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The largest singletons in weighted set partitions and its applications

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Recently, Deutsch and Elizalde studied the largest fixed points of permutations. Motivated by their work, we consider the analogous problems in weighted set partitions. Let $A_{n,k}(\mathbf{t})$ denote the total weight of partitions on $[n+1] = \{1, 2, \ldots, n+1\}$ with the largest singleton $\{k+1\}$. In this paper, explicit formulas for $A_{n,k}(\mathbf{t})$ and many combinatorial identities involving $A_{n,k}(\mathbf{t})$ are obtained by umbral operators and combinatorial methods. In particular, the permutation case leads to an identity related to tree enumerations, namely,

$$\sum_{k=0}^{n} \binom{n}{k} D_{k+1}(n+1)^{n-k} = n^{n+1},$$

where D_k is the number of permutations of $[k]$ with no fixed points.

Keywords: Set partition, Bell polynomial, Permutation, Derangement.

1 Introduction

A *partition* of a set $[n] = \{1, 2, \ldots, n\}$ is a collection $\pi = \{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_r\}$ of nonempty and mutually disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. For a block $\mathbb{B}$, we denote by $|\mathbb{B}|$ the size of the block $\mathbb{B}$, that is the number of the elements in the block $\mathbb{B}$. A block $\mathbb{B}$ will be called *singleton* if $|\mathbb{B}| = 1$. If $\{k\}$ is a singleton of a partition, we denote it by k for short. If $|\mathbb{B}| = j$, we assign a weight t_j for $\mathbb{B}$. The weight $w(\pi)$ of a partition π is defined to be the product of the weight of each block of π.

It is well known that the weight of partitions of $[n]$ with r blocks is the partial Bell polynomial $\mathcal{B}_{n,r}(t_1, t_2, \ldots)$ [3] on the variables $\{t_j\}_{j\geq 1}$, that is

$$\mathcal{B}_{n,r}(t_1, t_2, \ldots) = \sum_{\kappa_n(r)} \frac{n!}{r_1! r_2! \cdots r_n!} \left(\frac{t_1}{1!}\right)^{r_1} \left(\frac{t_2}{2!}\right)^{r_2} \cdots \left(\frac{t_n}{n!}\right)^{r_n},$$

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where the summation $\kappa_n(r)$ is over all the nonnegative integer solutions of $r_1 + r_2 + \cdots + r_n = r$ and $r_1 + 2r_2 + \cdots + nr_n = n$. The total weight for partitions of $[n]$ is the complete Bell polynomial

$$\mathcal{Y}_n(\mathbf{t}) = \mathcal{Y}_n\big(t_1, t_2, \dots\big) = \sum_{r=0}^{n} \mathcal{B}_{n,r}\big(t_1, t_2, \dots\big),$$

which has the exponential generating function

$$\mathcal{Y}(\mathbf{t}; x) = \sum_{n\geq 0} \mathcal{Y}_n\big(t_1, t_2, \dots\big)\frac{x^n}{n!} = \exp\big(\sum_{j\geq 1} t_j \frac{x^j}{j!}\big).$$

Let $\mathbb{A}_{n,k}$ denote the set of partitions of $[n+1]$ with the largest singleton $k+1$. Let $A_{n,k}(\mathbf{t})$ denote the total weight of partitions in $\mathbb{A}_{n,k}$. Clearly,

$$A_{n,0}(\mathbf{t}) = t_1\mathcal{Y}_n\big(0, t_2, \dots\big) \ \text{ and } \ A_{n,n}(\mathbf{t}) = t_1\mathcal{Y}_n\big(t_1, t_2, \dots\big),$$

where $\mathcal{Y}_n\big(0, t_2, \dots\big)$ is the weight of partitions of $[n]$ without singletons.

Recently, Deutsch and Elizalde [4] studied the largest fixed points of permutations, which is the special case when $t_j = (j-1)!$ for $j \geq 1$. Later, Sun and Wu [17] considered the largest singletons in set partitions, which is the special case when $t_j = 1$ for $j \geq 1$.

In this paper we will investigate the largest singletons in weighted set partitions generally. The next section is devoted to studying the properties of $A_{n,k}(\mathbf{t})$, involving its explicit formulas and many combinatorial identities for $A_{n,k}(\mathbf{t})$. In the third section, we consider the permutation case, i.e., the special case when $t_j = (j-1)!$ for $j \geq 1$, and derive a surprising identity analogous to the Riordan identity related to tree enumerations.

