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The largest singletons in weighted set partitions and its applications

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Recently, Deutsch and Elizalde studied the largest fixed points of permutations. Motivated by their work, we consider the analogous problems in weighted set partitions. Let $A_{n,k}(t)$ denote the total weight of partitions on $[n+1] = \{1,2,\ldots,n+1\}$ with the largest singleton $\{k+1\}$. In this paper, explicit formulas for $A_{n,k}(t)$ and many combinatorial identities involving $A_{n,k}(t)$ are obtained by umbral operators and combinatorial methods. In particular, the permutation case leads to an identity related to tree enumerations, namely,

$$\sum_{k=0}^{n} \binom{n}{k} D_{k+1}(n+1)^{n-k} = n^{n+1},$$

where $D_k$ is the number of permutations of $[k]$ with no fixed points.

**Keywords:** Set partition, Bell polynomial, Permutation, Derangement.

1 Introduction

A partition of a set $[n] = \{1,2,\ldots,n\}$ is a collection $\pi = \{B_1,B_2,\ldots,B_r\}$ of nonempty and mutually disjoint subsets of $[n]$, called blocks, whose union is $[n]$. For a block $B$, we denote by $|B|$ the size of the block $B$, that is the number of the elements in the block $B$. A block $B$ will be called singleton if $|B| = 1$. If $\{k\}$ is a singleton of a partition, we denote it by $k$ for short. If $|B| = j$, we assign a weight $t_j$ for $B$. The weight $w(\pi)$ of a partition $\pi$ is defined to be the product of the weight of each block of $\pi$.

It is well known that the weight of partitions of $[n]$ with $r$ blocks is the partial Bell polynomial $B_{n,r}(t_1,t_2,\ldots)$ on the variables $\{t_j\}_{j\ge 1}$, that is

$$B_{n,r}(t_1,t_2,\ldots) = \sum_{\alpha\in\mathcal{R}(r)} \frac{n!}{r_1!r_2!\cdots r_n!} \left(\frac{t_1}{1!}\right)^{r_1} \left(\frac{t_2}{2!}\right)^{r_2} \cdots \left(\frac{t_n}{n!}\right)^{r_n},$$

where $\alpha\in\mathcal{R}(r)$ is a partition of $r$.

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where the summation $\kappa_n(r)$ is over all the nonnegative integer solutions of $r_1 + r_2 + \cdots + r_n = r$ and $r_1 + 2r_2 + \cdots + nr_n = n$. The total weight for partitions of $[n]$ is the complete Bell polynomial

$$\mathcal{Y}_n(t) = \mathcal{Y}_n(t_1, t_2, \ldots) = \sum_{r=0}^{n} B_{n,r}(t_1, t_2, \ldots),$$

which has the exponential generating function

$$\mathcal{Y}(t; x) = \sum_{n \geq 0} \mathcal{Y}_n(t_1, t_2, \ldots) \frac{x^n}{n!} = \exp \left( \sum_{j \geq 1} \frac{t_j x^j}{j!} \right).$$

Let $A_{n,k}$ denote the set of partitions of $[n+1]$ with the largest singleton $k + 1$. Let $A_{n,k}(t)$ denote the total weight of partitions in $A_{n,k}$. Clearly,

$$A_{n,0}(t) = t_1 \mathcal{Y}_n(0, t_2, \ldots) \quad \text{and} \quad A_{n,n}(t) = t_1 \mathcal{Y}_n(t_3, t_2, \ldots),$$

where $\mathcal{Y}_n(0, t_2, \ldots)$ is the weight of partitions of $[n]$ without singletons.

Recently, Deutsch and Elizalde [4] studied the largest fixed points of permutations, which is the special case when $t_j = (j - 1)!$ for $j \geq 1$. Later, Sun and Wu [17] considered the largest singletons in set partitions, which is the special case when $t_j = 1$ for $j \geq 1$.

