

On Weakly Distinguishing Graph Polynomials

Johann A. Makowsky¹

Vsevolod Rakita²

¹ Technion - Israel Institute of Technology, Department of Computer Science

² Technion - Israel Institute of Technology, Department of Mathematics

received 1st Nov. 2018, revised 5th Mar. 2019, accepted 25th Mar. 2019.

A univariate graph polynomial $P(G; X)$ is weakly distinguishing if for almost all finite graphs G there is a finite graph H with $P(G; X) = P(H; X)$. We show that the clique polynomial and the independence polynomial are weakly distinguishing. Furthermore, we show that generating functions of induced subgraphs with property C are weakly distinguishing provided that C is of bounded degeneracy or treewidth. The same holds for the harmonious chromatic polynomial.

Keywords: Graph Theory, Graph Polynomials

1 Introduction and Outline

Throughout this paper we consider only simple (i.e. finite, undirected loopless graphs without parallel edges), vertex labelled graphs. Let P be a graph polynomial. A graph G is P -unique if every graph H with $P(G; X) = P(H; X)$ is isomorphic to G . A graph H is a P -mate of G if $P(G; X) = P(H; X)$ but H is not isomorphic to G . In [14] P -unique graphs are studied for the Tutte polynomial $T(G; X, Y)$, the chromatic polynomial $\chi(G; X)$, the matching polynomial $m(G; X)$ and the characteristic polynomial $\text{char}(P; X)$.

A statement holds for almost all graphs if the proportion of graphs of order n for which it holds, tends to 1, when n tends to infinity. A graph polynomial P is *almost complete* if almost all graphs G are P -unique, and it is *weakly distinguishing* if almost all graphs G have a P -mate. In [3] it is conjectured that almost all graphs are χ -unique and T -unique, in other words, both $\chi(G; X)$ and $T(G; X, Y)$ are almost complete. There are plenty of trivial graph polynomials which are weakly distinguishing, like $X^{|V(G)|}$ or $X^{|E(G)|}$. However, one might expect that the prominent graph polynomials from the literature are not weakly distinguishing. Here we show that various non-trivial graph polynomials are still weakly distinguishing.

The degree polynomial $\text{Deg}(G; x)$ of a graph G is the generating function of the degree sequence of G . A graph G is *Deg-unique*, also called in the literature a *unigraph*, if it is determined by its degree sequence. An updated discussion on how to recognize unigraphs can be found in [4].

A simple counting argument gives:

Theorem 1. *Almost all graphs G have a Deg-mate.*

The *Independence and Clique polynomials* of a graph $G = (V(G), E(G))$ contain much information about G . Both were first studied in [11]. For a more recent survey on the independence polynomial see [13].

Theorem 2. *The independence and clique polynomials are weakly distinguishing.*

The proof uses estimates for the independence number $\alpha(G)$ and the clique number $\omega(G)$ for random graphs, (see [2] and [7]) together with a counting argument.

This theorem can be generalized:

Definition 1. Let \mathcal{C} be a graph property. We say that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is an *independence (clique) function for \mathcal{C}* if for every graph $G \in \mathcal{C}$, the graph G has an independent set (clique) of size $f(|V(G)|)$.

Denote by $\hat{\mathcal{C}}$ the class of complement graphs \bar{G} of graphs $G \in \mathcal{C}$, and $P_{\mathcal{C}}(G; X) = \sum_{A \subset V(G): G[A] \in \mathcal{C}} X^{|A|}$, and $P_{\hat{\mathcal{C}}}(G; X) = \sum_{A \subset V(G): G[A] \in \hat{\mathcal{C}}} X^{|A|}$.