2 The properties of $A_{n,k}(\mathbf{t})$

According to the definition of $A_{n,k}(\mathbf{t})$, for any weighted partition π of $[n+1]$ with the largest singleton $k+1$, if k is also a singleton, delete the singleton $k+1$ and subtracting one from all the entries larger than $k+1$, we obtain a partition of $[n]$ with the largest singleton k. This contributes the weight $t_1 A_{n-1,k-1}(\mathbf{t})$; if k is not a singleton, exchange k and $k+1$, we obtain a partition of $[n+1]$ with the largest singleton k. This contributes the weight $A_{n,k-1}(\mathbf{t})$. Consequently, we obtain a recurrence for $n, k \geq 1$,

$$A_{n,k}(\mathbf{t}) = A_{n,k-1}(\mathbf{t}) + t_1 A_{n-1,k-1}(\mathbf{t}) \tag{1}$$

with the initial conditions $A_{n,0}(\mathbf{t}) = t_1\mathcal{Y}_n\big(0, t_2, \dots\big)$ for $n \geq 0$.

Lemma 2.1 *The bivariate exponential generating function for $A_{n+k,k}(\mathbf{t})$ is given by*

$$A(\mathbf{t}; x, y) = \sum_{n,k\geq 0} A_{n+k,k}(\mathbf{t})\frac{x^n}{n!}\frac{y^k}{k!} = t_1 e^{-xt_1}\mathcal{Y}(\mathbf{t}; x+y).$$

Proof: Define

$$A_k(\mathbf{t}; x) = \sum_{n\geq 0} A_{n+k,k}(\mathbf{t})\frac{x^n}{n!}.$$

Clearly, $A_0(\mathbf{t};x) = t_1 e^{-xt_1}\mathcal{Y}(\mathbf{t};x)$. From (1), one can derive that

$$A_k(\mathbf{t};x) = t_1 A_{k-1}(\mathbf{t};x) + \frac{\partial}{\partial x} A_{k-1}(\mathbf{t};x),$$

which produces

$$A_k(\mathbf{t};x) = (t_1 + \frac{\partial}{\partial x})A_{k-1}(\mathbf{t};x) = (t_1 + \frac{\partial}{\partial x})^k A_0(\mathbf{t};x).$$

Then

$$\begin{aligned} A(\mathbf{t};x,y) &= \sum_{k\geq 0} A_k(\mathbf{t};x)\frac{y^k}{k!} = \sum_{k\geq 0} \frac{y^k(t_1+\frac{\partial}{\partial x})^k}{k!} A_0(\mathbf{t};x) \\ &= e^{yt_1 + y\frac{\partial}{\partial x}} t_1 e^{-xt_1}\mathcal{Y}(\mathbf{t};x) = t_1 e^{yt_1} e^{y\frac{\partial}{\partial x}} e^{-xt_1}\mathcal{Y}(\mathbf{t};x) \\ &= t_1 e^{yt_1} e^{-(x+y)t_1}\mathcal{Y}(\mathbf{t};x+y) = t_1 e^{-xt_1}\mathcal{Y}(\mathbf{t};x+y). \end{aligned}$$

This completes the proof. □

Theorem 2.2 *For any integers $n, m \geq 0$ and any indeterminate λ, there hold*

$$\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,m+k}(\mathbf{t}) = \sum_{k=0}^{n} \binom{n+\lambda}{k}\binom{n+\lambda-k-1}{n-k} A_{m+k,m}(\mathbf{t}) t_1^{n-k}, \tag{2}$$

$$\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,m+k}(\mathbf{t}) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+\lambda}{k} \mathcal{Y}_{m+k}(\mathbf{t}) t_1^{n-k+1}. \tag{3}$$

Proof: With the umbra $\mathbf{Y_t}$, given by $\mathbf{Y_t}^n = \mathcal{Y}_n(\mathbf{t})$, $\mathcal{Y}(\mathbf{t};x)$ may be written as $\mathcal{Y}(\mathbf{t};x) = e^{\mathbf{Y_t}x}$. (See, for example, [7, 12, 13]). Then, by Lemma 2.1, we have

$$A(\mathbf{t};x,y) = t_1 e^{\mathbf{Y_t}(x+y) - t_1 x} = t_1 e^{(\mathbf{Y_t}-t_1)x} e^{\mathbf{Y_t}y}.$$

