Some results on the palette index of graphs[*]

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Given a proper edge coloring φ of a graph G, we define the palette $S_G(v, \varphi)$ of a vertex $v \in V(G)$ as the set of all colors appearing on edges incident with v. The palette index $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G. In this paper we give various upper and lower bounds on the palette index of G in terms of the vertex degrees of G, particularly for the case when G is a bipartite graph with small vertex degrees. Some of our results concern (a, b)-biregular graphs; that is, bipartite graphs where all vertices in one part have degree a and all vertices in the other part have degree b. We conjecture that if G is (a, b)-biregular, then $\check{s}(G) \leq 1 + \max\{a, b\}$, and we prove that this conjecture holds for several families of (a, b)-biregular graphs. Additionally, we characterize the graphs whose palette index equals the number of vertices.

Keywords: edge coloring, palette index, cyclic interval edge coloring

1   Introduction

Given an edge coloring φ of a graph G, we define the *palette* $S_G(v, \varphi)$ (or just $S(v, \varphi)$) of a vertex $v \in V(G)$ as the set of all colors appearing on edges incident with v. The *palette index* $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G. This notion was introduced quite recently by Horňák et al. (2014) and has so far primarily been studied for the case of regular graphs.

Denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree and the chromatic index of a graph G, respectively. By Vizing's well-known edge coloring theorem $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ for every graph G. In the former case G is said to be *Class 1*, and in the latter case G is *Class 2*.

Trivially, $\check{s}(G) = 1$ if and only G is a regular Class 1 graph, and by Vizing's edge coloring theorem it holds that if G is regular and Class 2, then $3 \leq \check{s}(G) \leq \Delta(G) + 1$; the case $\check{s}(G) = 2$ is not possible, as proved in Horňák et al. (2014).

Since computing the chromatic index of a given graph is $\mathcal{NP}$-complete, as proved in Leven and Galil (1983), determining the palette index of a given graph is $\mathcal{NP}$-complete, even for 3-regular graphs. Note further that this in fact means that even determining if a given graph has palette index 1 is an $\mathcal{NP}$-complete problem. Nevertheless, in Horňák et al. (2014) it was proved that the palette index of a cubic Class 2 graph

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is 3 or 4 according to whether the graph has a perfect matching or not. Bonvicini and Mazzuoccolo (2016) investigated 4-regular graphs; they proved that $\check{s}(G) \in \{3, 4, 5\}$ if G is 4-regular and Class 2, and that all these values are in fact attained.

Vizing's edge coloring theorem yields an upper bound on the palette index of a general graph G with maximum degree Δ and no isolated vertices, namely that $\check{s}(G) \leq 2^{\Delta+1} - 2$. However, this is probably far from being tight. Indeed, Avesani et al. (2018) described an infinite family of multigraphs whose palette index grows asymptotically as Δ^2; it is an open question whether there are such examples without multiple edges. Furthermore, they suggested to prove that there is a polynomial $p(\Delta)$ such that for any graph with maximum degree Δ, $\check{s}(G) \leq p(\Delta)$. In fact, they suggested that such a polynomial is quadratic in Δ. We thus arrive at the following conjecture:

Conjecture 1.1. *There is a constant C, such that for any graph G with maximum degree Δ, $\check{s}(G) \leq C\Delta^2$.*

Very little is known about the palette index of non-regular graphs. Bonisoli et al. (2017) studied the palette index of trees, and quite recently Horňák and Hudák (2018) completely determined the palette index of complete bipartite graphs $K_{a,b}$ with $a \leq 5$.

In this note we study the palette index of some families of non-regular graphs. Before outlining the results of this paper, let us briefly consider a connection to another kind of edge coloring.

An *interval t-coloring* of a graph G is a proper t-edge coloring such that for every vertex v of G the colors of the edges incident with v form an interval of consecutive integers; if we also add the condition that color 1 is considered as consecutive of color t, then we get a *cyclic interval t-coloring*. Note that any graph G with an interval coloring admits a cyclic interval $\Delta(G)$-coloring (by taking all colors modulo $\Delta(G)$).

