$. We now partition $\mathcal{R}$ into subsets $X_1,\dots,X_j$ of size at most k such that j is as small as possible. Since $\ell_1 = k+1$, for each $i = 1,\dots,j$, the set X_i does not dominate $\mathcal{L}$ in G_1, and so there exists a $y_i \in \mathcal{L} \setminus N_1(X_i)$ which therefore totally dominates X_i in G_2. Hence, $\{y_1,\dots,y_j\}$ totally dominates $\mathcal{R}$ in G_2. Since $j = \lceil |\mathcal{R}|/k \rceil$ and $|\mathcal{R}| = s$, we deduce that $r_2 \le \lceil s/k \rceil < s/k+1$.

(c) Suppose $k \le 3$. Then, $\gamma_t^1 = r_1+\ell_1 \le 6$. By Theorem 3, $\gamma_t^2 \le 2|V_2|/3 = 4s/3$. Hence, $\gamma_t^1\gamma_t^2 \le 6(4s/3) = 8s$. We may therefore assume that $k \ge 4$. By Part (b), we may assume $4 < s/k+1$ and thus $k < s/3$. This completes the proof of Claim B. $_{(\Box)}$

By Claims A and B we now have that $r_1 = 2$, $\ell_1 = k+1$, $r_2 < s/k+1$, and $\ell_2 \leq s-k$. Hence, by Lemma 5(b), we have

$$\gamma_t^1 \gamma_t^2 < (k+3)(s/k+s-k+1). \tag{3}$$

We proceed with the following claim which introduces an assumption we can make on the size of s and improves the bounds in Claim B(b) and Claim B(c).

CLAIM C: We may assume that the following hold.
(a) $s \geq 20$.
(b) $5 \leq r_2 < s/k+1$.
(c) $4 \leq k < s/4$.

PROOF. **(a)** Suppose $s = 19$. Then by Claim B(c) we have $k \in \{4,5,6\}$. Substituting these values into Equation (3) we get $\gamma_t^1\gamma_t^2 < 145.25$, $\gamma_t^1\gamma_t^2 < 150.4$, and $\gamma_t^1\gamma_t^2 < 154.5$, respectively. In each case $\gamma_t^1\gamma_t^2 < 156 = \lfloor (s+6)^2/4 \rfloor$, and thus we may assume $s \neq 19$. Suppose now that $s \leq 18$. By Claim B(c) we have $k \in \{4,5\}$. If $k = 4$ then

$$\begin{array}{rcll} \gamma_t^1\gamma_t^2 & < & (7)(s/4+s-3) & \text{(substituting into Equation (3))} \\ & = & 8s + 3s/4 - 21 & \\ & < & 8s & \text{(since } 3s/4 \leq 3(18)/4 < 21\text{).} \end{array}$$

Hence we may assume that $k = 5$. But now we have

$$\begin{array}{rcll} \gamma_t^1\gamma_t^2 & < & (8)(s/5+s-4) & \text{(substituting into Equation (3))} \\ & < & 8s & \text{(since } s/5 \leq 18/5 < 4\text{).} \end{array}$$

Hence we may assume that $s \geq 20$.

(b) By Claim B(b) we have $4 \leq r_2 < s/k+1$. If $r_2 = 4$ then

$$\begin{array}{rcll} \gamma_t^1\gamma_t^2 & = & (\ell_1+r_1)(\ell_2+r_2) & \text{(by Lemma 5(b))} \\ & \leq & (k+3)(s-k+4) & \text{(by Claim B(a)).} \end{array}$$

We note that $(k+3)(s-k+4)$ is a parabola as a function of k which achieves its maximum at $k = (s+1)/2$ and is therefore strictly increasing on the interval $[4, s/3]$. Therefore, since $4 \leq k < s/3$ (by Claim B(c)), we have

$$\begin{array}{rcl} \gamma_t^1\gamma_t^2 & < & (s/3+3)(s-s/3+4) \\ & = & 2s^2/9 + 10s/3 + 12 \\ & = & (s+6)^2/4 - s^2/36 + s/3 + 3. \end{array}$$