When comparing the coefficient of $\frac{x^n y^k}{n!k!}$, $A_{n+k,k}(\mathbf{t})$ can be represented umbrally as

$$A_{n+k,k}(\mathbf{t}) = t_1 \mathbf{Y_t}^k (\mathbf{Y_t} - t_1)^n. \tag{4}$$

Let $[x^n]f(x)$ denote the coefficient of x^n in the formal power series $f(x)$. Then we get

$$\begin{aligned} &\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,m+k}(\mathbf{t}) \\ &= \sum_{k=0}^{n} (-1)^k \binom{-\lambda}{k} t_1 \mathbf{Y_t}^{m+k} (\mathbf{Y_t}-t_1)^{n-k} \\ &= t_1 \mathbf{Y_t}^m (\mathbf{Y_t}-t_1)^n \sum_{k=0}^{n} \binom{-\lambda}{k} \Big(-\frac{\mathbf{Y_t}}{\mathbf{Y_t}-t_1}\Big)^k \end{aligned}$$

$$
\begin{aligned}
&= t_1\mathbf{Y}_{\mathbf{t}}^m(\mathbf{Y}_{\mathbf{t}}-t_1)^n\sum_{k=0}^{n}[x^k]\Big(1-\frac{x\mathbf{Y}_{\mathbf{t}}}{\mathbf{Y}_{\mathbf{t}}-t_1}\Big)^{-\lambda}\\
&= t_1\mathbf{Y}_{\mathbf{t}}^m(\mathbf{Y}_{\mathbf{t}}-t_1)^n[x^n]\frac{1}{1-x}\Big(1-\frac{x\mathbf{Y}_{\mathbf{t}}}{\mathbf{Y}_{\mathbf{t}}-t_1}\Big)^{-\lambda}\\
&= t_1\mathbf{Y}_{\mathbf{t}}^m(\mathbf{Y}_{\mathbf{t}}-t_1)^n[x^n]\frac{1}{(1-x)^{\lambda+1}}\Big(1-\frac{x}{(1-x)}\frac{t_1}{(\mathbf{Y}_{\mathbf{t}}-t_1)}\Big)^{-\lambda}\\
&= t_1\mathbf{Y}_{\mathbf{t}}^m(\mathbf{Y}_{\mathbf{t}}-t_1)^n[x^n]\sum_{k=0}^{n}\binom{-\lambda}{n-k}\frac{x^{n-k}}{(1-x)^{n+\lambda-k+1}}\Big(-\frac{t_1}{\mathbf{Y}_{\mathbf{t}}-t_1}\Big)^{n-k}\\
&= \sum_{k=0}^{n}(-1)^k\binom{-(n+\lambda-k+1)}{k}\binom{-\lambda}{n-k}t_1\mathbf{Y}_{\mathbf{t}}^m(\mathbf{Y}_{\mathbf{t}}-t_1)^k(-t_1)^{n-k}\\
&= \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{n+\lambda-k-1}{n-k}A_{m+k,m}(\mathbf{t})t_1^{n-k},
\end{aligned}
$$

which proves (2).

By the identity

$$\binom{n}{k}\binom{k}{i}=\binom{n}{i}\binom{n-i}{k-i},$$

and Vandermonde's convolution identity

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}=\binom{a+b}{n},$$

we have

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{k+\lambda-1}{k}A_{n+m,m+k}(\mathbf{t})\\
&\quad= \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{-\lambda}{n-k}t_1\mathbf{Y}_{\mathbf{t}}^m(\mathbf{Y}_{\mathbf{t}}-t_1)^k(-t_1)^{n-k}\\
&\quad= \sum_{k=0}^{n}\binom{n+\lambda}{k}\binom{-\lambda}{n-k}\sum_{i=0}^{k}\binom{k}{i}t_1\mathbf{Y}_{\mathbf{t}}^{m+i}(-t_1)^{n-i}\\
&\quad= \sum_{i=0}^{n}t_1\mathbf{Y}_{\mathbf{t}}^{m+i}(-t_1)^{n-i}\sum_{k=i}^{n}\binom{n+\lambda}{k}\binom{-\lambda}{n-k}\binom{k}{i}\\
&\quad= \sum_{i=0}^{n}\binom{n+\lambda}{i}t_1\mathbf{Y}_{\mathbf{t}}^{m+i}(-t_1)^{n-i}\sum_{k=i}^{n}\binom{-\lambda}{n-k}\binom{n+\lambda-i}{k-i}\\
&\quad= \sum_{i=0}^{n}\binom{n+\lambda}{i}t_1\mathbf{Y}_{\mathbf{t}}^{m+i}(-t_1)^{n-i}
\end{aligned}
$$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n+\lambda}{k} \mathcal{Y}_{m+k}(\mathbf{t}) t_1^{n-k+1},$$

which proves (3). □

The case $\lambda = 0$ in (3) yields an explicit formula for $A_{n+m,m}(\mathbf{t})$.