In this paper we will investigate the largest singletons in weighted set partitions generally. The next section is devoted to studying the properties of $A_{n,k}(t)$, involving its explicit formulas and many combinatorial identities for $A_{n,k}(t)$. In the third section, we consider the permutation case, i.e., the special case when $t_j = (j - 1)!$ for $j \geq 1$, and derive a surprising identity analogous to the Riordan identity related to tree enumerations.

2 The properties of $A_{n,k}(t)$

According to the definition of $A_{n,k}(t)$, for any weighted partition $\pi$ of $[n+1]$ with the largest singleton $k + 1$, if $k$ is also a singleton, delete the singleton $k + 1$ and subtracting one from all the entries larger than $k + 1$, we obtain a partition of $[n]$ with the largest singleton $k$. This contributes the weight $t_1 A_{n-1,k-1}(t)$; if $k$ is not a singleton, exchange $k$ and $k + 1$, we obtain a partition of $[n+1]$ with the largest singleton $k$. This contributes the weight $A_{n,k-1}(t)$. Consequently, we obtain a recurrence for $n, k \geq 1$,

$$A_{n,k}(t) = A_{n,k-1}(t) + t_1 A_{n-1,k-1}(t) \quad (1)$$

with the initial conditions $A_{n,0}(t) = t_1 \mathcal{Y}_n(0, t_2, \ldots)$ for $n \geq 0$.

Lemma 2.1 The bivariate exponential generating function for $A_{n+k,k}(t)$ is given by

$$A(t; x, y) = \sum_{n,k \geq 0} A_{n+k,k}(t) \frac{x^n y^k}{n! k!} = t_1 e^{-xt_1} \mathcal{Y}(t; x + y).$$

Proof: Define

$$A_k(t; x) = \sum_{n \geq 0} A_{n+k,k}(t) \frac{x^n}{n!}.$$
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Clearly, \( A_0(t; x) = t_1 e^{-xt_1} \mathcal{Y}(t; x) \). From (1), one can derive that

\[
A_k(t; x) = t_1 A_{k-1}(t; x) + \frac{\partial}{\partial x} A_{k-1}(t; x),
\]

which produces

\[
A_k(t; x) = (t_1 + \frac{\partial}{\partial x}) A_{k-1}(t; x) = (t_1 + \frac{\partial}{\partial x})^k A_0(t; x).
\]

Then

\[
A(t; x, y) = \sum_{k \geq 0} A_k(t; x) \frac{y^k}{k!} = \sum_{k \geq 0} \frac{y^k (t_1 + \frac{\partial}{\partial x})^k}{k!} A_0(t; x)
= e^{yt_1+\frac{\partial}{\partial x} t_1 e^{-xt_1} \mathcal{Y}(t; x)} = t_1 e^{yt_1} e^{y\frac{\partial}{\partial x} e^{-xt_1} \mathcal{Y}(t; x)}
= t_1 e^{yt_1} e^{-(x+y)t_1} \mathcal{Y}(t; x + y) = t_1 e^{-xt_1} \mathcal{Y}(t; x + y).
\]

This completes the proof. □

**Theorem 2.2** For any integers \( n, m \geq 0 \) and any indeterminate \( \lambda \), there hold

\[
\sum_{k=0}^{n} \binom{k + \lambda - 1}{k} A_{n+m,m+k}(t) = \sum_{k=0}^{n} \binom{n + \lambda}{k} \binom{n + \lambda - k - 1}{n - k} A_{m+k,m}(t) t_1^{n-k},
\]

\[
\sum_{k=0}^{n} \binom{k + \lambda - 1}{k} A_{n+m,m+k}(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n + \lambda}{k} f_{m+k}(t) t_1^{n-k+1}.
\]

**Proof:** With the umbra \( Y_t \), given by \( Y^n_t = Y_n(t) \), \( \mathcal{Y}(t; x) \) may be written as \( \mathcal{Y}(t; x) = e^{xt}. \) (See, for example, [7][12][13]). Then, by Lemma 2.1, we have

\[
A(t; x, y) = t_1 e^{Y_t(x+y)-t_1 x} = t_1 e^{Y_t-x} e^{Y_t y}.
\]

