A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. In this paper we study Nordhaus-Gaddum-type results for total domination. We examine the sum and product of $\gamma_t(G_1)$ and $\gamma_t(G_2)$ where $G_1 \oplus G_2 = K(s, s)$, and $\gamma_t$ is the total domination number. We show that the maximum value of the sum of the total domination numbers of $G_1$ and $G_2$ is $2s + 4$, with equality if and only if $G_1 = sK_2$ or $G_2 = sK_2$, while the maximum value of the product of the total domination numbers of $G_1$ and $G_2$ is $\max\{8s, \left\lfloor \frac{(s + 6)^2}{4} \right\rfloor\}$.

Keywords: Nordhaus-Gaddum; Total domination; Relative complement

1 Introduction

In this paper, we continue the study of total domination in graphs which was introduced by Cockayne, Dawes, and Hedetniemi [Cockayne et al. (1980)]. A total dominating set, abbreviated TDS, of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex is adjacent to a vertex in $S$. Every graph without isolated vertices has a TDS, since $S = V(G)$ is such a set. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of $G$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [Haynes et al. (1998a,b)]. A recent survey of total domination in graphs can be found in Henning (2009).

In 1956 the original paper [Nordhaus and Gaddum (1956)] by Nordhaus and Gaddum appeared. In it they gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters; see, for example, [Chartrand and Mitchem (1971); Füredi et al. (2005)]. Nordhaus-Gaddum inequalities involving domination parameters in graphs have been studied in several papers; see, for example, [Arumugan and Thraivaswamy (1996); Cockayne et al. (1980); Erfang et al. (2004); Favaron et al. (2010); Goddard and Henning (2003); Goddard et al.].
In both inequalities, equality holds if and only if $G$ or $\bar{G}$ consists of disjoint copies of $K_2$.

For an overview of Nordhaus-Gaddum inequalities for domination-related parameters we refer the reader to Chapter 10 in the domination book by Haynes, Hedetniemi, and Slater [Haynes et al. (1998a)]

Plesnık [Plesnık (1978)] was the first to extend Nordhaus and Gaddum’s results to the case where the complete graph is factored into several factors. Goddard, Henning and Swart [Goddard et al. (1992)] continued this approach and considered the domination number and $G_1 \oplus G_2 \oplus G_3 = K_n$. They also looked at another variation on Nordhaus-Gaddum type results in which they extended the concept by considering $G_1 \oplus G_2 = K(s, s)$ rather than $G_1 \oplus G_2 = K_n$. (If $G$ and $H$ are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.)

In this paper, we focus our attention on Nordhaus-Gaddum type results for the total domination number. In particular, we establish upper bounds on the sums and products of the total domination numbers of $G_1$ and $G_2$, where $G_1 \oplus G_2 = K(s, s)$, and neither $G_1$ nor $G_2$ contains isolated vertices.

### 1.1 Notation

For notation and graph theory terminology we in general follow [Haynes et al. (1998a)]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its open neighborhood is the set $N(S) = \cup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. For subsets $S, T \subseteq V$, the set $S$ totally dominates the set $T$ if $T \subseteq N(S)$, while $S$ dominates $T$ if $T \subseteq N[S]$. In particular, if $S$ dominates $V$, then $S$ is called a dominating set in $G$. The minimum degree among the vertices of $G$ is denoted by $\delta(G)$. A cycle on $n$ vertices is denoted by $C_n$ and a path on $n$ vertices by $P_n$, while a complete graph on $n$ vertices is denoted by $K_n$. If $G$ is a disjoint union of $k$ copies of a graph $F$, we write $G = kF$. For disjoint subsets $X$ and $Y$ of $V$, we let $G[X, Y]$ denote the set of all edges of $G$ between $X$ and $Y$.