Corollary 2.3 *For any integers $n, m \geq 0$, there holds*

$$A_{n+m,m}(\mathbf{t}) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t_1^{n-k+1} \mathcal{Y}_{m+k}(\mathbf{t}). \tag{5}$$

Corollary 2.4 *For any integers $n, m \geq 0$, there hold*

$$\sum_{k=0}^{n} A_{n+m,m+k}(\mathbf{t}) = \frac{t_1 \mathcal{Y}_{n+m+1}(\mathbf{t}) - A_{n+m+1,m}(\mathbf{t})}{t_1}, \tag{6}$$

$$\sum_{k=0}^{n} (k+1) A_{n+m,m+k}(\mathbf{t}) = \frac{A_{n+m+2,m}(\mathbf{t}) - t_1 \mathcal{Y}_{n+m+2}(\mathbf{t}) + (n+2) t_1^2 \mathcal{Y}_{n+m+1}(\mathbf{t})}{t_1^2}, \tag{7}$$

$$\sum_{k=0}^{n} (n-k+1) A_{n+m,m+k}(\mathbf{t}) = \frac{t_1 \mathcal{Y}_{n+m+2}(\mathbf{t}) - A_{n+m+2,m}(\mathbf{t}) - (n+2) t_1 A_{n+m+1,m}(\mathbf{t})}{t_1^2}. \tag{8}$$

Proof: By combining (5) with n replaced by $n+1$ (resp. $n+2$) and with the case $\lambda = 1$ (resp. $\lambda = 2$) in (3), we obtain (6) (resp. (7)). Moreover, (8) can be easily obtained from (6) and (7). □

Theorem 2.5 *For any integers $n, m, k \geq 0$, there holds*

$$A_{n+m+k,m+k}(\mathbf{t}) = \sum_{j=0}^{m} \binom{m}{j} t_1^{m-j} A_{n+k+j,k}(\mathbf{t}). \tag{9}$$

Proof: Here we provide a combinatorial proof. For any $\pi \in \mathbb{A}_{n+m+k,m+k}$, suppose that π has exactly $m-j$ singletons in $\{k+1, \ldots, k+m\}$, which contributes the weight t_1^{m-j}, and there are $\binom{m}{j}$ ways to do this. The remaining j elements in $\{k+1, \ldots, k+m\}$ can not be singletons in π. These j elements can be regarded as the roles that greater than $m+k+1$, so the remaining $n+k+j+1$ elements can be partitioned with the largest singleton $m+k+1$, these cases contribute the weight $A_{n+k+j,k}(\mathbf{t})$. Thus the total weight of such partitions is $\binom{m}{j} t_1^{m-j} A_{n+k+j,k}(\mathbf{t})$. Summing up all the possible cases yields (9). □

Theorem 2.6 *For any integers $n, m \geq 0$ and any indeterminate y, there hold*

$$\sum_{k=0}^{n} \binom{n}{k} A_{n+m,m+k}(\mathbf{t}) y^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \mathcal{Y}_{m+k}(\mathbf{t}) (y+1)^k t_1^{n-k+1}, \tag{10}$$

$$\sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(\mathbf{t}) y^{n-k} = t_1 \sum_{k=0}^{n} \binom{n}{k} \mathcal{Y}_{m+k}(\mathbf{t}) (y - t_1)^{n-k}. \tag{11}$$

Proof: By (4), we have

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}A_{n+m,m+k}(\mathbf{t})y^k &= \sum_{k=0}^{n}\binom{n}{k}t_1\mathbf{Y}_{\mathbf{t}}^{m+k}(\mathbf{Y}_{\mathbf{t}}-t_1)^{n-k}y^k \\
&= t_1\mathbf{Y}_{\mathbf{t}}^{m}((y+1)\mathbf{Y}_{\mathbf{t}}-t_1)^n \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(y+1)^k\mathbf{Y}_{\mathbf{t}}^{m+k}t_1^{n-k+1} \\
&= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(y+1)^k\mathcal{Y}_{m+k}(\mathbf{t})t_1^{n-k+1},
\end{aligned}$$

which proves (10). Similarly, (11) can be obtained, but here we provide a combinatorial proof.

Let $\mathbb{X}_{n,m} = \bigcup_{j=0}^{n}\mathbb{X}_{n,m,k}$ and $\mathbb{X}_{n,m,k}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is an $(n-k)$-subset of $[m+2, n+m+1] = \{m+2, \ldots, n+m+1\}$, and each element of $\mathbb{S}$ is colored by t_1 or $y-t_1$;
- π is a partition of the set $[n+m+1]-\mathbb{S}$ with the largest singleton $m+1$, and each element of $[n+m+1]-\mathbb{S}$ is only colored by 1.