But by Part (a) we have $s > 18$, and so $s^2/36 > 18s/36 = s/2 = s/3 + s/6 > s/3 + 3$. Thus, $-s^2/36 + s/3 + 3 < 0$ and the above inequality chain reduces to $\gamma_t^1\gamma_t^2 < (s+6)^2/4$, and hence $\gamma_t^1\gamma_t^2 \leq \lfloor (s+6)^2/4 \rfloor$. We may therefore assume that $r_2 \neq 4$, and so $5 \leq r_2 < s/k+1$.

(c) By Claim B(c) we have $k \geq 4$. By Part (b), we may assume $5 < s/k+1$ and thus $k < s/4$. This completes the proof of Claim C. (□)

We return to the proof of Theorem 7. Multiplying out Equation (3) yields

$$\gamma_t^1\gamma_t^2 < s + 3s/k + (k+3)(s-k+1). \tag{4}$$

We note that since $k \geq 4$ we have that $3s/k \leq 3s/4$. Furthermore, $(k+3)(s-k+1)$ is a parabola as a function of k which achieves its maximum at $k = (s-2)/2$ and is therefore strictly increasing on the interval $[4, s/4]$. Using this information and the fact that $4 \leq k < s/4$ (by Claim C(c)) in Equation (4), we get

$$\begin{array}{rcl} \gamma_t^1\gamma_t^2 & < & s + 3s/4 + (s/4+3)(3s/4+1) \\ & = & 3s^2/16 + 17s/4 + 3 \\ & = & (s+6)^2/4 - s^2/16 + 5s/4 - 6. \end{array}$$

But by Claim C(a) we have $s \geq 20$ and so $s^2/16 \geq 20s/16 = 5s/4$. Thus, $-s^2/16 + 5s/4 - 6 < 0$ and the above inequality chain reduces to $\gamma_t^1\gamma_t^2 < (s+6)^2/4$. We conclude that $\gamma_t^1\gamma_t^2 \leq \lfloor (s+6)^2/4 \rfloor$. This completes the proof of Theorem 7. $\square$

That the bound of Theorem 7 is essentially best possible, may be seen as follows. For $s \geq 2$, by taking $G_1 = sK_2$ we note that $\gamma_t(G_1) = 2s$ and $\gamma_t(G_2) = 4$, whence $\gamma_t(G_1)\gamma_t(G_2) = 8s$. Let $s \geq 4$ and let $K(s,s)$ have partite sets A and B. Partition A (B) into two sets, one of size $\lfloor s/2 \rfloor$, say A_1 (B_1), and one of size $\lceil s/2 \rceil$, say A_2 (B_2). Form the edge set of G_1 from the set of all edges between A and B_1 by removing (the edges of) a matching between A_1 and B_1 and inserting a matching between A_2 and B_2. If S is a TDS of G_1, then $A_2 \subset S$ in order to totally dominate B_2, while $|S \cap B_1| \geq 2$ in order to totally dominate A_1. Hence, $\gamma_t(G_1) \geq \lceil s/2 \rceil + 2$. However adding any two vertices of B_1 to the set A_2 produces a TDS of G_1. Consequently, $\gamma_t(G_1) = \lceil s/2 \rceil + 2$. Similarly, $\gamma_t(G_2) = \lfloor s/2 \rfloor + 2$. Thus, $\gamma_t(G_1)\gamma_t(G_2) = (\lceil s/2 \rceil + 2)(\lfloor s/2 \rfloor + 2) = \lfloor (s/2+2)^2 \rfloor = \lfloor (s+4)^2/4 \rfloor$.

We remark that $\lfloor (s+6)^2/4 \rfloor \geq 8s$ for $s \geq 18$.

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