Theorem 3. *Let \mathcal{C} be a graph property that has an independence or a clique function f that satisfies that for all $n \in \mathbb{N}$, $f(n) \geq n/a$ for some fixed $a \in \mathbb{N}$. Then $P_{\mathcal{C}}$ is weakly distinguishing.*

This applies to the following cases:

- A graph G is k -degenerate if every induced subgraph of G has a vertex of degree at most k . It is easy to see that every k -degenerate graph G of order n has an independent set of size $\left\lceil \frac{n}{k+1} \right\rceil$.
- Among the k -degenerate graphs we find the graphs of treewidth at most k , graphs of degree at most k , and planar graphs.
- A k -colourable graphs G has an independent set of size at least $\left\lceil \frac{n}{k} \right\rceil$.
- Let \mathcal{C} be a graph property. A function $\gamma : V(G) \rightarrow [k]$ is a \mathcal{C} -colouring if every color class induces a graph in \mathcal{C} . Such coloring were studied in [9]. If we assume that \mathcal{C} has an independence (clique) function $g(n)$, then the graphs which are \mathcal{C} -colorable with at most k colors have an independence (clique) function $f(n) = \left\lceil \frac{g(n)}{k} \right\rceil$.

Therefore, for \mathcal{C} one of the properties above, the graph polynomials $P_{\mathcal{C}}(G; X)$ are all weakly distinguishing.

A harmonious colouring of G with at most k colors is a proper colouring of G such that every pair of colors occurs at most once along an edge. Let $\chi_{\text{harm}}(G; k)$ count the number of harmonious colourings of G . It was observed in [12] that $\chi_{\text{harm}}(G; k)$ is a polynomial in k .

Theorem 4. *Almost all graphs G have a χ_{harm} -mate.*

The status of P -uniqueness remains open for $T(G; X, Y)$, $\chi(G; X)$, $m(G; X)$ and $\text{char}(G; X)$.

2 Preliminaries

Let $G = (V, E)$ be a graph. Denote by $\mathcal{G}(n)$ the set of all non-isomorphic graphs with n vertices, and by \mathcal{G} the set of all non-isomorphic graphs.

Fact 5 ([10]). $|\mathcal{G}(n)| \approx \frac{2 \binom{n}{2}}{n!}$ for a sufficiently large n .

Let $P : \mathcal{G} \rightarrow \mathbb{Z}[x]$ be a graph polynomial. For a graph G , we say two non-isomorphic graphs G and H are called P -mates if $P(G) = P(H)$.

Denote by $U_P(n)$ the set of P unique graphs with n vertices, by $\beta_P(n)$ the number of polynomials in $\mathbb{Z}[x]$ such that there is a graph of order less or equal to n that maps to that polynomial, and by $\beta_{P,\mathcal{C}}(n)$ the number of polynomials in $\mathbb{Z}[x]$ such that there is a graph of order less or equal to n in \mathcal{C} that maps to that polynomial.

We denote by K_n the clique of size n , and by I_n the edgeless graph of size n .

Let G be a graph, and $A \subseteq V(G)$. The induced subgraph of A in G , denoted $G[A]$, is the graph with vertex set A , and for $v, u \in A$, $(u, v) \in E(G[A])$ iff $(u, v) \in E(G)$.

Definition 2. For a graph polynomial P , we say P is weakly distinguishing if $\lim_{n \rightarrow \infty} \frac{|U_P(n)|}{|\mathcal{G}(n)|} = 0$. For a family of graphs \mathcal{C} we say that P is weakly distinguishing on \mathcal{C} if $\lim_{n \rightarrow \infty} \frac{|U_P(n) \cap \mathcal{C}|}{|\mathcal{G}(n) \cap \mathcal{C}|} = 0$

We wish to consider a particular type of graph polynomials:

Definition 3. Let Q be a graph property. For all graphs G , define $P_Q(G; x) = \sum_{A \subseteq V(G): G[A] \in Q} X^{|A|}$.

3 The Degree Polynomial

Definition 4. For a graph $G = (V, E)$ of order n and $v \in V$, denote by $deg(v)$ the degree of v . Define the Degree polynomial of G to be $Deg(G, x) = \sum_{v \in V} x^{deg(v)}$.

Note that the degree of a vertex is bounded above by $n - 1$, so the degree of the polynomial $Deg(G, x)$ is at most $n - 1$. For every $0 \leq i \leq n - 1$, the coefficient of x^i in $Deg(G, x)$ is an integer number between 0 and n . Thus, we get

$$\beta_{Deg}(n) \leq (n + 1)^{n-1} \leq (n + 1)^n$$

Now we are ready to prove theorem 1:

Theorem 1: Almost all graphs G have a Dg -mate.