When comparing the coefficient of \( e^{\frac{x^n + k}{m + k}} \), \( A_{n+k,k}(t) \) can be represented umbrally as

\[
A_{n+k,k}(t) = t_1 Y_t^n(Y_t - t_1)^n.
\]

Let \( [x^n]f(x) \) denote the coefficient of \( x^n \) in the formal power series \( f(x) \). Then we get

\[
\sum_{k=0}^{n} \binom{k + \lambda - 1}{k} A_{n+m,m+k}(t)
= \sum_{k=0}^{n} (-1)^k \binom{-\lambda}{k} t_1 Y_t^{m+k}(Y_t - t_1)^{n-k}
= t_1 Y_t^n(Y_t - t_1)^n \sum_{k=0}^{n} \binom{-\lambda}{k} \left(-\frac{Y_t}{Y_t - t_1}\right)^k.
\]
\[= t_1 Y_t^n (Y_t - t_1)^a \sum_{k=0}^{n} [x^k] \left(1 - \frac{x Y_t}{Y_t - t_1}\right)^{-\lambda}\]
\[= t_1 Y_t^n (Y_t - t_1)^a \frac{1}{1 - x} \left(1 - \frac{x Y_t}{Y_t - t_1}\right)^{-\lambda}\]
\[= t_1 Y_t^n (Y_t - t_1)^a [x^n] \frac{1}{(1 - x)^{\lambda+1}} \left(1 - \frac{x}{(1 - x) (Y_t - t_1)}\right)^{-\lambda}\]
\[= t_1 Y_t^n (Y_t - t_1)^a \sum_{k=0}^{n} \left(\frac{-\lambda}{n-k}\right) x^{n-k} \frac{1}{(1 - x)^{n+\lambda-k+1}} (-\frac{t_1}{Y_t - t_1})^{n-k}\]
\[= \sum_{k=0}^{n} (-1)^k \left(\frac{n+\lambda-k+1}{k}\right) (-\frac{\lambda}{n-k}) t_1 Y_t^n (Y_t - t_1)^k (-t_1)^{n-k}\]
\[= \sum_{k=0}^{n} \left(\frac{n+\lambda}{k}\right) \left(\frac{n+\lambda-k-1}{n-k}\right) A_{m+k,m}(t) t_1^{n-k},\]

which proves (2).

By the identity
\[\binom{n}{k} \binom{k}{i} = \binom{n}{i} \binom{n-i}{k-i},\]

and Vandermonde’s convolution identity
\[\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n},\]

we have
\[\sum_{k=0}^{n} \binom{k+\lambda-1}{k} A_{n+m,m+k}(t)\]
\[= \sum_{k=0}^{n} \binom{n+\lambda}{k} \left(\frac{-\lambda}{n-k}\right) t_1 Y_t^n (Y_t - t_1)^k (-t_1)^{n-k}\]
\[= \sum_{k=0}^{n} \binom{n+\lambda}{k} \left(\frac{-\lambda}{n-k}\right) \sum_{i=0}^{k} \binom{k}{i} t_1 Y_t^{m+i} (-t_1)^{n-i}\]
\[= \sum_{i=0}^{n} t_1 Y_t^{m+i} (-t_1)^{n-i} \sum_{k=i}^{n} \binom{n+\lambda}{k} \left(\frac{-\lambda}{n-k}\right) \binom{k}{i}\]
\[= \sum_{i=0}^{n} \binom{n+\lambda}{i} t_1 Y_t^{m+i} (-t_1)^{n-i} \sum_{k=i}^{n} \binom{-\lambda}{n-k} \binom{n+\lambda-i}{k-i}\]
\[= \sum_{i=0}^{n} \binom{n+\lambda}{i} t_1 Y_t^{m+i} (-t_1)^{n-i}\]
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\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n+\lambda}{k} \gamma_{m+k}(t)t_{1}^{n-k+1}, \]

which proves (3).

The case \( \lambda = 0 \) in (3) yields an explicit formula for \( A_{n,m}(t) \).