### 2 Total Domination Number and Relative Complement

If $G$ is a subgraph of $H$, then the graph $H - E(G)$ is called the complement of $G$ relative to $H$. In [Goddard et al. (1992)], the authors determined the graphs $H$ with respect to which complements are always unique in the following sense: if $G_1$ and $G_2$ are isomorphic subgraphs of $H$, then their complements $H - E(G_1)$ and $H - E(G_2)$ are isomorphic.
Theorem 2 (Goddard et al. (1992)) Let $H$ be a graph without isolated vertices with respect to which complements are always unique. Then $H$ is one of the following: (a) $rK(1, s)$, (b) $rK_s$, (c) $K_s$, (d) $C_5$, or (e) $K(s, s)$, for some integers $r$ and/or $s$.

The above results suggest that the complete bipartite graph $K(s, s)$ is an obvious alternate to $K_n$ in Nordhaus-Gaddum results. In Goddard et al. (1992), the sums and products of $\psi(G_1)$ and $\psi(G_2)$ are examined where $G_1 \oplus G_2 = K(s, s)$, and $\psi$ is the independence, domination, or independent domination number, inter alia. In this section we take $\psi$ to be the total domination number.

2.1 Notation

For the rest of the section we shall assume that $K(s, s)$ has partite sets $\mathcal{L}$ and $\mathcal{R}$ (standing for “left” and “right”), and that $G_1 \oplus G_2 = K(s, s)$ where neither $G_1$ nor $G_2$ have an isolated vertex. For $G \subset K(s, s)$, let $\delta^L(G)$ denote the minimum degree of a vertex of $\mathcal{L}$. Define $\delta^R(G)$, $\Delta^L(G)$ and $\Delta^R(G)$ similarly. Further we shall abbreviate parameters by writing $\gamma_i^L$ for $\gamma_i(G_i)$, $\psi_i$ for $\psi(G_i)$ for parameters $\psi \neq \gamma_i$, $N_i(v)$ for $N_{G_i}(v)$, $E_i$ for $E(G_i)$, $V_i$ for $V(G_i)$, and $m_i$ for the size of $G_i$, for $i \in \{1, 2\}$.

Recall that for subsets $S, T \subseteq V$, the set $S$ totally dominates the set $T$ if $T \subseteq N(S)$. Let $G \subset K(s, s)$. A left total dominating set, abbreviated left TDS, of $G$ is a set of vertices that totally dominates $\mathcal{L}$; that is, a left TDS is a set $S$ of vertices of $G$ such that $\mathcal{L} \subseteq N(S)$. Necessarily a left TDS of $G$ belongs to $\mathcal{L}$. The minimum cardinality of a left TDS in $G$ is called the left domination number, denoted $\ell(G)$, of $G$. Similarly, a right total dominating set, abbreviated right TDS, of $G$ is a set of vertices that totally dominates $\mathcal{R}$ and the minimum cardinality of a right TDS in $G$ is the right domination number, denoted $r(G)$, of $G$. We note that a right TDS of $G$ belongs to $\mathcal{L}$.

2.2 Preliminary Results

Cockayne et al. (Cockayne et al. (1980)) obtained the following upper bound on the total domination number of a connected graph in terms of the order of the graph.

Theorem 3 (Cockayne et al. (1980)) If $G$ is a connected graph of order $n \geq 3$, then $\gamma_i(G) \leq 2n/3$.

We shall need the following observation.

Observation 4 Let $G_1 \oplus G_2 = K(s, s)$, and let $i \in \{1, 2\}$. Then the following hold.
(a) $\delta^L_i + \Delta^L_{s-i} = \delta^R_i + \Delta^R_{s-i} = s$.
(b) $\delta^L_s + \delta^R_{s-i} \leq s$.

Proof. Part (a) is immediate. Counting edges in $G_1$ and $G_2$, we have $m_i \geq s\delta^L_i$ and $m_{s-i} \geq s\delta^R_{s-i}$, and so $s^2 = m_i + m_{s-i} \geq s(\delta^L_i + \delta^R_{s-i})$, or, equivalently, $\delta^L_s + \delta^R_{s-i} \leq s$. □

The following lemma will prove to be useful.