As noted in Avesani et al. (2018), if a graph G with maximum degree Δ has an interval coloring, then $\check{s}(G) \leq \Delta^2 - \Delta + 1$. Moreover, this upper bound holds for graphs with a cyclic interval Δ-coloring (as implicit in the proof in Avesani et al. (2018)). In fact, it holds that for any graph G with maximum degree Δ, if G has a cyclic interval $C\Delta$-coloring, where C is some absolute constant, then the palette index of G is bounded by a quadratic polynomial in Δ. An example of a family of graphs with this property (which do not in general admit interval colorings) are complete multipartite graphs; such a graph G has a cyclic interval coloring with at most $2\Delta(G)$ colors, as proved in Asratian et al. (2018b). Since there are at most Δ different vertex degrees in a graph with maximum degree Δ, it follows that Conjecture 1.1 is true for every complete multipartite graph.

Proposition 1.2. *If G is a complete multipartite graph with maximum degree Δ, then $\check{s}(G) \leq 2\Delta^2$.*

We do not know of any cyclically interval colorable graph G that requires more than $2\Delta(G)$ colors for a cyclic interval coloring; thus we suggest that Conjecture 1.1 particularly holds for any graph with a cyclic interval coloring. Note further that it is in fact an open problem to determine if there is a graph G that requires more than $\Delta(G) + 1$ colors for a cyclic interval coloring (cf. Casselgren et al. (2018)).

In the following, we shall present some further upper bounds on the palette index based on connections with cyclic interval colorings; as it turns out, existence of cyclic interval colorings do in fact provide tight upper bounds on the palette index of some families of graphs. Furthermore, motivated by the connection with cyclic interval colorings, we consider the problem of determining the palette index of a natural generalization of regular bipartite graphs, namely so-called (a, b)-*biregular graphs*, i.e., bipartite graphs where all vertices in one part have degree a and all vertices in the other part have degree b. Note that

regular bipartite graphs trivially have cyclic interval colorings; it has been conjectured in Casselgren and Toft (2015) that this also holds for (a, b)-biregular graphs.

Conjecture 1.3. *Every (a, b)-biregular graph admits a cyclic interval $\max\{a, b\}$-coloring.*[(i)]

The general problem of determining the palette index of a given (a, b)-biregular graph is $\mathcal{NP}$-complete; this follows e.g. from the complexity result in Asratian and Casselgren (2007). We would like to suggest the following weakening of Conjecture 1.3, which is a strengthening of Conjecture 1.1 for biregular graphs.

Conjecture 1.4. *For any (a, b)-biregular graph G, $\check{s}(G) \leq 1 + \max\{a, b\}$.*

Note that the upper bound in Conjecture 1.4 is in general tight, since $\check{s}(G) = b + 1$ if G is $(1, b)$-biregular. However, as we shall see, the upper bound in Conjecture 1.4 can be slightly improved for some values of a and b.

Let us now outline the main results of this paper. We shall present several results towards Conjectures 1.1 and 1.4. In the next section we prove a general upper bound on the palette index of bipartite graphs and deduce that Conjecture 1.1 holds for bipartite graphs where all vertex degrees are in the set $\{1, 2, 3, 4, 2r - 4, 2r - 3, 2r - 2, 2r - 1, 2r\}$, for some $r \geq 1$. Additionally, we demonstrate that Conjecture 1.1 is true for general graphs G satisfying that $\Delta(G) - \delta(G) \leq 2$, where $\delta(G)$ denotes the minimum degree of G.

In Section 3 we consider bipartite graphs with small vertex degrees. In particular, we obtain sharp upper bounds on the palette indices of Eulerian bipartite graphs with maximum degree at most 6. We also determine the palette index of grids.

Section 4 concerns biregular graphs and Conjecture 1.4. We prove that this conjecture holds for all $(2, r)$-biregular and $(2r - 2, 2r)$-biregular graphs. Additionally, we establish that it holds for all (a, b)-biregular graphs such that

- $(a, b) \in \{(3, 6), (3, 9)\}$;

- $(a, b) \in \{(4, 6), (4, 8), (4, 12), (4, 16)\}$;

- $(a, b) \in \{(5, 10), (6, 9), (6, 12)\}$;

- $(a, b) \in \{(8, 12), (8, 16), (12, 16)\}$.

Finally, as mentioned above, $\check{s}(G) = 1$ if and only if G is regular and Class 1; in Section 5 we characterize the graphs whose palette index is at the opposite end of the spectrum; that is, we give a complete characterization of the graphs whose palette index equals the number of vertices.

2   General upper bounds

As noted earlier, Vizing's edge coloring theorem yields an upper bound of the palette index of a general graph, and König's edge coloring theorem shows that this general upper bound can be slightly improved for bipartite graphs: $\check{s}(G) \leq 2^\Delta - 1$ for any bipartite graph G with maximum degree Δ and no isolated vertices. In the following we shall give an improvement of this general upper bound for bipartite graphs. Throughout, we assume that all graphs in this section do not contain any isolated vertices.