**Corollary 2.3** For any integers \( n, m \geq 0 \), there holds

\[ A_{n+m,m}(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t_{1}^{n-k+1} \gamma_{m+k}(t). \]  

(5)

**Corollary 2.4** For any integers \( n, m \geq 0 \), there hold

\[ \sum_{k=0}^{n} A_{n+m,k+m}(t) = \frac{t_{1} \gamma_{n+m+1}(t) - A_{n+m+1,m}(t)}{t_{1}}, \]  

(6)

\[ \sum_{k=0}^{n} (k+1) A_{n+m,k+m}(t) = \frac{A_{n+m+2,m}(t) - t_{1} \gamma_{n+m+2}(t) + (n+2)t_{1}^{2} \gamma_{n+m+1}(t)}{t_{1}^{2}}, \]  

(7)

\[ \sum_{k=0}^{n} (n-k+1) A_{n+m,k+m}(t) = \frac{t_{1} \gamma_{n+m+2}(t) - A_{n+m+2,m}(t) - (n+2)t_{1} A_{n+m+1,m}(t)}{t_{1}^{2}}. \]  

(8)

**Proof:** By combining (5) with \( n \) replaced by \( n+1 \) (resp. \( n+2 \)) and with the case \( \lambda = 1 \) (resp. \( \lambda = 2 \)) in (3), we obtain (6) (resp. (7)). Moreover, (8) can be easily obtained from (6) and (7).

**Theorem 2.5** For any integers \( n, m, k \geq 0 \), there holds

\[ A_{n+m+k,m+k}(t) = \sum_{j=0}^{m} \binom{m}{j} t_{1}^{m-j} A_{n+k+j,k}(t). \]  

(9)

**Proof:** Here we provide a combinatorial proof. For any \( \pi \in A_{n+m+k,m+k} \), suppose that \( \pi \) has exactly \( m-j \) singletons in \( \{k+1, \ldots, k+m\} \), which contributes the weight \( t_{1}^{m-j} \), and there are \( \binom{m}{j} \) ways to do this. The remaining \( j \) elements in \( \{k+1, \ldots, k+m\} \) can not be singletons in \( \pi \). These \( j \) elements can be regarded as the roles that greater than \( m+k+1 \), so the remaining \( n+k+j \) elements can be partitioned with the largest singleton \( m+k+1 \), these cases contribute the weight \( A_{n+k+j,k}(t) \). Thus the total weight of such partitions is \( \binom{m}{j} t_{1}^{m-j} A_{n+k+j,k}(t) \). Summing up all the possible cases yields (9).

**Theorem 2.6** For any integers \( n, m \geq 0 \) and any indeterminate \( y \), there hold

\[ \sum_{k=0}^{n} \binom{n}{k} A_{n+m+k}(t)y^{k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \gamma_{m+k}(t)(y+1)^{k}t_{1}^{n-k+1}, \]  

(10)

\[ \sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(t)y^{n-k} = t_{1} \sum_{k=0}^{n} \binom{n}{k} \gamma_{m+k}(t)(y-t_{1})^{n-k}. \]  

(11)
Proof: By (4), we have

\[
\sum_{k=0}^{n} \binom{n}{k} A_{n+m,m+k}(t)y^k = \sum_{k=0}^{n} \binom{n}{k} t_1 Y^{m+k}_1((Y_t - t_1)^{n-k}y^k
\]

\[
= t_1 Y^{m+k}_1((y+1)Y_t - t_1)^n
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (y+1)^k Y^{m+k}_1 t_1^{n-k+1}
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (y+1)^k Y^{m+k}_1(t) t_1^{n-k+1},
\]

which proves (10). Similarly, (11) can be obtained, but here we provide a combinatorial proof.

Let \(X_{n,m} = \bigcup_{k=0}^{n} X_{n,m,k}\) and \(X_{n,m,k}\) denote the set of pairs \((\pi, S)\) such that

- \(S\) is an \((n-k)\)-subset of \([m+2, n+m+1]\) = \(\{m+2, \ldots, n+m+1\}\), and each element of \(S\) is colored by \(t_1\) or \(y-t_1\);

- \(\pi\) is a partition of the set \([n+m+1] - S\) with the largest singleton \(m+1\), and each element of \([n+m+1] - S\) is only colored by \(1\).