Lemma 5 Let $G_1 \oplus G_2 = K(s, s)$, where neither $G_1$ nor $G_2$ have an isolated vertex, and let $i \in \{1, 2\}$. Then the following hold.
(a) $2 \leq \ell_i, r_i \leq s$.
(b) $\gamma^L_i = \ell_i + r_i$.
(c) $\ell_i \leq 1 + \delta^R_{s-i}$ and $r_i \leq 1 + \delta^L_{s-i}$.
(d) $(\ell_i - 2)(\ell_{s-i} - 1) \leq \delta^R_{s-i} - 1$ and $(r_i - 2)(r_{s-i} - 1) \leq \delta^L_{s-i} - 1$.
(e) $\gamma_i^L \leq 2 + 2 \min\{\Delta^L_{s-i}, \Delta^R_{s-i}\}$.
Proof. (a) Part (a) follows from the observation that \(1 \leq \delta_r \leq \Delta_1 \leq s - 1\).

(b) Part (b) follows from the observation that every TDS of \(G_1\) can be partitioned into two sets, one contained in \(L\) that totally dominates \(R\) and the other contained in \(R\) that totally dominates \(L\).

(c) Let \(v \in R\) be a vertex of degree \(\Delta^R_1\) in \(G_1\). We now consider the set \(N_2(v)\) and note that \(|N_2(v)| = \delta_r^R\).

For each vertex \(u \in N_2(v)\), select an arbitrary neighbor \(u'\) in \(G_1\). Let \(U = \cup\{u'\}\), where the union is taken over all vertices \(u \in N_2(v)\). Then, \(U \cup \{v\}\) is a left TDS in \(G_1\), and so \(\ell_1 \leq |U| + 1 \leq |N_2(v)| + 1 = 1 + \delta_r^R\).

The other results follow by symmetry.

(d) Let \(v \in R\) have degree \(\Delta^R_1\) in \(G_1\) and let \(X = L \setminus N_1(v)\). We now partition \(X\) into subsets \(X_1, \ldots, X_k\) of size at most \(r_2 - 1\) such that \(k\) is as small as possible. For each \(i = 1, \ldots, k\), the set \(X_i\) does not dominate \(R\) in \(G_2\), and so there exists a \(y_i \in R \setminus N_2(X_i)\) which therefore totally dominates \(X_i\) in \(G_1\).

Hence, \(\{v, y_1, \ldots, y_k\}\) totally dominates \(L\) in \(G_1\). Since \(k = \lceil |X|/(r_2 - 1) \rceil\) and \(|X| = s - \Delta_1^R = \delta_r^R\), we deduce that \(\ell_1 \leq 1 + \lceil \delta_r^R/(r_2 - 1) \rceil \leq 2 + (\delta_r^R - 1)/(r_2 - 1)\), or, equivalently, \((\ell_1 - 2)(r_2 - 1) \leq \delta_r^R - 1\). The other results follow by symmetry.

(e) Let \(v \in R\) have degree \(\Delta^R_1\) in \(G_1\), and let \(u \in L \setminus X\). Further let \(Y = N_1(u)\) and \(Z = R \setminus (Y \cup \{v\})\). If some vertex \(y \in Y\) dominates \(X\) in \(G_1\), then the degree of \(y\) in \(G_1\) exceeds that of \(v\), a contradiction. Hence every \(y \in Y\) is adjacent to at least one vertex of \(X\) in \(G_2\), and so \(X\) totally dominates \(Y\) in \(G_2\). For each vertex \(x \in X\), select an arbitrary neighbor \(x'\) in \(G_2\). Let \(X' = \cup_{x \in X} \{x'\}\) and note that \(X'\) totally dominates \(X\) in \(G_2\) and \(|X'| \leq |X|\). Moreover, \(\{u, v\}\) totally dominates \(V \setminus (X \cup Y)\) in \(G_2\). Thus, \(X \cup Y \cup \{u, v\}\) is a TDS in \(G_2\), and so \(\gamma^2_2 \leq 2 + |X| = 2 + 2\Delta^R_2\). Similarly by choosing \(v \in L\) to have degree \(\Delta^L_1\) in \(G_1\), we have that \(\gamma^2_1 \leq 2 + 2\Delta^L_1\). Thus, \(\gamma^2 = 2 + 2 \min\{\Delta^R_1, \Delta^L_2\}\). Similarly, \(\gamma^1 = 2 + 2 \min\{\Delta^R_1, \Delta^L_2\}\).