Proof: Let $G = (V, E)$ be a graph with $|V(G)| = n$. We now evaluate:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{U_{Deg}(n)}{|\mathcal{G}(n)|} &\leq \lim_{n \rightarrow \infty} \frac{\beta_{Deg}(n)}{|\mathcal{G}(n)|} \leq \lim_{n \rightarrow \infty} \frac{n^n}{|\mathcal{G}(n)|} = \lim_{n \rightarrow \infty} \frac{(n + 1)^n n!}{2^{n(n-1)/2}} \\ &\leq \lim_{n \rightarrow \infty} \frac{(n + 1)^n \cdot (n + 1)^n}{2^{n(n-1)/2}} = \lim_{n \rightarrow \infty} \frac{(n + 1)^{2n}}{2^{n(n-1)/2}} = 0 \end{aligned}$$

□

4 A General Method for Proving Graph Polynomials are Weakly Distinguishing

We wish to apply the same idea used in proving the degree polynomial is weakly distinguishing to a large class of graph polynomials. We start with some lemmas. First, we show that if a graph polynomial P is weakly distinguishing on a large subset of \mathcal{G} , it is weakly distinguishing:

Lemma 6. Let P be a graph polynomial and \mathcal{C} a family of graphs such that $\lim_{n \rightarrow \infty} |\mathcal{C}(n)|/|\mathcal{G}(n)| = 1$. If $\lim_{n \rightarrow \infty} |U_P(n) \cap \mathcal{C}|/|\mathcal{G}(n)| = 0$ then P is weakly distinguishing.

Proof:

$$\frac{|U_P(n)|}{|\mathcal{G}(n)|} = \frac{|U_P(n) \cap \mathcal{C}| + |U_P(n) \cap (\mathcal{G}(n) - \mathcal{C})|}{|\mathcal{G}(n)|} = \frac{|U_P(n) \cap \mathcal{C}|}{|\mathcal{G}(n)|} + \frac{|U_P(n) \cap (\mathcal{G}(n) - \mathcal{C})|}{|\mathcal{G}(n)|}$$

When taking the limit, note that the left term in the sum converges to 0 by assumption, so it remains to evaluate:

$$\lim_{n \rightarrow \infty} \frac{|U_P(n) \cap (\mathcal{G} - \mathcal{C})|}{|\mathcal{G}(n)|} \leq \lim_{n \rightarrow \infty} \frac{|\mathcal{G}(n) - \mathcal{C}|}{|\mathcal{G}(n)|} = 0$$

□

Lemma 7. *Let $f : \mathbb{N} \rightarrow \mathbb{R}$. If $f(n) \leq (\log n)^{O(1)}$, then asymptotically*

$$\binom{n}{f(n)}^{f(n)} \leq \left(\frac{n}{f(n)}\right)^{f(n)+nf(n)} \frac{1}{(2\pi)^{f(n)/2} \cdot n^{f(n)/2}}$$

Proof: By applying the Stirling approximation $k! = \sqrt{2\pi k} \left(\frac{k}{e}\right)^k$ we evaluate:

$$\begin{aligned} \binom{n}{f(n)}^{f(n)} &= \left(\frac{n!}{f(n)!(n-f(n))!}\right)^{f(n)} \\ &\approx \left(\frac{\sqrt{2\pi n}(n/e)^n}{\sqrt{2\pi f(n)}(f(n)/e)^{f(n)} \sqrt{2\pi(n-f(n))}((n-f(n))/e)^{(n-f(n))}}\right)^{f(n)} \\ &= \left(\frac{n^n}{(f(n))^{f(n)}(n-f(n))^{n-f(n)}} \cdot \frac{\sqrt{n}}{\sqrt{2\pi f(n)}(n-f(n))}\right)^{f(n)} \\ &\leq \left(\frac{n}{2\pi f(n) \cdot f(n)}\right)^{f(n)/2} \cdot \frac{n^{nf(n)}}{f(n)^{nf(n)}} \end{aligned}$$

where the inequality is due to $f(n) \leq n - f(n)$ for a sufficiently large n .