Let \(Y_{n,m} = \bigcup_{k=0}^{n} Y_{n,m,k}\) and \(Y_{n,m,k}\) denote the set of pairs \((\pi, S)\) such that

- \(S\) is an \((n-k)\)-subset of \([m+2, n+m+1]\) and each element of \(S\) is only colored by \(y-t_1\);

- \(\pi\) is a partition of the set \([n+m+1] - S\) such that \(m+1\) must be a singleton, and each element of \([n+m+1] - S\) is only colored by \(1\).

The weight of \((\pi, S)\) is defined to be the product of the weight of \(\pi\) and the color of each element of \([n+m+1]\). Clearly, the weights of \(X_{n,m}\) and \(Y_{n,m}\) are counted respectively by the left and right sides of (11).

Given any pair \((\pi, S)\) in \(X_{n,m}\), \(S\) can be partitioned into two parts \(S_1\) and \(S_2\) such that each element of \(S_1\) is colored by \(y-t_1\) and each element of \(S_2\) is colored by \(t_1\). Regard each element of \(S_2\) as a singleton which is weighted by \(t_1\) and colored by \(1\), together with \(\pi\), we obtain a partition \(\pi_1\) of \([n+m+1] - S_1\) such that \(m+1\) is always a singleton. Then the pair \((\pi_1, S_1)\) lies in \(Y_{n,m}\).

Conversely, for any pair \((\pi_1, S_1)\) in \(Y_{n,m}\), let \(S\) denote the union of \(S_1\) and the singletons of \(\pi_1\) greater than \(m+1\), then \(\pi_1\) can be partitioned into two parts \(\pi\) and \(\pi'\) such that \(\pi\) is a partition of \([n+m+1] - S\) with the largest singleton \(m+1\) and \(\pi'\) is the singletons of \(\pi_1\) greater than \(m+1\). By regarding \(\pi'\) as a subset of \([m+2, n+m+1]\) in which each element is colored by \(t_1\), together with \(S_1\). Then we obtain an \((n-k)\)-subset of \([m+2, n+m+1]\) for some \(k\) such that each element of \(S\) is colored by \(t_1\) or \(y-t_1\). Then the pair \((\pi, S)\) lies in \(X_{n,m}\).

Clearly we find a bijection between \(X_{n,m}\) and \(Y_{n,m}\), which proves (11). \(\Box\)

The cases \(y = -1\) in (10) and \(y = t_1\) in (11) lead to
Corollary 2.7 For any integers \( n, m \geq 0 \), there hold
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_{n+m,m+k}(t) = \mathcal{Y}_m(t) t_1^{n+1},
\]
\[
\sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(t) t_1^{n-k-1} = \mathcal{Y}_m(t).
\]

The case \( y = \frac{\mu t_1}{y+1} \) in (11), together with (10) generates the following result which has a combinatorial interpretation.

Corollary 2.8 For any integers \( n, m \geq 0 \), there holds
\[
\sum_{k=0}^{n} \binom{n}{k} A_{m+k,m}(t)(y+1)^k(yt_1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} A_{n+m,m+k}(t)y^k. \tag{12}
\]

Proof: Let \( X^*_{n,m} = \bigcup_{j=0}^{n} X^*_{n,m,k} \) and \( X^*_{n,m,k} \) denote the set of pairs \( (\pi, S) \) such that
- \( \pi \) is a partition of the set \([n + m + 1]\) containing at least the singleton \( m + 1 \);
- \( S \) is an \((n-k)\)-subset of \([m + 2, n + m + 1]\) which is also the set of singletons of \( \pi \) greater than \( m + 1 \), each element of \( S \) is only colored by \( y \) and each element of \([m + 2, n + m + 1] - S \) is colored by \( 1 \) or \( y \);
- each element of \([m + 1]\) is only colored by \( 1 \).