2.3 The Sum

In this section, we look at \(G_1 \oplus G_2 = K(s, s)\) and the upper bound on \(\gamma_1(G_1) + \gamma_1(G_2)\), where neither \(G_1\) nor \(G_2\) have an isolated vertex. We shall establish:

**Theorem 6** If \(G_1 \oplus G_2 = K(s, s)\) where neither \(G_1\) nor \(G_2\) has an isolated vertex, then \(\gamma_1(G_1) + \gamma_1(G_2) \leq 2s + 4\), with equality if and only if \(G_1 = sK_2\) or \(G_2 = sK_2\).

**Proof.** That the upper bound of \(2s + 4\) on the sum \(\gamma^1 + \gamma^2\) is sharp, may be seen by taking \(G_1 = sK_2\), in which case \(\gamma^1 = 2s\) and \(\gamma^2 = 4\), or, by symmetry, taking \(G_2 = sK_2\).

We now prove the upper bound holds. Renaming sets, if necessary, we may assume without loss of generality that \(r_1 = \min\{\ell_1, r_1, \ell_2, r_2\}\). By Lemma 5(a), \(r_1 \geq 2\). If \(r_1 = 2\), then we have that

\[
\gamma^1 + \gamma^2 \leq r_1 + \ell_1 + \gamma^2 \leq 2 + (1 + \delta_r^R) + (2 + 2\Delta^R_1) = 5 + s + \Delta^R_1. \tag{by Lemma 5(b) and (c) and Observation 4}
\]

Suppose, further, that \(\gamma^1 + \gamma^2 = 2s + 4\) (and still \(r_1 = 2\)). Then equality occurs throughout the above inequality chain. In particular, this requires that \(\gamma^2 = 2 + 2\Delta^R_1\) and \(\Delta^R_1 = s - 1\). But then \(\gamma^2 = 2s\). This is only possible if \(G_2 = sK_2\).

Hence we may assume that \(r_1 \geq 3\), for otherwise the desired result follows. By Lemma 5(c), \(r_1 \leq 1 + \delta_r^R\), implying that \(\delta_r^R \geq 2\). Similarly, since \(\ell_1 \geq r_1 \geq 3\), we note that \(\delta_r^R \geq 2\). Since \(r_2 \geq r_1 \geq 3\),
Lemma 5(d) implies that \( \ell_1 \leq 2 + (\delta_2^R - 1)/(r_2 - 1) \leq 2 + (\delta_2^R - 1)/2 \). Similarly, \( r_1 \leq 2 + (\delta_2^L - 1)/(\ell_2 - 1) \leq 2 + (\delta_2^L - 1)/2 \). By Lemma 5(b), \( \gamma_1^l = \ell_1 + r_1 \), and so
\[
\gamma_1^l \leq 3 + (\delta_2^R + \delta_2^L)/2. \tag{1}
\]

By Lemma 5(e), \( \gamma_2^l \leq 2 + 2 \min\{\Delta_1^R, \Delta_1^L\} \leq 2 + \Delta_1^R + \Delta_1^L \). By Observation 4 we note that \( \Delta_1^R = s - \delta_2^R \) and \( \Delta_1^L = s - \delta_2^L \). Hence,
\[
\gamma_2^l \leq 2 + 2s - \delta_2^R - \delta_2^L. \tag{2}
\]

Adding Equation (1) and Equation (2), we have that \( \gamma_1^l + \gamma_2^l \leq 2s + 5 - (\delta_2^R + \delta_2^L)/2 \). However as observed earlier, \( r_1 \geq 3 \) implies that \( \delta_2^R \geq 2 \) and \( \delta_2^L \geq 2 \). Hence, \( \gamma_1^l + \gamma_2^l \leq 2s + 5 - 2 < 2s + 4 \). \( \square \)