[(i)] See e.g. Asratian et al. (2018b), for further information on the status of this conjecture.

We shall need a classic result from factor theory. A 2-*factor* of a multigraph G (where loops are allowed) is a 2-regular spanning subgraph of G.

Theorem 2.1. *(Petersen's Theorem). Let G be a $2r$-regular multigraph (where loops are allowed). Then G has a decomposition into edge-disjoint 2-factors.*

For a graph G, denote by $D(G)$ the set of all degrees in G, and by $D^{\mathrm{odd}}(G)$ ($D^{\mathrm{even}}(G)$) the set of all odd (even) degrees in G. A graph is *even (odd)* if all vertex degrees of the graph are even (odd).

Theorem 2.2. *If G is an even bipartite graph, then*

$$\check{s}(G) \leq \sum_{d \in D(G)} \binom{\frac{\Delta(G)}{2}}{\frac{d}{2}}.$$

Proof: For the proof, we construct a new multigraph $G^\star$ as follows: for each vertex $u \in V(G)$ of degree $2k$, we add $\frac{\Delta(G)}{2} - k$ loops at u $\left(1 \leq k < \frac{\Delta(G)}{2}\right)$. Clearly, $G^\star$ is a $\Delta(G)$-regular multigraph. By Petersen's theorem, $G^\star$ can be represented as a union of edge-disjoint 2-factors $F_1, \ldots, F_{\frac{\Delta(G)}{2}}$. By removing all loops from 2-factors $F_1, \ldots, F_{\frac{\Delta(G)}{2}}$ of $G^\star$, we obtain that the resulting graph G is a union of edge-disjoint even subgraphs $F'_1, \ldots, F'_{\frac{\Delta(G)}{2}}$. Since G is bipartite, for each i $\left(1 \leq i \leq \frac{\Delta(G)}{2}\right)$, F'_i is a collection of even cycles in G, and we can properly color the edges of F'_i alternately with colors $2i - 1$ and $2i$; the obtained coloring α is a proper edge coloring of G with colors $1, \ldots, \Delta(G)$.

Now, if $u \in V(G)$ and $d_G(u) = 2k$, then there are k even subgraphs $F'_{i_1}, F'_{i_2}, \ldots, F'_{i_k}$ such that $d_{F'_{i_1}}(u) = d_{F'_{i_2}}(u) = \cdots = d_{F'_{i_k}}(u) = 2$, and thus $S_G(u, \alpha) = \{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k\}$. This implies that for vertices $u \in V(G)$ with $d_G(u) = 2k$, we have at most $\binom{\frac{\Delta(G)}{2}}{k}$ distinct palettes in the coloring α. $\qquad\square$

In the next two sections, we shall see that Theorem 2.2 can in fact be used to deduce sharp upper bounds on the palette index of some classes of bipartite graphs.

From a given bipartite graph G we can construct an even supergraph G' by taking two vertex-disjoint copies G_1 and G_2 of G and for every odd-degree vertex of G_1 joining it by an edge with its copy in G_2. By applying the preceding proposition to G' we immediately obtain the following.

Corollary 2.3. *If G is a bipartite graph, then*

$$\check{s}(G) \leq \sum_{d \in D^{odd}(G)} \binom{\left\lceil \frac{\Delta(G)}{2} \right\rceil}{\frac{d+1}{2}} \times (d+1) + \sum_{d \in D^{even}(G)} \binom{\left\lceil \frac{\Delta(G)}{2} \right\rceil}{\frac{d}{2}}.$$

Proof: Consider the graph G' defined above, and a proper edge coloring α of G' defined as in the proof of Theorem 2.2. For each palette $S_{G'}(v, \alpha)$ in G', where $v \in D^{odd}(G)$, there are at most $(d_G(v) + 1)$ possible palettes in the restriction of α to G. $\qquad\square$

Using Corollary 2.3, we deduce an improvement of the general upper bound $2^{\Delta(G)} - 1$ on the palette index of any bipartite graph.