Let $\mathbb{Y}_{n,m} = \bigcup_{k=0}^{n}\mathbb{Y}_{n,m,k}$ and $\mathbb{Y}_{n,m,k}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is an $(n-k)$-subset of $[m+2, n+m+1]$ and each element of $\mathbb{S}$ is only colored by $y-t_1$;
- π is a partition of the set $[n+m+1]-\mathbb{S}$ such that $m+1$ must be a singleton, and each element of $[n+m+1]-\mathbb{S}$ is only colored by 1.

The weight of $(\pi, \mathbb{S})$ is defined to be the product of the weight of π and the color of each element of $[n+m+1]$. Clearly, the weights of $\mathbb{X}_{n,m}$ and $\mathbb{Y}_{n,m}$ are counted respectively by the left and right sides of (11).

Given any pair $(\pi, \mathbb{S}) \in \mathbb{X}_{n,m}$, $\mathbb{S}$ can be partitioned into two parts $\mathbb{S}_1$ and $\mathbb{S}_2$ such that each element of $\mathbb{S}_1$ is colored by $y-t_1$ and each element of $\mathbb{S}_2$ is colored by t_1. Regard each element of $\mathbb{S}_2$ as a singleton which is weighted by t_1 and colored by 1, together with π, we obtain a partition π_1 of $[n+m+1]-\mathbb{S}_1$ such that $m+1$ is always a singleton. Then the pair $(\pi_1, \mathbb{S}_1)$ lies in $\mathbb{Y}_{n,m}$.

Conversely, for any pair $(\pi_1, \mathbb{S}_1) \in \mathbb{Y}_{n,m}$, let $\mathbb{S}$ denote the union of $\mathbb{S}_1$ and the singletons of π_1 greater than $m+1$, then π_1 can be partitioned into two parts π and π' such that π is a partition of $[n+m+1]-\mathbb{S}$ with the largest singleton $m+1$ and π' is the singletons of π_1 greater than $m+1$. By regarding π' as a subset of $[m+2, n+m+1]$ in which each element is colored by t_1, together with $\mathbb{S}_1$. Then we obtain an $(n-k)$-subset of $[m+2, n+m+1]$ for some k such that each element of $\mathbb{S}$ is colored by t_1 or $y-t_1$. Then the pair $(\pi, \mathbb{S})$ lies in $\mathbb{X}_{n,m}$.

Clearly we find a bijection between $\mathbb{X}_{n,m}$ and $\mathbb{Y}_{n,m}$, which proves (11). $\square$

The cases $y = -1$ in (10) and $y = t_1$ in (11) lead to

Corollary 2.7 *For any integers $n, m \geq 0$, there hold*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n+m,m+k}(\mathbf{t}) = \mathcal{Y}_m(\mathbf{t})t_1^{n+1},$$
$$\sum_{k=0}^{n}\binom{n}{k}A_{m+k,m}(\mathbf{t})t_1^{n-k-1} = \mathcal{Y}_{m+n}(\mathbf{t}).$$

The case $y = \frac{yt_1}{y+1}$ in (11), together with (10) generates the following result which has a combinatorial interpretation.

Corollary 2.8 *For any integers $n, m \geq 0$, there holds*

$$\sum_{k=0}^{n}\binom{n}{k}A_{m+k,m}(\mathbf{t})(y+1)^k(yt_1)^{n-k} = \sum_{k=0}^{n}\binom{n}{k}A_{n+m,m+k}(\mathbf{t})y^k. \tag{12}$$

Proof: Let $\mathbb{X}^*_{n,m} = \bigcup_{j=0}^{n}\mathbb{X}^*_{n,m,k}$ and $\mathbb{X}^*_{n,m,k}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- π is a partition of the set $[n+m+1]$ containing at least the singleton $m+1$;
- $\mathbb{S}$ is an $(n-k)$-subset of $[m+2, n+m+1]$ which is also the set of singletons of π greater than $m+1$, each element of $\mathbb{S}$ is only colored by y and each element of $[m+2, n+m+1]-\mathbb{S}$ is colored by 1 or y;
- each element of $[m+1]$ is only colored by 1.

Let $\mathbb{Y}^*_{n,m} = \bigcup_{k=0}^{n}\mathbb{Y}^*_{n,m,k}$ and $\mathbb{Y}^*_{n,m,k}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is a k-subset $\{i_1, i_2, \ldots, i_k\}$ of $[m+2, n+m+1]$ in increasing order, each element of $\mathbb{S}$ is only colored by y and each element of $[n+m+1]-\mathbb{S}$ is only colored by 1;
- π is a partition of the set $[n+m+1]$ such that i_k must be the largest singleton if $\mathbb{S}$ is not empty and $m+1$ must be the largest singleton if $\mathbb{S}$ is empty;
- each element of $[m+2, n+m+1]-\mathbb{S}$ must not be a singleton.