2.4 The Product

In this section, we look at \( G_1 \oplus G_2 = K(s, s) \) and the upper bound on \( \gamma(t)(G_1)\gamma(t)(G_2) \), where neither \( G_1 \) nor \( G_2 \) has an isolated vertex. We shall establish:

**Theorem 7** If \( G_1 \oplus G_2 = K(s, s) \) where neither \( G_1 \) nor \( G_2 \) has an isolated vertex, then \( \gamma(t)(G_1)\gamma(t)(G_2) \leq \max\{8s, [(s + 6)^2]/4\} \).

**Proof.** That the upper bound of 8s on the product \( \gamma_1^1 \gamma_2^2 \) is achievable, may be seen by taking \( G_1 = sK_2 \) or \( G_2 = sK_2 \). We remark that if \( G_1 \cong B_{12} \), where \( B_{12} \) is the bipartite cubic graph of order \( n = 12 \) shown in Figure 1, then \( G_2 \cong B_{12} \) and \( \gamma_1^1 \gamma_2^2 = 36 = [(s + 6)^2]/4 \). This shows that the upper bound of \( [(s + 6)^2]/4 \) on the product \( \gamma_1^1 \gamma_2^2 \) is achievable.

![Fig. 1: The bipartite cubic graph B_{12}.](image)

If \( \gamma_1^1 = 4 \), then since \( \gamma_2^2 \leq 2s \), we have that \( \gamma_1^1 \gamma_2^2 \leq 8s \). Further if \( \gamma_1^1 \gamma_2^2 = 8s \), then \( \gamma_2^2 = 2s \) which is only possible if \( G_2 = sK_2 \). Similarly, if \( \gamma_2^2 = 4 \), then \( \gamma_1^1 \gamma_2^2 \leq 8s \), with equality if and only if \( G_1 = sK_2 \). Hence we may assume that \( \gamma_1^1 \geq 5 \) and \( \gamma_2^2 \geq 5 \), for otherwise the desired result follows. In particular, this implies that both \( G_1 \) and \( G_2 \) are connected. Renaming sets, if necessary, we may assume without loss of generality that \( r_1 = \min\{\ell_1, r_1, \ell_2, r_2\} \). By Lemma 5(a), \( r_1 \geq 2 \).

We proceed further with the following claim.

**Claim A:** If \( r_1 \geq 3 \), then \( \gamma_1^1 \gamma_2^2 \leq [(s + 6)^2]/4 \).

**Proof.** An identical argument as in the proof of Theorem 6 shows that Equation (1) holds; that is, \( \gamma_1^1 \leq 3 + (\delta_2^R + \delta_2^L)/2 \). By a symmetric argument, \( \gamma_2^2 \leq 3 + (\delta_2^R + \delta_2^L)/2 \). Adding these two inequalities yields \( \gamma_1^1 + \gamma_2^2 \leq 6 + (\delta_2^R + \delta_2^L + \delta_2^R + \delta_2^L)/2 \). Hence, by Observation 4(b), \( \gamma_1^1 + \gamma_2^2 \leq s + 6 \). Since the geometric
mean is at most the arithmetic mean, we note that \( \sqrt{\frac{1}{\ell_1} + \frac{1}{\ell_2}} \leq (\gamma_1^1 + \gamma_1^2)/2 \), and so \( \gamma_1^1 \gamma_1^2 \leq [(s + 6)^2/4] \). This completes the proof of Claim A. (c)

By Claim A we may assume that \( r_1 = 2 \). We now let \( \ell_1 = k + 1 \) and continue with the following claim.

CLAIM B: We may assume that the following hold.

(a) \( \ell_2 \leq s - k \).
(b) \( 4 \leq r_2 < s/k + 1 \).
(c) \( 4 \leq k < s/3 \).