Corollary 2.4. *For any bipartite graph G, $\check{s}(G) \leq (\Delta(G) + 2)2^{\lceil \Delta(G)/2 \rceil}$.*

As noted above, the palette index of a regular Class 1 graph is 1. We note that Corollary 2.3 implies that Conjecture 1.1 holds for bipartite graphs that are "almost regular" in the sense that if G is a bipartite graph where all vertex degrees are in the set $\{1, 2, 3, 4, 2r - 4, 2r - 3, 2r - 2, 2r - 1, 2r\}$, for some $r \geq 4$, then G satisfies Conjecture 1.1. For general graphs, a slightly weaker proposition is true.

Proposition 2.5. *If a graph G satisfies that $\Delta(G) - \delta(G) \leq 2$, then $\check{s}(G) \leq \Delta^2(G) + \Delta(G) + 1$.*

The proof of this proposition is along the same lines as the proof of Theorem 5.9 in Asratian et al. (2018a); for the sake of completeness, we provide a brief sketch here.

Proof (sketch): If $\Delta(G) - \delta(G) \leq 1$, or G is Class 1, then the proposition clearly holds; indeed if G is Class 1, then $\check{s}(G) \leq \binom{\Delta(G)}{2} + \Delta(G) + 1 \leq \Delta^2(G) + \Delta(G) + 1$.

So assume that $\Delta(G) = \delta(G) + 2$, and that G is Class 2. Set $k = \Delta(G)$ and denote by V_i the set of vertices in G that have degree i.

Let M be a maximum matching of $G[V_k]$. Set $H = G - M$. Note that in H no two vertices of degree k in H are adjacent, so by a well-known result due to Fournier (1973), H is Class 1. Let M' be a minimum matching in H covering all vertices of degree k in H; such a matching exists since H is Class 1. Note that the graph $J = H - M'$ has maximum degree at most $k - 1$. Let M'' be a maximum matching in J_{k-1}, where J_{k-1} is the subgraph of J induced by the vertices of degree $k - 1$ in J. Let $\hat{M} = M \cup M' \cup M''$. The rest of the proof is based on the following two claims, the proofs of which are omitted (for details, see Asratian et al. (2018a)).

Claim 1. *The subgraph of G induced by $\hat{M}$ is 2-edge-colorable.*

Claim 2. *The graph $G - \hat{M}$ is $(k-1)$-edge-colorable.*

Let ψ be a proper $(k-1)$-edge coloring of $G - \hat{M}$ using colors $1, \ldots k - 1$, and let φ be a proper 2-edge coloring of the subgraph of G induced by $\hat{M}$ using colors k and $k + 1$. Denote by α the edge coloring of G obtained by taking the two edge colorings ψ and φ together.

Since a vertex of degree $k - 2$ in G is incident with at most one edge from $\hat{M}$, there are $2\binom{k-1}{k-3} + (k-1)$ possible palettes under α; a vertex of degree $k - 1$ in G is incident with at most one edge from $\hat{M}$ and thus there are most $2(k-1) + 1$ possible palettes under α; a vertex of degree k in G is incident with one or two edges from $\hat{M}$ and thus there are at most $2 + (k-1)$ possible palettes. $\square$

Finally, let us remark that every graph where all vertex degrees are in the set $\{1, 2, r - 2, r - 1, r\}$, for some $r \geq 5$, also satisfies Conjecture 1.1.

3 Bipartite graphs with small vertex degrees

In this section we consider bipartite graphs with small vertex degrees. As above, throughout this section we assume that all graphs do not contain any isolated vertices. We begin this section by noting some immediate implications of Theorem 2.2.

Corollary 3.1. *If G is an Eulerian bipartite graph with $\Delta(G) = 4$, then $\check{s}(G) \leq 3$.*

If G is bipartite, Eulerian, has maximum degree 4, and there is a vertex of degree 4 in G which is adjacent to at least three vertices of degree two, then $\check{s}(G) \geq 3$; for instance $\check{s}(K_{2,4}) \geq 3$, so the upper bound in Corollary 3.1 is sharp.

Corollary 3.2. *If G is a bipartite graph with $\Delta(G) = 4$, then $\check{s}(G) \leq 11$. Moreover, if G has no pendant vertices, then $\check{s}(G) \leq 7$.*

Proof: Starting from two copies of G, we can create an Eulerian bipartite graph G' with maximum degree 4 containing G as a subgraph. Let φ be a proper 4-edge coloring of G' constructed as in the proof of Theorem 2.2, and let us consider the restriction of this edge coloring to G. Vertices of degree 4 all have the same palette, vertices of degree 2 in G have at most 2 distinct possible palettes; vertices of degree 3 in G have at most 4 distinct palettes, and similarly for vertices of degree 1.   □

We note that the preceding corollary is sharp, which follows by considering a disjoint union of $K_{1,4}$, $K_{2,4}$, and $K_{3,4}$: the palette indices of these graphs are 5, 3 and 5, respectively, as observed in Horňák and Hudák (2018); in fact, in any proper edge coloring of this graph the vertices of degree 1 have four distinct palettes, vertices of degree 2 have at least two distinct palettes, vertices of degree three have four different palettes, and vertices of degree four have at least one palette. Hence, the palette index of the disjoint union of these complete bipartite graphs is at least 11.