The weight of $(\pi, \mathbb{S})$ is defined to be the product of the weight of π and the colors of all elements in $[n+m+1]$. Clearly, any $(\pi, \mathbb{S}) \in \mathbb{X}^*_{n,m}$ can be obtained as follows. First choose an $(n-k)$-subset $\mathbb{S}$ of $[m+2, n+m+1]$, there are $\binom{n}{k}$ ways to do this. Regard each element of $\mathbb{S}$ as a singleton with color y. Then color each element of $[m+2, n+m+1]-\mathbb{S}$ by 1 or y, namely, each element of $[m+2, n+m+1]-\mathbb{S}$ is colored by $y+1$. Now partitioning $[n+m+1]-\mathbb{S}$ such that the largest singleton is $m+1$, together with the $n-k$ singletons formed form $\mathbb{S}$, we get the partition π of $[n+m+1]$ such that $m+1$ must be a singleton; Hence the total weight of pairs $(\pi, \mathbb{S}) \in \mathbb{X}^*_{n,m}$ is just the left hand side of (12).

Similarly, the total weight of pairs $(\pi, \mathbb{S}) \in \mathbb{Y}^*_{n,m}$ is just the right hand side of (12) if regarding each element of $[m+2, n+m+1]-\mathbb{S}$ as the role greater than i_k when $\mathbb{S} \neq \emptyset$.

Now we can construct a bijection φ between $\mathbb{X}^*_{n,m}$ and $\mathbb{Y}^*_{n,m}$ which preserves the weights. For any $(\pi, \mathbb{S}) \in \mathbb{X}^*_{n,m}$, let $\mathbb{S}_1$ denote the set of elements of $[n+m+1]$ with colors y. Clearly, $\mathbb{S}$ is a subset of $\mathbb{S}_1$.

Assume that $\mathbb{S}_1 = \{i_1, i_2, \ldots, i_k\}$ for some $0 \leq k \leq n$ in increasing order. If $\mathbb{S}_1$ is the empty set $\emptyset$, which implies that $\mathbb{S} = \emptyset$ and all elements of $[n+m+1]$ are colored by 1, it is obvious that $(\pi, \emptyset) \in \mathbb{Y}^*_{n,m}$. Then define $\varphi(\pi, \emptyset) = (\pi, \emptyset)$. If $\mathbb{S}_1$ is not the empty set, exchanging $m+1$ and i_k in π, we obtain a partition π_1, it is easily to verify that $(\pi_1, \mathbb{S}_1) \in \mathbb{Y}^*_{n,m}$ and has the same weight as $(\pi, \mathbb{S})$. Then define $\varphi(\pi, \mathbb{S}) = (\pi_1, \mathbb{S}_1)$.

Conversely, for any $(\pi_1, \mathbb{S}_1) \in \mathbb{Y}^*_{n,m}$, if $\mathbb{S}_1 = \emptyset$, so π_1 has the largest singleton $m+1$, then $(\pi_1, \emptyset) \in \mathbb{X}^*_{n,m}$ and define $\varphi^{-1}(\pi_1, \emptyset) = (\pi_1, \emptyset)$. If $\mathbb{S}_1 \neq \emptyset$, assume that $\mathbb{S}_1 = \{i_1, i_2, \ldots, i_k\}$ for some $1 \leq k \leq n$ in increasing order, let $\mathbb{S}$ denote the set of all the elements in $\mathbb{S}_1$ such that each forms a singleton of π_1. Now exchanging $m+1$ and i_k in π_1, we obtain a partition π, it is easy verifiable that $(\pi, \mathbb{S}) \in \mathbb{X}^*_{n,m}$ which has the same weight as $(\pi_1, \mathbb{S}_1)$. Then define $\varphi^{-1}(\pi_1, \mathbb{S}_1) = (\pi, \mathbb{S})$.

Clearly, φ is indeed a bijection between $\mathbb{X}^*_{n,m}$ and $\mathbb{Y}^*_{n,m}$, which proves (12). $\square$

3 The special case for permutations

When the parameter $\mathbf{t}$ in $A_{n,k}(\mathbf{t})$ takes some special value, that is to assign a special structure to each block of partitions of $[n+1]$. For example, the case $\mathbf{t} = (1^0, 2^1, 3^2, \ldots)$ indicates that each block of partitions is assigned by a (rooted and labeled) tree structure, such partitions are equivalent to labeled forests; The case $\mathbf{t} = (1, 1, 0, \ldots)$ leads to involutions on $[n+1]$.