PROOF. (a) Let \( X \) be a minimum left TDS of \( G_1 \) and note that \( X \subseteq \mathcal{R} \) and \( |X| = \ell_1 \). Let \( X = \{x_1, \ldots, x_{\ell_1} \} \). By the minimality of \( X \), every vertex \( x \in X \) has at least one neighbor in \( G_1 \) which has no other neighbor in \( X \) except for \( x \). For each \( x_i \in X \), let \( y_i \) be such a neighbor, and so \( N_1(y_i) \cap X = \{x_i\} \).

Let \( Y = \{y_1, \ldots, y_{\ell_1} \} \) and note that \( Y \subseteq \mathcal{L} \) and that \( |X| = |Y| = \ell_1 \). Furthermore, \( Y \setminus \{y_i\} \subseteq N_2(x_i) \) for all \( i \in \{1, \ldots, \ell_1\} \). Let \( Y' = \mathcal{L} \setminus Y = \{y'_1, \ldots, y'_{\ell_1} \} \).

Suppose there are no edges between \( X \) and \( Y' \) in \( G_2 \), that is \( G_2[X,Y'] = \emptyset \). Since \( G_2 \) is connected, there must exist vertices \( y_{j_1} \in X \) and \( y'_{j_2} \in Y' \) with a common neighbor, \( x' \) say, in \( G_2 \). Necessarily, \( x' \in \mathcal{R} \setminus X \). Now, for each \( i \in \{1, \ldots, s - \ell_1 \} \setminus \{j_2\} \), we let \( x'_i \in N_2(y'_i) \), and let \( x''_{j_2} = x' \). Let

\[
X' = \bigcup_{i=1}^{s-\ell_1} \{x'_i\}
\]

and note that \( |X'| \leq s - \ell_1 \). But now \( Y \setminus \{y_{j_1}\} \subseteq N_2(x_{j_1}) \) and \( Y' \cup \{y_{j_1}\} \subseteq N_2(X') \), and so \( X' \cup \{x_{j_1}\} \) is a left TDS of \( G_2 \). Hence, \( \ell_2 \leq |X' \cup \{x_{j_1}\}| \leq s - \ell_1 + 1 = s - k \), as desired.

Hence we may assume that \( G_2[X,Y'] \neq \emptyset \). Thus for some \( k_1 \in \{1, \ldots, \ell_1\} \) and some \( k_2 \in \{1, \ldots, s - \ell_1\} \), we have \( x_{k_1}, y'_{k_2} \in E_2 \). For each \( i \in \{1, \ldots, s - \ell_1\} \setminus \{k_2\} \), we let \( x''_i \in N_2(y'_j) \), and let \( x''_{k_2} = x_{k_1} \). Let

\[
X'' = \bigcup_{i=1}^{s-\ell_1} \{x''_i\}
\]

and note that \( |X''| \leq s - \ell_1 \). Let \( k_3 \in \{1, \ldots, \ell_1\} \setminus \{k_2\} \). Then, \( Y \setminus \{y_{k_3}\} \subseteq N_2(x_{k_3}) \) and \( Y' \cup \{y_{k_3}\} \subseteq N_2(X'') \), and so \( X'' \cup \{x_{k_3}\} \) is a left TDS of \( G_2 \). Hence, \( \ell_2 \leq |X'' \cup \{x_{k_3}\}| \leq s - \ell_1 + 1 = s - k \), as desired.

(b) Suppose \( r_2 \leq 3 \). Recall that \( \ell_1 = k + 1 \), \( r_1 = 2 \), and by part (a), \( \ell_2 \leq s - k \). Therefore, \( \gamma_1^1 \gamma_1^2 = (\ell_1 + r_1)(\ell_2 + r_2) \leq (k + 3)(s - k + 3) \). Since the geometric mean is at most the arithmetic mean, we have that \( \sqrt{(k + 3)(s - k + 3)} \leq (s + 6)/2 \), and so \( \gamma_1^1 \gamma_1^2 \leq [(s + 6)^2/4] \). Hence, we may assume that \( r_2 \geq 4 \). We now partition \( \mathcal{R} \) into subsets \( X_1, \ldots, X_j \) of size at most \( k \) such that \( j \) is as small as possible.