From Corollary 3.2 we deduce an upper bound on the palette index of bipartite graphs with maximum degree 5.

Corollary 3.3. *If G is a bipartite graph with $\Delta(G) = 5$, then $\check{s}(G) \leq 23$. Moreover, if G has a perfect matching, then $\check{s}(G) \leq 12$.*

Proof: Let M be minimal matching in G covering all vertices of degree 5; such a matching exists e.g. by König's edge coloring theorem. By Corollary 3.2, $G - M$ has a proper edge coloring with 4 colors and at most 11 distinct palettes; by assigning a new color 5 to all edges of M, we obtain a proper edge coloring of G with at most 23 distinct palettes, because for any palette in $G - M$, we obtain at most 2 different palettes in G, and additionally, the palette $\{5\}$.

The second part follows by applying Corollary 3.2 to the graph $G - M'$, where M' is a perfect matching in G.   □

For Eulerian bipartite graphs with maximum degree six we have the following immediate consequence of Theorem 2.2.

Corollary 3.4. *If G is an Eulerian bipartite graph with $\Delta(G) = 6$, then $\check{s}(G) \leq 7$.*

Consider a graph that is the disjoint union of $K_{2,6}$ and $K_{4,6}$. Horňák and Hudák (2018) proved that $\check{s}(K_{2,6}) = 4$, and $\check{s}(K_{4,6}) = 4$, which, as above, implies that the upper bound in the preceding corollary is sharp.

Note further that the preceding corollary shows that Conjecture 1.4 holds for $(4, 6)$-biregular graphs.

For Eulerian bipartite graphs G with maximum degree 8, Theorem 2.2 implies that $\check{s}(G) \leq 15$. Using a result from Asratian et al. (2018b) we deduce that in fact a better upper bound holds:

Proposition 3.5. *If G is an Eulerian bipartite graph with maximum degree 8, then $\check{s}(G) \leq 13$.*

The proof is omitted since it immediately follows from the proof of Theorem 3 in Asratian et al. (2018b).

Our final result in this section concerns a particular family of bipartite graphs. The grids $G(m, n)$ are Cartesian products of paths on m and n vertices, respectively. Here, we determine the exact value of the palette index of $G(m, n)$.

Theorem 3.6. *For any $m, n \geq 2$,*

$$\check{s}(G(m, n)) = \begin{cases} 1, & \text{if } m = n = 2, \\ 2, & \text{if } \min\{m, n\} = 2 \text{ and } \max\{m, n\} \geq 3, \\ 3, & \text{if } m, n \geq 3 \text{ and } mn \text{ is even}, \\ 5, & \text{if } m, n \geq 3 \text{ and } mn \text{ is odd}. \end{cases}$$

Proof: Let $V(G(m, n)) = \left\{ v_j^{(i)} : 1 \leq i \leq m, 1 \leq j \leq n \right\}$ and

$$E(G(m, n)) = \left\{ v_j^{(i)} v_{j+1}^{(i)} : 1 \leq i \leq m, 1 \leq j \leq n - 1 \right\} \cup \left\{ v_j^{(i)} v_j^{(i+1)} : 1 \leq i \leq m - 1, 1 \leq j \leq n \right\}.$$

First we show that if mn is even, then

$$\check{s}(G(m, n)) = \begin{cases} 1, & \text{if } m = n = 2, \\ 2, & \text{if } \min\{m, n\} = 2 \text{ and } \max\{m, n\} \geq 3, \\ 3, & \text{if } m, n \geq 3 \text{ and } mn \text{ is even}. \end{cases}$$

Trivially, $\check{s}(G(2, 2)) = \check{s}(C_4) = 1$. So, without loss of generality we may assume that $\max\{m, n\} \geq 3$ and m is even. Define an edge coloring α of $G(m, n)$ as follows:

(1) for $i = 1, \ldots, m$, $j = 1, \ldots, n - 1$, let

$$\alpha\left(v_j^{(i)} v_{j+1}^{(i)} \right) = \begin{cases} 2, & \text{if } j \text{ is odd}, \\ 1, & \text{if } j \text{ is even}; \end{cases}$$

(2) for $i = 1, \ldots, \frac{m}{2}$, $j = 1, \ldots, n - 1$, let

$$\alpha\left(v_j^{(2i-1)} v_j^{(2i)} \right) = \begin{cases} 1, & \text{if } j = 1, \\ 3, & \text{otherwise}; \end{cases}$$

(3) for $i = 1, \ldots, \frac{m}{2} - 1$, $j = 1, \ldots, n$, let

$$\alpha\left(v_j^{(2i)} v_j^{(2i+1)} \right) = \begin{cases} 3, & \text{if } j = 1 \text{ or } j = n, \\ 4, & \text{otherwise}; \end{cases}$$

(4) for $i = 1, \ldots, \frac{m}{2}$, let

$$\alpha\left(v_n^{(2i-1)} v_n^{(2i)} \right) = \begin{cases} 2, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}, \end{cases}$$

It is easy to see that α is proper edge coloring of $G(m, n)$ with colors $1, 2, 3, 4$, such that for each vertex $v \in V(G(m, n))$, $S(v, \alpha) \in \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. This shows that if $\max\{m, n\} \geq 3$ and mn is even, then

$$\check{s}(G(m,n)) = \left\{ \begin{array}{ll} 2, & \text{if } \min\{m,n\} = 2 \text{ and } \max\{m,n\} \geq 3, \\ 3, & \text{if } m,n \geq 3 \text{ and } mn \text{ is even.} \end{array} \right.$$

Next we consider the case $m, n \geq 3$ and mn is odd. We first prove the upper bound, i.e. that $\check{s}(G(m,n)) \leq 5$. Without loss of generality we may assume that $m \leq n$. Let us first show that $\check{s}(G(3,n)) \leq 5$.

Define an edge coloring β of $G(3,n)$ as follows:

1) for $i = 1, 2, 3$, $j = 1, \ldots, n-1$, let

$$\beta \left(v_j^{(i)} v_{j+1}^{(i)} \right) = \left\{ \begin{array}{ll} 2, & \text{if } i = 1 \text{ and } j \text{ is odd,} \\ 1, & \text{if } i = 1 \text{ and } j \text{ is even,} \\ 2, & \text{if } i = 2 \text{ and } j \text{ is odd,} \\ 4, & \text{if } i = 2 \text{ and } j \text{ is even,} \\ 4, & \text{if } i = 3 \text{ and } j \text{ is odd,} \\ 2, & \text{if } i = 3 \text{ and } j \text{ is even;} \end{array} \right.$$

2) $j = 2, \ldots, n-1$, let

$$\beta \left(v_j^{(1)} v_j^{(2)} \right) = 3 \text{ and } \beta \left(v_j^{(2)} v_j^{(3)} \right) = 1;$$

3) $\beta \left(v_1^{(1)} v_1^{(2)} \right) = \beta \left(v_n^{(2)} v_n^{(3)} \right) = 1$, $\beta \left(v_n^{(1)} v_n^{(2)} \right) = 2$ and $\beta \left(v_1^{(2)} v_1^{(3)} \right) = 3$.

It is not difficult to see that β is proper edge coloring of $G(3,n)$ with colors $1, 2, 3, 4$ such that for each vertex $v \in V(G(3,n))$, $S(v,\beta) \in \{\{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$.

If $m \geq 5$, then we define a proper edge coloring of $G(m,n)$ in the following way: let $H = G(m,n) - \left\{ v_i^{(m-3)} v_i^{(m-2)} : 1 \leq i \leq n \right\}$. The graph H consists of two components H_1 and H_2, where H_1 is isomorphic to $G(m-3,n)$, and H_2 is isomorphic to $G(3,n)$. Let α' be a proper edge coloring of H_1 corresponding to the coloring α of $G(m-3,n)$ defined above, and let β' be a proper edge coloring of H_2 corresponding to the edge coloring β of $G(3,n)$ defined above. Suppose further that these edge colorings are chosen in such a way that vertices $v_1^{(m-3)}, v_2^{(m-3)}, \ldots, v_n^{(m-3)}$ of H_1 have the same palettes as vertices $v_1^{(m-2)}, v_2^{(m-2)}, \ldots, v_n^{(m-2)}$ of H_2. Thus, by coloring all edges of $G(m,n)$ with one endpoint in H_1 and one endpoint in H_2 with color 4, we obtain a proper edge coloring of $G(m,n)$ with 5 palettes; thus $\check{s}(G(m,n)) \leq 5$.