In this section, we just present an interesting specialization, but leave others to inclined readers. Consider the special case when $\mathbf{t} = (0!, 1!, 2!, \ldots)$, that is to assign a cycle structure to each block of partitions, such partitions is equivalent to permutations. Let $P_{n,k} = A_{n,k}(\mathbf{t})$ with $\mathbf{t} = (0!, 1!, 2!, \ldots)$, i.e., $P_{n,k}$ is the number of permutations of $[n+1]$ with the largest fixed point $k+1$. From (5) and (9), one has the explicit formulas for $P_{n,k}$

$$P_{n+k,k} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (k+j)! = \sum_{j=0}^{k} \binom{k}{j} D_{n+j}.$$

Clearly, $P_{n,n} = n! = \mathcal{Y}_n(0!, 1!, 2!, \ldots)$ and $P_{n,0} = D_n = \mathcal{Y}_n(0, 1!, 2!, \ldots)$, where D_n is the derangement number of $[n]$, i.e., the number of permutations of $[n]$ without fixed points. See Table 1 for some small values of $P_{n,k}$.

n/k	0	1	2	3	4	5	6
0	1						
1	0	1					
2	1	1	2				
3	2	3	4	6			
4	9	11	14	18	24		
5	44	53	64	78	96	120	
6	265	309	362	426	504	600	720

Table 1. The values of $P_{n,k}$ for n and k up to 6.

In fact $\{P_{n,k}\}_{n\geq k\geq 0}$ forms the difference table introduced by Euler, which has been investigated in depth in derangement theory [2, 5, 6, 8, 9]. Chen [1] gave two other interpretations for $P_{n,k}$ using k-relative derangements on $[n]$ and skew derangements from $[n]$ to $\{-k+1,\ldots,-1,0,1,\ldots,n-k\}$ for $0\leq k\leq n$. Moreover, Chen established a bijection between these two settings. Recently, Deutsch and Elizalde [4] gave a new interpretation of derangement number D_{n+2} as the sum of the values of the largest fixed points of all non-derangements of length $n+1$, namely,

$$\sum_{k=0}^{n}(k+1)P_{n,k}=D_{n+2},$$

which is the special case of (7) when $\mathbf{t}=(0!,1!,2!,\ldots)$ and $m=0$.

Next, we can explore some new relations between $P_{n,k}$ and other classical sequences such as Bell numbers or Fibonacci numbers.

Example 3.1 *By Lemma 2.1, one can derive the bivariate exponential generating function for $P_{n+k,k}$, i.e.,*

$$P(x,y)=\sum_{n,k\geq 0}P_{n+k,k}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^{-x}}{1-x-y}.$$

Extracting the coefficient of $\frac{x^n}{n!}$ in $P(x,x^2)$, we have

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{2k}{k}k!P_{n-k,k}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k!F_k,$$

where F_k is the k-th Fibonacci number defined by $\frac{1}{1-x-x^2}=\sum_{k\geq 0}F_kx^k$.

Example 3.2 *When $\mathbf{t}=(0!,1!,2!,\ldots)$, (10) and (11) reduce to*

$$\sum_{k=0}^{n}\binom{n}{k}P_{n+m,m+k}y^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(m+k)!(y+1)^k, \tag{13}$$

$$\sum_{k=0}^{n}\binom{n}{k}P_{m+k,m}y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}(m+k)!(y-1)^{n-k}. \tag{14}$$

It should be noted that (13) and (14) have close relations to the (re-normalized) Charlier polynomials $C_n(u,v)$ [7] defined by

$$C_n(u,v)=\sum_{k=0}^{n}\binom{n}{k}(u)_kv^{n-k},$$

where $(u)_k=u(u+1)\cdots(u+k-1)$ for $k\geq 1$ and $(u)_0=1$. In fact (13) is $\frac{(y+1)^n}{m!}C_n(m+1,-\frac{1}{y+1})$ and (14) is equal to $\frac{1}{m!}C_n(m+1,y-1)$.