Since \( \ell_1 = k + 1 \), for each \( i = 1, \ldots, j \), the set \( X_i \) does not dominate \( \mathcal{L} \) in \( G_1 \), and so there exists a \( y_i \in \mathcal{L} \setminus N_1(X_i) \) which therefore totally dominates \( X_i \) in \( G_2 \). Hence, \( \{y_1, \ldots, y_j\} \) totally dominates \( \mathcal{R} \) in \( G_2 \). Since \( j = \lceil |\mathcal{R}|/k \rceil \) and \( |\mathcal{R}| = s \), we deduce that \( r_2 \leq \lceil s/k \rceil < s/k + 1 \).

(c) Suppose \( k \leq 3 \). Then, \( \gamma_1^1 = r_1 + \ell_1 \leq 6 \). By Theorem 3, \( \gamma_1^2 = 2|V_2|/3 = 4s/3 \). Hence, \( \gamma_1^1 \gamma_1^2 \leq 6(4s/3) = 8s \). We may therefore assume that \( k \geq 4 \). By Part (b), we may assume \( 4 < s/k + 1 \) and thus \( k < s/3 \). This completes the proof of Claim B. (c)
By Claims A and B we now have that $r_1 = 2$, $\ell_1 = k + 1$, $r_2 < s/k + 1$, and $\ell_2 \leq s - k$. Hence, by Lemma 5(b), we have

$$\gamma_1^2 \gamma_2^2 < (k + 3)(s/k + s - k + 1). \quad (3)$$

We proceed with the following claim which introduces an assumption we can make on the size of $s$ and improves the bounds in Claim B(b) and Claim B(c).

CLAIM C: We may assume that the following hold.
(a) $s \geq 20$.
(b) $5 \leq r_2 < s/k + 1$.
(c) $4 \leq k < s/4$.

PROOF. (a) Suppose $s = 19$. Then by Claim B(c) we have $k \in \{4, 5, 6\}$. Substituting these values into Equation (3) we get $\gamma_1^2 \gamma_2^2 < 145.25$, $\gamma_1^2 \gamma_2^2 < 150.4$, and $\gamma_1^2 \gamma_2^2 < 154.5$, respectively. In each case $\gamma_1^2 \gamma_2^2 < 156 = [(s + 6)^2/4]$, and thus we may assume $s \neq 19$. Suppose now that $s \leq 18$. By Claim B(c) we have $k \in \{4, 5\}$. If $k = 4$ then

$$\gamma_1^2 \gamma_2^2 < (7)(s/4 + s - 3) \quad \text{(substituting into Equation (3))}$$

$$= 8s + 3s/4 - 21$$

$$< 8s \quad \text{(since } 3s/4 \leq 3(18)/4 < 21).$$

Hence we may assume that $k = 5$. But now we have

$$\gamma_1^2 \gamma_2^2 < (8)(s/5 + s - 4) \quad \text{(substituting into Equation (3))}$$

$$< 8s \quad \text{(since } s/5 \leq 18/5 < 4).$$

Hence we may assume that $s \geq 20$.

(b) By Claim B(b) we have $4 \leq r_2 < s/k + 1$. If $r_2 = 4$ then

$$\gamma_1^2 \gamma_2^2 = (\ell_1 + r_1)(\ell_2 + r_2) \quad \text{(by Lemma 5(b))}$$

$$\leq (k + 3)(s - k + 4) \quad \text{(by Claim B(a)).}$$

We note that $(k + 3)(s - k + 4)$ is a parabola as a function of $k$ which achieves its maximum at $k = (s + 1)/2$ and is therefore strictly increasing on the interval $[4, s/3]$. Therefore, since $4 \leq k < s/3$ (by Claim B(c)), we have