We now turn to the lower bound. Since $m, n \geq 3$ and mn is odd, the graph $G(m,n)$ contains vertices of degree 2, 3 and 4; hence $\check{s}(G(m,n)) \geq 3$.

Next, we prove that $\check{s}(G(m,n)) \geq 4$. Let γ be a proper edge coloring of $G(m,n)$ with three distinct palettes. This implies that for each vertex $v \in V(G(m,n))$ with degree four, we have $S(v,\gamma) = \{a,b,c,d\}$. Let M_a, M_b, M_c and M_d be the color classes of γ corresponding to the colors a, b, c and d. Now, there are precisely $(m-2)(n-2)$ vertices of degree four in $G(m,n)$, and since $(m-2)(n-2)$ is an odd number, the edges with colors a, b, c and d cannot only be incident with vertices of degree four. This implies that for each color $x \in \{a,b,c,d\}$, there exists an edge e_x with color x joining vertices with

degrees 4 and 3. Thus, all colors a, b, c and d appear in palettes of vertices of degree 3, which implies that $\check{s}(G(m, n)) \geq 4$.

Finally, we show that if mn is odd, then $\check{s}(G(m, n)) = 5$. Suppose, to the contrary, that $\check{s}(G(m, n)) = 4$, and let ϕ be a proper edge coloring of $G(m, n)$ with four distinct palettes. Throughout the rest of the proof, denote by M_i the color class i under ϕ, i.e., the set of edges with color i under ϕ.

Let us first prove that the number of 3-element palettes under ϕ is at least two. Since there are at most two palettes of size 4, the set A of colors appearing in palettes of size 4 satisfies $4 \leq |A| \leq 8$. Moreover, A clearly has a partition $\{A_1, A_2, A_3, A_4\}$ such that $1 \leq |A_i| \leq 2$, and each palette of size 4 contains exactly one color from A_i, $i = 1, 2, 3, 4$. Furthermore, since mn is odd, there is an odd number of vertices of degree 4 in $G(m, n)$. Therefore, for every $i \in \{1, 2, 3, 4\}$, there is a color $a_i \in A_i$ and an edge colored a_i that joins vertices of degree 3 and 4. We thus conclude that each of the colors a_1, a_2, a_3, a_4 appears in a palette of size 3, and it follows that the number of palettes of size 3 is at least two.

Now, since there are at least two palettes of size 3, there must be exactly one palette of size 4 and one palette of size 2. Without loss of generality we assume that for each vertex $v \in V(G(m, n))$ with degree four, we have $S(v, \phi) = \{1, 2, 3, 4\}$, and for each color $x \in \{1, 2, 3, 4\}$, there exists an edge e_x with color x joining vertices with degrees 4 and 3. Thus, all colors $1, 2, 3$ and 4 appear in palettes of vertices of degree three.

Since two distinct palettes occur at vertices of degree three, at most six colors $1, \ldots, 6$ are used in the coloring ϕ. Suppose first that disjoint palettes occurs at vertices of degree three. If three colors from $\{1, 2, 3, 4\}$ appear in one such palette, i.e., if for each vertex $v \in V(G(m, n))$ with degree three, either, say, $S(v, \phi) = \{1, 2, 3\}$ or $S(v, \phi) = \{4, 5, 6\}$, then since both m and n are odd, vertices of degree two only have one possible palette under ϕ, and neither of colors 5 and 6 appear at vertices of degree 4, this implies that all vertices with degree three have the same palette, which is a contradiction. If instead two colors from $\{1, 2, 3, 4\}$ appear in both palettes, e.g. if for each vertex $v \in V(G(m, n))$ with degree three, either $S(v, \phi) = \{1, 2, 5\}$ or $S(v, \phi) = \{3, 4, 6\}$, then, again, this implies that all vertices with degree three have the same palette, which is a contradiction.

Suppose now instead that the two distinct palettes at vertices of degree three contain exactly one common color. We first consider the case when this common color is in $\{1, 2, 3, 4\}$. Assume, without loss of generality, that this color is 3, and consider the color class M_3. The edges in M_3 either cover all vertices of the graph or all vertices except those with degree two; but this is impossible, since mn and $mn - 4$ are both odd numbers.