Let \( Y^*_{n,m} = \bigcup_{k=0}^{n} Y^*_{n,m,k} \) and \( Y^*_{n,m,k} \) denote the set of pairs \( (\pi, S) \) such that
- \( S \) is a \( k \)-subset \( \{i_1, i_2, \ldots, i_k\} \) of \([m + 2, n + m + 1]\) in increasing order, each element of \( S \) is only colored by \( y \) and each element of \([n + m + 1] - S \) is only colored by \( 1 \);
- \( \pi \) is a partition of the set \([n + m + 1]\) such that \( i_k \) must be the largest singleton if \( S \) is not empty and \( m + 1 \) must be the largest singleton if \( S \) is empty;
- each element of \([m + 2, n + m + 1] - S \) must not be a singleton.

The weight of \((\pi, S)\) is defined to be the product of the weight of \( \pi \) and the colors of all elements in \([n + m + 1]\). Clearly, any \((\pi, S) \in X^*_{n,m}\) can be obtained as follows. First choose an \((n-k)\)-subset \( S \) of \([m + 2, n + m + 1]\), there are \( \binom{n}{k} \) ways to do this. Regard each element of \( S \) as a singleton with color \( y \). Then color each element of \([m + 2, n + m + 1] - S \) by \( 1 \) or \( y \), namely, each element of \([m + 2, n + m + 1] - S \) is colored by \( y + 1 \). Now partitioning \([n + m + 1] - S \) such that the largest singleton is \( m + 1 \), together with the \( n - k \) singletons formed form \( S \), we get the partition \( \pi \) of \([n + m + 1]\) such that \( m + 1 \) must be a singleton; Hence the total weight of pairs \((\pi, S) \in X^*_{n,m}\) is just the left hand side of (12).

Similarly, the total weight of pairs \((\pi, S) \in Y^*_{n,m}\) is just the right hand side of (12) if regarding each element of \([m + 2, n + m + 1] - S \) as the role greater than \( i_k \) when \( S \neq \emptyset \).

Now we can construct a bijection \( \varphi \) between \( X^*_{n,m} \) and \( Y^*_{n,m} \) which preserves the weights. For any \((\pi, S) \in X^*_{n,m}\), let \( S_1 \) denote the set of elements of \([n + m + 1]\) with colors \( y \). Clearly, \( S \) is a subset of \( S_1 \).
Assume that \( S_1 = \{i_1, i_2, \ldots, i_k\} \) for some \( 0 \leq k \leq n \) in increasing order. If \( S_1 \) is the empty set \( \emptyset \), which implies that \( S = \emptyset \) and all elements of \([n + m + 1]\) are colored by 1, it is obvious that \((\pi, \emptyset) \in \mathcal{Y}_{n,m}^\ast\). Then define \( \varphi(\pi, \emptyset) = (\pi, \emptyset) \). If \( S_1 \) is not the empty set, exchanging \( m + 1 \) and \( i_k \) in \( \pi \), we obtain a partition \( \pi_1 \), it is easily to verify that \((\pi_1, S_1) \in \mathcal{Y}_{n,m}^\ast\) and has the same weight as \((\pi, S)\). Then define \( \varphi(\pi, S) = (\pi_1, S_1) \).

Conversely, for any \((\pi_1, S_1) \in \mathcal{Y}_{n,m}^\ast\), if \( S_1 = \emptyset \), so \( \pi_1 \) has the largest singleton \( m + 1 \), then \((\pi_1, \emptyset) \in \mathcal{Y}_{n,m}^\ast\) and define \( \varphi^{-1}(\pi_1, \emptyset) = (\pi_1, \emptyset) \). If \( S_1 \neq \emptyset \), assume that \( S_1 = \{i_1, i_2, \ldots, i_k\} \) for some \( 1 \leq k \leq n \) in increasing order, let \( \mathcal{S} \) denote the set of all the elements in \( S_1 \) such that each forms a singleton of \( \pi_1 \). Now exchanging \( m + 1 \) and \( i_k \) in \( \pi_1 \), we obtain a partition \( \pi \), it is easy verifiable that \((\pi, S) \in \mathcal{X}_{n,m}^\ast\) which has the same weight as \((\pi_1, S_1)\). Then define \( \varphi^{-1}(\pi_1, S_1) = (\pi, S) \).