$$\gamma_1^2 \gamma_2^2 < (s/3 + 3)(s - s/3 + 4)$$

$$= 2s^2/9 + 10s/3 + 12$$

$$= (s + 6)^2/4 - s^2/36 + s/3 + 3.$$ 

But by Part (a) we have $s > 18$, and so $s^2/36 > 18s/36 = s/2 = s/3 + s/6 > s/3 + 3$. Thus, $-s^2/36 + s/3 + 3 < 0$ and the above inequality chain reduces to $\gamma_1^2 \gamma_2^2 < (s + 6)^2/4$, and hence $\gamma_1^2 \gamma_2^2 \leq [(s + 6)^2/4]$. We may therefore assume that $r_2 \neq 4$, and so $5 \leq r_2 < s/k + 1$.

(c) By Claim B(c) we have $k \geq 4$. By Part (b), we may assume $5 < s/k + 1$ and thus $k < s/4$. This completes the proof of Claim C.

We return to the proof of Theorem 7. Multiplying out Equation (3) yields

$$\gamma_1^2 \gamma_2^2 < s + 3s/k + (k + 3)(s - k + 1). \quad (4)$$
We note that since \( k \geq 4 \) we have that \( 3s/k \leq 3s/4 \). Furthermore, \((k+3)(s-k+1)\) is a parabola as a function of \( k \) which achieves its maximum at \( k = (s-2)/2 \) and is therefore strictly increasing on the interval \([4, s/4]\). Using this information and the fact that \( 4 \leq k < s/4 \) (by Claim C(c)) in Equation (4), we get
\[
\gamma_1^t \gamma_2^t < s + 3s/4 + (s/4 + 3)(3s/4 + 1) \\
= 3s^2/16 + 17s/4 + 3 \\
= (s + 6)^2/4 - s^2/16 + 5s/4 - 6.
\]
But by Claim C(a) we have \( s \geq 20 \) and so \( s^2/16 \geq 20s/16 = 5s/4 \). Thus, \(-s^2/16 + 5s/4 - 6 < 0\) and the above inequality chain reduces to \( \gamma_1^t \gamma_2^t < (s + 6)^2/4 \). We conclude that \( \gamma_1^t \gamma_2^t \leq \lfloor (s + 6)^2/4 \rfloor \). This completes the proof of Theorem 7. \( \square \)

That the bound of Theorem 7 is essentially best possible, may be seen as follows. For \( s \geq 2 \), by taking \( G_1 = sK_2 \) we note that \( \gamma_t(G_1) = 2s \) and \( \gamma_t(G_2) = 4 \), whence \( \gamma_t(G_1) \gamma_t(G_2) = 8s \). Let \( s \geq 4 \) and let \( K(s,s) \) have partite sets \( A \) and \( B \). Partition \( A \) \((B)\) into two sets, one of size \([s/2], 1\) \([s/2], \) and one of size \([s/2], 2\), say \( A_1 \) \([B_1], \) and \( A_2 \) \([B_2]. \) Form the edge set of \( G_1 \) from the set of all edges between \( A \) and \( B_1 \) by removing (the edges of) a matching between \( A_1 \) and \( B_1 \) and inserting a matching between \( A_2 \) and \( B_2 \). If \( S \) is a TDS of \( G_1 \), then \( A_2 \subseteq S \) in order to totally dominate \( B_2 \), while \( |S \cap B_1| \geq 2 \) in order to totally dominate \( A_1 \). Hence, \( \gamma_t(G_1) \geq [s/2] + 2 \). However adding any two vertices of \( B_1 \) to the set \( A_2 \) produces a TDS of \( G_1 \). Consequently, \( \gamma_t(G_1) = [s/2] + 2 \). Similarly, \( \gamma_t(G_2) = [s/2] + 2 \). Thus, \( \gamma_t(G_1) \gamma_t(G_2) = ([s/2] + 2)[[s/2] + 2] = [[s/2] + 2]^2 = [[s + 4]^2] / 4 \).

We remark that \( [[s + 4]^2] / 4 \) \( \geq 8s \) for \( s \geq 18 \).

References

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Nordhaus-Gaddum Type Results for Total Domination