$$\left(\frac{n}{2\pi f(n) \cdot f(n)}\right)^{f(n)/2} \cdot \frac{n^{nf(n)}}{f(n)^{nf(n)}} = \left(\frac{n}{f(n)}\right)^{f(n)+nf(n)} \frac{1}{(2\pi)^{f(n)/2} \cdot n^{f(n)/2}}$$

□

Our main tool for proving graph polynomials are weakly distinguishing is theorem 3, which provides a sufficient condition for a polynomial P_Q to be weakly distinguishing. This condition is given in terms of independence and clique functions (see definition 1). We will prove theorem 3 using the following theorems:

Theorem 8 (Frieze [7]). For a graph G , denote by $\alpha(G)$ the size of the largest independent set of vertices in G . Then for almost all graphs of order n , $\alpha(G) \approx 4 \log \frac{n}{2}$

Theorem 9 (Erdős and Bollobás [2]). For almost all graphs G of order n , $\omega(G) \approx \frac{2}{\log 2} \cdot \log n$

We are now ready to prove theorem 3:

Theorem 3: Let Q be a graph property that has an independence or a clique function f that satisfies that for all $n \in \mathbb{N}$, $f(n) \geq n/a$ for some fixed $a \in \mathbb{N}$. Then P_Q is weakly distinguishing.

Proof: Assume f is an independence function. Set $\epsilon = 1/10$ and let $\mathcal{C} = \{G : \alpha(G) \leq 4 \log \frac{n}{2} + \epsilon\}$. By theorem 8, almost all graphs are in \mathcal{C} . Note that if $G \in \mathcal{C}$, and H is an induced subgraph of G with $H \in Q$, then there is an independent set of size $\frac{|V(H)|}{a}$ in H , and hence in G , and so $|V(H)| \leq 4 \log \frac{n}{2} + \epsilon$.

This implies that $P_Q(G, x) = \sum_{k=1}^{4 \log \frac{|V(G)|}{2} + \epsilon} b_k x^k$ with $0 \leq b_k \leq \binom{n}{k}$ for all k , and so

$$\beta_{P_Q, \mathcal{C}}(n) \leq \left(4 \log \frac{n}{2} + \epsilon \right)^{4 \log \frac{|V(G)|}{2} + \epsilon}$$

hence by lemmas 6 and 7, P_Q is weakly distinguishing.

If f is a clique function, the proof is similar using theorem 9. □

5 Applications of the Method

5.1 The Clique and Independence Polynomials

Definition 5. Let G be a graph. For $i \in \mathbb{N}$, denote $c_i(G) = |\{A \subseteq V(G) : G[A] \cong K_i\}|$. The clique polynomial of G , $Cl(G, x)$ is defined to be $Cl(G, x) = 1 + \sum_{i=1}^{\infty} c_i(G)x^i$. Note that this is a graph polynomial, and that the sum in the definition is finite. The clique number of G , denoted $\omega(G)$, is the degree of the clique polynomial (i.e. this is the size of the largest clique subgraph of G).

Definition 6. Let G be a graph. For $i \in \mathbb{N}$, denote $s_i(G) = |\{A \subseteq V(G) : G[A] \cong I_i\}|$. The independence polynomial of G , $Ind(G, x)$ is defined to be $Ind(G, x) = 1 + \sum_{i=1}^{\infty} s_i(G)x^i$. Note that this is a graph polynomial, and that the sum in the definition is finite. The independence number of G , denoted $\alpha(G)$, is the degree of the independence polynomial (i.e. this is the size of the largest independent set in G).

Theorem 2 is now a direct corollary of theorem 3:

Theorem 2: Almost all graphs G have an Ind -mate and a Cl -mate.

Proof: For the independence polynomial, note that $Ind(G, x) = P_Q(G; x)$, where Q is the property consisting of edgeless graphs. Note that the identity function on \mathbb{N} is an independence function for Q , and clearly it satisfies the condition in theorem 3 for $a = 1$. Hence the independence polynomial is weakly distinguishing.