Recall that, by (4), $P_{n,k}$ can be represented umbrally as

$$P_{n,k} = \mathbf{P}^k(\mathbf{P}-1)^{n-k},$$

where $\mathbf{P} = \mathbf{Y_t}$ with $\mathbf{t} = (0!, 1!, 2!, \ldots)$. In particular, $D_n = (\mathbf{P}-1)^n$ and $n! = \mathbf{P}^n$. Hence, the case $y = \mathbf{P}-1$ in (13) and the case $y = \mathbf{P}$ in (14) generate

$$\sum_{k=0}^{n}\binom{n}{k}P_{n+m,m+k}D_k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(m+k)!k!,$$
$$\sum_{k=0}^{n}\binom{n}{k}P_{m+k,m}(n-k)! = \sum_{k=0}^{n}\binom{n}{k}(m+k)!D_{n-k}.$$

With the Bell umbra $\mathbf{B}$ [7, 12, 13], given by $\mathbf{B} = \mathbf{Y_t}$ with $\mathbf{t} = (1, 1, 1, \ldots)$, the Bell number can be written as $B_n = \mathbf{B}^n$ and $\mathbf{B}^{n+1} = (\mathbf{B}+1)^n$. Then the case $y = \mathbf{B}$ in (13) and the case $y = \mathbf{B}+1$ in (14) generate

$$\sum_{k=0}^{n}\binom{n}{k}P_{n+m,m+k}B_k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(m+k)!B_{k+1},$$
$$\sum_{k=0}^{n}\binom{n}{k}P_{m+k,m}B_{n-k+1} = \sum_{k=0}^{n}\binom{n}{k}(m+k)!B_{n-k}.$$

Using the Riordan identity [3, 11, P173],

$$\sum_{k=0}^{n}\binom{n}{k}(k+1)!(n+1)^{n-k} = (n+1)^{n+1},$$

the case in (13) with $m = 1$ and $y = -\frac{n+2}{n+1}$ and the case in (14) with $m = 1$ and $y = n+2$ generate respectively

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P_{n+1,k+1}(n+2)^k(n+1)^{n-k} = (n+1)^{n+1},$$
$$\sum_{k=0}^{n}\binom{n}{k}(D_k + D_{k+1})(n+2)^{n-k} = (n+1)^{n+1}, \tag{15}$$

where we use the relation $P_{k+1,1} = D_k + D_{k+1}$. By the well-known recurrence $D_{k+2} = (k+1)(D_k + D_{k+1})$ for derangement numbers D_k, together with $D_1 = 0$, after routine computation, (15) is equivalent to

$$\sum_{k=0}^{n}\binom{n}{k}D_{k+1}(n+1)^{n-k} = n^{n+1}, \tag{16}$$

which was also obtained by Riordan [10]. In a forthcoming paper [18], using functional digraph theory, we will give a combinatorial interpretation for a more general identity involving the Riordan identity and (16) as special cases.

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References

[1] W. Y. C. Chen, *The skew, relative, and classical derangements*, Discrete Mathematics 160 (1996), 235-239.

[2] R. J. Clarke, G.-N. Han, and J. Zeng, *A combinatorial interpretation of the Seidel generation of q-derangement numbers*, Annals of Combinatorics 4 (1997), 313-327.

[3] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Dordrecht-Holland, 1974.

[4] E. Deutsch and S. Elizalde, *The largest and the smallest fixed points of permutations*, Europ. J. Combin., 31 (5) (2010), 1404-1409.

[5] D. Dumont, *Matrices d'Euler-Seidel*, Séminaire Lotharingien de Combinatoire, B05c (1981), http://cartan.u-strasbg.fr/~slc.

[6] D. Dumont and A. Randrianarivony, *Dérangements et nombres de Genocchi*, Disc. Math., 132 (1994), 37-49.

[7] I. M. Gessel, *Applications of the classical umbral calculus*, Algebra Universalis, 49 (2003), 397-434.

[8] F. Rakotondrajao, *k-Fixed-points permutations*, Integers: Electr. J. Combin. Numb. Theo. 7 (2007), #A36.

[9] F. Rakotondrajao, *On Euler's difference table*, Proc. FPSAC'07, Nankai University, Tianjin, China (2007).

[10] J. Riordan, *Enumeration of linear graphs for mappings of finite sets*, Ann. Math. Statist. 33 (1962), 178-185.

[11] J. Riordan, *Forests of labeled trees*, J. Combinatorial Theory 5 (1968), 90-103.

[12] S. Roman, *The Umbral Calculus*, Academic Press, Orlando, FL, 1984.

[13] S. Roman and G.-C. Rota, *The umbral calculus*, Adv. Math. 27 (1978), 95-188.

[14] G. C. Rota, *The number of partitions of a set*, Amer. Math. Monthly 71 (1964), 498-504.

[15] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences.

[16] R. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, Cambridge, 1997.

[17] Y. Sun and X. Wu, *The largest singletons of set partitions*, Europ. J. Combinatorics, 32 (2011), 369-382.

[18] Y. Sun and J. Zhuang, *λ-factorials of n*, Electr. J. Combinatorics 17(1) (2010), #169.