Clearly, \( \varphi \) is indeed a bijection between \( \mathcal{X}_{n,m}^\ast \) and \( \mathcal{Y}_{n,m}^\ast \), which proves (12). \( \square \)

3 The special case for permutations

When the parameter \( t \) in \( A_{n,k}(t) \) takes some special value, that is to assign a special structure to each block of partitions of \([n + 1]\). For example, the case \( t = (1^0, 2^1, 3^2, \ldots) \) indicates that each block of partitions is assigned by a (rooted and labeled) tree structure, such partitions are equivalent to labeled forests; The case \( t = (1, 1, 0, \ldots) \) leads to involutions on \([n + 1]\).

In this section, we just present an interesting specialization, but leave others to inclined readers. Consider the special case when \( t = (0!, 1!, 2!, \ldots) \), that is to assign a cycle structure to each block of partitions, such partitions is equivalent to permutations. Let \( P_{n,k} = A_{n,k}(t) \) with \( t = (0!, 1!, 2!, \ldots) \), i.e., \( P_{n,k} \) is the number of permutations of \([n + 1]\) with the largest fixed point \( k + 1 \). From (5) and (9), one has the explicit formulas for \( P_{n,k} \):

\[
P_{n+k,k} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (k+j)! = \sum_{j=0}^{k} \binom{k}{j} D_{n+j}.
\]

Clearly, \( P_{n,n} = n! = \mathcal{Y}_n(0!, 1!, 2!, \ldots) \) and \( P_{n,0} = D_n = \mathcal{Y}_n(0, 1!, 2!, \ldots) \), where \( D_n \) is the derangement number of \([n]\), i.e., the number of permutations of \([n]\) without fixed points. See Table 1 for some small values of \( P_{n,k} \).

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<td>426</td>
<td>504</td>
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</tbody>
</table>

Table 1. The values of \( P_{n,k} \) for \( n \) and \( k \) up to 6.
In fact, \( \{P_{n,k}\}_{n \geq k \geq 0} \) forms the difference table introduced by Euler, which has been investigated in depth in derangement theory \([2, 5, 6, 8, 9]\). Chen \([1]\) gave two other interpretations for \( P_{n,k} \) using \( k \)-relative derangements on \([n]\) and skew derangements from \([n]\) to \( \{-k+1, \ldots, -1, 0, 1, \ldots, n-k\} \) for \( 0 \leq k \leq n \). Moreover, Chen established a bijection between these two settings. Recently, Deutsch and Elizalde \([4]\) gave a new interpretation of derangement number \( D_{n+2} \) as the sum of the values of the largest fixed points of all non-derangements of length \( n+1 \), namely,

\[
\sum_{k=0}^{n} (k+1)P_{n,k} = D_{n+2},
\]

which is the special case of \([7]\) when \( t = (0!, 1!, 2!, \ldots) \) and \( m = 0 \).

Next, we can explore some new relations between \( P_{n,k} \) and other classical sequences such as Bell numbers or Fibonacci numbers.

**Example 3.1** By Lemma 2.1, one can derive the bivariate exponential generating function for \( P_{n,k} \), i.e.,

\[
P(x, y) = \sum_{n,k \geq 0} P_{n+k,k} \frac{x^n y^k}{n! k!} = \frac{e^{-x}}{1 - x - y}.
\]

Extracting the coefficient of \( \frac{x^n}{n!} \) in \( P(x, x^2) \), we have

\[
\frac{1}{n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} k!P_{n-k,k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!F_k,
\]

where \( F_k \) is the \( k \)-th Fibonacci number defined by

\[
\binom{1}{x} = \sum_{k \geq 0} F_k x^k.
\]