Similarly, for the clique polynomial note that $Cl(G, x) = P_Q(G; x)$, where Q is the property of complete graphs. Note that the identity function on \mathbb{N} is a clique function for Q , and clearly it satisfies the condition in theorem 3 for $a = 1$. Hence the clique polynomial is weakly distinguishing. □

5.2 Generating Functions

Theorem 3 can be applied to many graph classes to produce weakly distinguishing graph polynomials. Of particular interest are k -degenerate classes and amongst them classes of bounded treewidth.

For a graph G , and $v \in V(G)$ denote by $N_G(v)$ the closed neighbourhood of v in G , i.e. $N_G(v) = \{v\} \cup \{u \in V(G) : \{v, u\} \in E(G)\}$

Definition 7. For $k \in \mathbb{N}$, a graph G is said to be k -degenerate if every induced subgraph of G has a vertex of degree at most k .

The following propositions 9.1, 10.1 and lemma 10 are well known results about degenerate graphs and treewidth. For completeness, we include their proofs:

Proposition 9.1. A graph $G = (V, E)$ is k degenerate if and only if there is a enumeration $\{v_1, v_2, \dots, v_n\} = V$ such that for all $1 \leq i \leq n$ the degree of v_i in the subgraph of G induced by $V - \{v_1, v_2, \dots, v_{i-1}\}$ is at most k .

Proof: Let $G = (V, E)$ be a k degenerate graph. From the definition, there is a vertex $v \in V$ with degree at most k . Denote this vertex by v_1 . Define v_i inductively: from the definition, the subgraph induced by $V - \{v_1, \dots, v_{i-1}\}$ has a vertex with degree at most k . Define v_i to be this vertex. Clearly, the enumeration $\{v_1, \dots, v_n\} = V$ has the desired property.

Conversely, let $\{v_1, \dots, v_n\}$ an enumeration as in the theorem, and let H be a subgraph of G induced by $U \subseteq V$. Denote $u = v_i$ the vertex in H who's index in the enumeration is the smallest. Note that the degree of u in $G[U \cup \{v_j | j \geq i\}]$ is at most k , and H is a subgraph of $G[U \cup \{v_j | j \geq i\}]$, hence the degree of u in H is at most k . So G is k degenerate, as required. \square

Lemma 10. A graph with treewidth at most k has a vertex with degree at most k .

Proof: Let G be a graph, and (T, X) a tree decomposition of G with width k . Note that T has a leaf, and there is a vertex v in the bag corresponding to this leaf that is not in the bag corresponding to its neighbour. Thus every neighbour of v in G is in the same bag. But the bag is of size at most $k + 1$, so v is of degree at most k . \square

Proposition 10.1. A graph G with treewidth at most k is k degenerate.

Proof: Let H be an induced subgraph of G . If $H = G$, then H has a vertex of degree at most k by the previous lemma. If H is a proper subgraph, note that H has treewidth at most k , so H has a vertex of degree at most k . So G is k degenerate. \square

The following proposition shows that a k degenerate graph has a large independent set:

Proposition 10.2. Every k degenerate graph G has an independent set of size $\left\lceil \frac{|V|}{k+1} \right\rceil$.

Proof: Let $G = (V, E)$ be a k degenerate graph, and $\{v_1, \dots, v_n\}$ be an enumeration as in proposition 9.1. Let $I_0 = \emptyset$, and $H_0 = G$. We construct an independent set inductively as follows. There exists $1 \leq l \leq n$ and an increasing sequence i_1, i_2, \dots, i_l in $\{1, 2, \dots, n\}$ with $i_1 = 1$ such that for all $1 \leq j \leq l$

$$\begin{aligned} I_j &= I_{j-1} \cup \{v_{i_j}\} \\ H_j &= G[V(H_{j-1}) - N_G(v_{i_j})] \end{aligned}$$

with I_l an independent set in G and $l \geq \left\lceil \frac{n}{k+1} \right\rceil$.

Indeed, $I_1 = \{v_1\}$ and $H_1 = G[V(G) - N_G(v_1)]$. Clearly, I_1 is an independent set, and note that $|V(H_1)| \geq n - (k+1)$ and no vertex in H_1 is a neighbour of the vertex in I_1 . Now, given an independent set I_j and an induced subgraph H_j of G such that $|V(H_j)| \geq n - j(k+1)$ and no vertex of H_j is a neighbour of a vertex in I_j , select $v_{i_{j+1}} \in V(H_j)$ with minimal index. Now I_{j+1} is an independent set, no vertex of H_{j+1} is a neighbour of a vertex in I_{j+1} , and since $\deg_{H_j}(v_{i_{j+1}}) \leq k$, $|V(H_{j+1})| \geq |V(H_j)| - (k+1) \geq n - (j+1)(k+1)$.

The induction stops when no more vertices can be selected, i.e. when $V(H_l) = \emptyset$. From the induction, we have that $0 = |V(H_l)| \geq n - l(k+1)$ and hence $l \geq \left\lceil \frac{n}{k+1} \right\rceil$ as required. \square

Combining this proposition with theorem 3, we can show that many non trivial graph polynomials are weakly distinguishing:

Corollary 10.1. *Fix $k \in \mathbb{N}$, and let Q be a class of k degenerate graphs. Then P_Q is weakly distinguishing.*

Corollary 10.2. *Fix $k \in \mathbb{N}$, and let Q be a class of graphs with treewidth at most k . Then P_Q is weakly distinguishing.*

6 The Harmonious and k -Harmonious Polynomials

Definition 8. For a graph G , a harmonious colouring in k colours is a function $f : V(G) \rightarrow [k]$ such that f is a proper colouring, and for all $i, j \in [k]$, $G[f^{-1}(i) \cup f^{-1}(j)]$ has at most one edge. Denote by $\chi_{harm}(G, \lambda)$ the number of λ harmonious colourings of G . Then χ_{harm} is a polynomial in λ , as shown in [12] and [8]. χ_{harm} is called the harmonious polynomial.

For more on the harmonious polynomial, see [5]. Theorem 4 was observed without proof in [5].

Theorem 4: Almost all graphs G have a χ_{harm} -mate.

Proof: Let \mathcal{C} be the class of graphs G that have the property that for every two vertices $v, u \in V(G)$, there is a vertex $w \in V(G)$ such that w is a neighbour of both v and u . This property is one of Gaifman's extension axioms, and hence from Fagin's proof of the 0/1-law for first order logic, almost all graphs are in \mathcal{C} (see [6] for details).

Note that any harmonious colouring of a graph $G \in \mathcal{C}$ of order n has to assign a different colour to each vertex of G , and so for $\lambda \in \mathbb{N}$ the evaluation of the harmonious polynomial of G at λ is $\chi_{harm}(G, \lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)$. Since this is true for every $\lambda \in \mathbb{N}$, by interpolation it is true for every $\lambda \in \mathbb{R}$, and so every two graphs in \mathcal{C} of the same order have the same harmonious polynomial. Hence, there is an n_0 such that all graphs in \mathcal{C} of order greater than n_0 have an χ_{harm} -mate. Thus the harmonious polynomial is weakly distinguishing. \square

This result can be easily generalised.

Definition 9. For a fixed $k \in \mathbb{N}$ and a graph G , we say that a proper colouring of G with λ colours $f : V(G) \rightarrow [\lambda]$ is k -harmonious if for every $S \subseteq [\lambda]$ such that $|S| = k$, S appears as the colour set of a

clique in the graph at most once, i.e. if $\{v_1, v_2, \dots, v_k\}, \{u_1, \dots, u_k\} \subseteq V(G)$ induce complete graphs of size k and $f(\{v_1, v_2, \dots, v_k\}) = f(\{u_1, \dots, u_k\})$, then $\{v_1, v_2, \dots, v_k\} = \{u_1, \dots, u_k\}$.

For $\lambda \in \mathbb{N}$ define $h_k(G, \lambda) = |\{f : V(G) \rightarrow [\lambda] : f \text{ is proper and } k\text{-harmonious}\}|$. $h_k(G, \lambda)$ is a polynomial in λ (again, by [12]). We will prove that h_k is weakly distinguishing.