**Example 3.2** When \( t = (0!, 1!, 2!, \ldots) \), \([10]\) and \([11]\) reduce to

\[
\sum_{k=0}^{n} \binom{n}{k} P_{n+m,m+k} y^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (m+k)!(y+1)^k, \quad (13)
\]

\[
\sum_{k=0}^{n} \binom{n}{k} P_{m+k,m} y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (m+k)!(y-1)^{n-k}. \quad (14)
\]

It should be noted that \([13]\) and \([14]\) have close relations to the (re-normalized) Charlier polynomials \( C_n(u, v) \) \([7]\) defined by

\[
C_n(u, v) = \sum_{k=0}^{n} \binom{n}{k} (u)_{k}v^{n-k},
\]

where \( (u)_k = u(u+1) \cdots (u+k-1) \) for \( k \geq 1 \) and \( (u)_0 = 1 \). In fact \([13]\) is \( \frac{(y+1)^n}{m!}C_n(m+1, -\frac{1}{y+1}) \) and \([14]\) is equal to \( \frac{1}{m!}C_n(m+1, y-1) \).
Recall that, by (4), \( P_{n,k} \) can be represented umbrally as
\[ P_{n,k} = P_k(\mathbf{P} - 1)^{n-k}, \]
where \( \mathbf{P} = \mathbf{Y}_t \) with \( t = (0!, 1!, 2!, \ldots) \). In particular, \( D_n = (\mathbf{P} - 1)^n \) and \( n! = \mathbf{P}^n \). Hence, the case \( y = \mathbf{P} - 1 \) in (13) and the case \( y = \mathbf{P} \) in (14) generate
\[
\sum_{k=0}^{n} \binom{n}{k} P_{n+m,m+k}D_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (m+k)!k!,
\]
\[
\sum_{k=0}^{n} \binom{n}{k} P_{m+k,m}(n-k)! = \sum_{k=0}^{n} \binom{n}{k} (m+k)!D_{n-k}.
\]

With the Bell umbra \( \mathbf{B} \) given by \( \mathbf{B} = \mathbf{Y}_t \) with \( t = (1, 1, 1, \ldots) \), the Bell number can be written as \( B_n = \mathbf{B}^n \) and \( \mathbf{B}^{n+1} = (\mathbf{B} + 1)^n \). Then the case \( y = \mathbf{B} \) in (13) and the case \( y = \mathbf{B} + 1 \) in (14) generate
\[
\sum_{k=0}^{n} \binom{n}{k} P_{n+m,m+k}B_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (m+k)!B_{k+1},
\]
\[
\sum_{k=0}^{n} \binom{n}{k} P_{m+k,m}B_{n-k+1} = \sum_{k=0}^{n} \binom{n}{k} (m+k)!B_{n-k}.
\]

Using the Riordan identity \([3][11], \text{P}173]\).
\[
\sum_{k=0}^{n} \binom{n}{k} (k+1)! \binom{1}{n-k} = (n+1)^{n+1},
\]
the case in (13) with \( m = 1 \) and \( y = -\frac{n+2}{n+1} \) and the case in (14) with \( m = 1 \) and \( y = n+2 \) generate respectively
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{n+1,k+1}(n+2)^k(n+1)^{n-k} = (n+1)^{n+1},
\]
\[
\sum_{k=0}^{n} \binom{n}{k} (D_k + D_{k+1})(n+2)^{n-k} = (n+1)^{n+1},
\]
(15)

where we use the relation \( P_{k+1,1} = D_k + D_{k+1} \). By the well-known recurrence \( D_{k+2} = (k+1)(D_k + D_{k+1}) \) for derangement numbers \( D_k \), together with \( D_1 = 0 \), after routine computation, (15) is equivalent to
\[
\sum_{k=0}^{n} \binom{n}{k} D_{k+1}(n+1)^{n-k} = n^{n+1},
\]
(16)

which was also obtained by Riordan \([10]\). In a forthcoming paper \([18]\), using functional digraph theory, we will give a combinatorial interpretation for a more general identity involving the Riordan identity and (16) as special cases.
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References