We start with a lemma:

Lemma 11. *Let \mathcal{C} be the class of graphs G with the property that for every two vertices $v, u \in V(G)$ there are vertices $w_1, \dots, w_k \in V(G)$ such that $\{u, w_1, \dots, w_k\}$ and $\{v, w_1, \dots, w_k\}$ induce a complete graph. Then almost all graphs are in \mathcal{C} .*

For convenience, we restate and prove the lemma in probabilistic language:

Lemma 12. *Fix $p \in (0, 1)$ and let $G \in \mathcal{G}(n, p)$ (i.e. G is a graph with n vertices and every edge is in the graph with probability p , independently of the others). Then $\lim_{n \rightarrow \infty} \mathbb{P}(G \in \mathcal{C}) = 1$*

Proof: For a graph G , denote by $Y_k(G)$ the number of k cliques in G . For fixed $u, v \in V(G)$, denote by $G_{u,v}$ the subgraph of G induced by the common neighbours of v and u . Note that $\mathbb{E}(|V(G_{u,v})|) = (n-2)p^2$, so from the multiplicative Chernoff bound,

$$\mathbb{P}[|V(G_{u,v})| \leq \frac{1}{9}(n-2)p^2] \leq \left(\frac{e^{-9/10}}{(1/9)^{1/9}} \right)^{(n-2)p^2} < e^{-\frac{1}{200}(n-2)p^2}$$

Hence, from the union bound

$$\mathbb{P}[\exists u, v \in V(G) \text{ s.t. } |V(G_{u,v})| \leq \frac{1}{9}(n-2)p^2] \leq \binom{n}{2} e^{-\frac{1}{200}(n-2)p^2}$$

Denote this number $r_p(n)$, and note that $\lim_{n \rightarrow \infty} r_p(n) = 0$.

Next, assume that $\forall u, v \in V(G)$, $|V(G_{u,v})| > \frac{1}{9}(n-2)p^2$. Fix $u, v \in V(G)$. Then $G_{u,v}$ is a random graph with more than $\frac{1}{9}(n-2)p^2$ vertices, and hence $\mathbb{P}[Y_k(G_{u,v}) < 1] \leq \mathbb{P}[Y_k(G') < 1]$ where $G' \in \mathcal{G}(\frac{1}{9}(n-2)p^2, p)$. From theorem 2 in [1], we have:

$$\mathbb{P}[Y_k(G') < 1] \leq \mathbb{P} \left[Y_k(G') \leq \frac{9}{10} \left(\frac{1}{9}(n-2)p^2 \right)^{3/2} \right] \leq \exp \left[-(1/100 + \alpha \sqrt{\frac{1}{9}(n-2)p^2}) \right]$$

for some constant $\alpha > 0$. Hence, if we denote $\mathcal{A} = \{G : \exists u, v \in V(G) \text{ s.t. } |V(G_{u,v})| \leq \frac{1}{9}(n-2)p^2\}$, from the union bound we have:

$$\mathbb{P}[\exists u, v \in V(G) \text{ s.t. } Y_k(G_{u,v}) < 1 | G \notin \mathcal{A}] \leq \binom{n}{2} \exp \left[-(1/100 + \alpha \sqrt{\frac{1}{9}(n-2)p^2}) \right]$$

Denote the right side of this inequality by $r'(n)$, and note that $r'(n) \rightarrow 0$ as $n \rightarrow \infty$.

To conclude, we have that

$$\begin{aligned} \mathbb{P}[G \notin \mathcal{C}] &= \mathbb{P}[G \in \mathcal{A} \cap \mathcal{C}^c] + \mathbb{P}[G \in \mathcal{A}^c \cap \mathcal{C}^c] \leq \\ &\leq \mathbb{P}[G \in \mathcal{A}] + \mathbb{P}[G \in \mathcal{A}^c] \mathbb{P}[G \in \mathcal{C}^c | G \in \mathcal{A}^c] = r_p(n) + (1 - r_p(n))r'_p(n) \end{aligned}$$

and both these terms tend to 0 as n tends to infinity. \square

We can now prove:

Theorem 13. *For a fixed $k \in \mathbb{N}$, h_k is weakly distinguishing.*

Proof: Let \mathcal{C} be the same as in the lemma. Similarly to the previous theorem, note that if $G \in \mathcal{C}$, then for $\lambda \in \mathbb{N}$ any k -harmonious colouring $f : V(G) \rightarrow [\lambda]$ of G must assign a different colour to every vertex (otherwise, if $v, u \in V(G)$ are such that $f(v) = f(u)$, and $w_1, \dots, w_k \in V(G)$ are such that $\{u, w_1, \dots, w_k\}$ and $\{v, w_1, \dots, w_k\}$ induce a complete graph in G , then $f(\{u, w_1, \dots, w_k\}) = f(\{v, w_1, \dots, w_k\})$, and thus $h_k(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1)$). By the same reasoning as in the proof of theorem 4, h_k is weakly distinguishing. \square

7 Conclusion

We have shown for many graph properties Q that the polynomials P_Q are weakly distinguishing, including the well studied clique and independence polynomials. We have also shown that the harmonious and k -harmonious polynomials are weakly distinguishing.

Our results relied on the fact that the number of polynomials that can be the P_Q polynomial of a graph is small, and on the fact that almost all graphs have properties that imply their harmonious and k -harmonious polynomials are trivial. This does not seem to be the case for the Tutte and the chromatic polynomials, so the original question of whether they, as well as the characteristic and matching polynomials, are weakly distinguishing remains open.

Problem. *Find a graph property Q such that the fraction of P_Q unique graphs is strictly positive.*

Problem. *Let P be a graph polynomial, G, H two graphs, and write $G \sim_P H$ if $P(G) = P(H)$. What can be said about the sizes of the equivalence classes of \sim_P ?*

References

- [1] Béla Bollobás. The chromatic number of random graphs. *Combinatorica*, 8(1):49–55, 1988.
- [2] Béla Bollobás and Paul Erdős. Cliques in random graphs. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 80, pages 419–427. Cambridge University Press, 1976.
- [3] Béla Bollobás, Luke Pebody, and Oliver Riordan. Contraction–deletion invariants for graphs. *Journal of Combinatorial Theory, Series B*, 80(2):320–345, 2000.
- [4] Alessandro Borri, Tiziana Calamoneri, and Rossella Petreschi. Recognition of unigraphs through superposition of graphs. In *International Workshop on Algorithms and Computation*, pages 165–176. Springer, 2009.

- [5] Ewa Drgas-Burchardt and Katarzyna Gibek. Harmonious colourings of graphs. *Discrete Applied Mathematics*, 217:175–184, 2017.
- [6] Ronald Fagin. Probabilities on finite models. *The Journal of Symbolic Logic*, 41(1):50–58, 1976.
- [7] Alan M. Frieze. On the independence number of random graphs. *Discrete Mathematics*, 81(2):171–175, 1990.
- [8] Benny Godlin, Tomer Kotek, and Johann A. Makowsky. Evaluations of graph polynomials. *WG*, 8:183–194, 2008.
- [9] Ivan Gutman and Frank Harary. Generalizations of the matching polynomial. *Utilitas Mathematicae*, 24:97–106, 1983.
- [10] Frank Harary and Edgar M. Palmer. *Graphical enumeration*. Elsevier, 2014.
- [11] Cornelis Hoede and Xueliang Li. Clique polynomials and independent set polynomials of graphs. *Discrete Mathematics*, 125(1-3):219–228, 1994.
- [12] Tomer Kotek, Johann A. Makowsky, and Boris Zilber. On counting generalized colorings. *Model Theoretic Methods in Finite Combinatorics*, 558:207–242.
- [13] Vadim E. Levit and Eugen Mandrescu. The independence polynomial of a graph—a survey. In *Proceedings of the 1st International Conference on Algebraic Informatics*, volume 233254. Aristotle Univ. Thessaloniki Thessaloniki, 2005.
- [14] Marc Noy. Graphs determined by polynomial invariants. *Theoretical Computer Science*, 307(2):365–384, 2003.