Suppose now instead that the common color of the different palettes of vertices of degree three is not in $\{1, 2, 3, 4\}$. We assume that this common color is 5, and since all colors in $\{1, 2, 3, 4\}$ appear on edges incident with vertices of degree 3, we may assume, that for each vertex $v \in V(G(m, n))$ with degree three, either $S(v, \phi) = \{1, 2, 5\}$ or $S(v, \phi) = \{3, 4, 5\}$. This means that the color class M_5 covers all vertices of the cycle C of $G(m, n)$ containing all vertices of degree 3 and 2 in $G(m, n)$, because any path in $G(m, n)$ between vertices of degree 2, whose intermediate vertices all have degree 3, has even length. Now, since all vertices with degree two have the same palette, we may assume that for each vertex $v \in V(G(m, n))$ with degree two, $S(v, \phi) = \{a, b\}$. Since color 5 appears at each vertex of C, we obtain that $a = 5$. Without loss of generality we may assume that $b = 1$. Let us now consider the color class M_3. Clearly,

$$|M_3| = \frac{1}{2} \left((m - 2)(n - 2) + l \right),$$

where l is the number of vertices of C with the palette $\{3, 4, 5\}$. Since $(m - 2)(n - 2)$ is odd, we get that

l is odd too. Let r_3 and r_4 be the number of edges of C with colors 3 and 4, respectively. Now we can count the number of vertices of C with the palette $\{3, 4, 5\}$ using r_3 and r_4. Since, all vertices of degree two have the palette $\{1, 5\}$, and color 5 does not appear on any edge incident with a vertex of degree four, $l = 2r_3 + 2r_4$; but this contradicts the fact that l is odd.

Finally, let us consider the case when the two distinct palettes at vertices of degree three contain two common colors. Suppose without loss of generality that for each vertex $v \in V(G(m, n))$ with degree three, either $S(v, \phi) = \{1, 2, 3\}$ or $S(v, \phi) = \{2, 3, 4\}$. Let us consider vertices with degree two in $G(m, n)$; all such vertices v have the same palette $S(v, \phi) = \{a, b\}$. If $\{a, b\} \cap \{2, 3\} \neq \emptyset$, then the color class M_a (or M_b) is a perfect matching of $G(m, n)$, which is a contradiction. So, we may assume that $\{a, b\} = \{1, 4\}$. Let us consider the color class M_2. Clearly,

$$|M_2| = \frac{1}{2}\left((m - 2)(n - 2) + k + l\right),$$

where k is the number of vertices of C with the palette $\{1, 2, 3\}$, and l is the number of vertices of C with the palette $\{2, 3, 4\}$. Since $(m - 2)(n - 2)$ is odd, we get that $k + l$ is odd too. On the other hand, it is easy to see that $k + l = 2(m - 2 + n - 2)$, which is a contradiction. $\square$

4   Biregular graphs

In this section we consider (a, b)-biregular graphs. Our primary aim here is to show that Conjecture 1.4 holds for several families of biregular graphs.

König's edge coloring theorem implies that $\check{s}(G) \leq 1 + \binom{b}{a}$ for every (a, b)-biregular graph G where $a \leq b$. In particular, this implies that if G is $(b - 1, b)$-biregular or $(1, b)$-biregular, then $\check{s}(G) \leq 1 + b$, which means that Conjecture 1.4 holds for all such graphs. In fact, the latter family of graphs show that the upper bound in Conjecture 1.4 is in general sharp.

The next lemma will be used frequently.

Lemma 4.1. *If G is an (a, b)-biregular graph with $a < b$, then $\check{s}(G) \geq 1 + \lceil \frac{b}{a} \rceil$.*

Proof: Let G be an (a, b)-biregular $(a < b)$ graph with bipartition (X, Y) so that $a|X| = b|Y|$. Consider an arbitrary proper edge coloring of G. Since any palette of size a appears on at most $|Y|$ vertice, the number of palettes of size a is bounded from below by $\left\lceil \frac{|X|}{|Y|} \right\rceil = \lceil \frac{b}{a} \rceil$. This implies that $\check{s}(G) \geq 1 + \lceil \frac{b}{a} \rceil$. $\square$

The smallest (a, b)-biregular graph is the complete bipartite graph $K_{a,b}$; the lower bound in the preceding lemma was obtained in Horňák and Hudák (2018) for the case of complete bipartite graphs. Furthermore, for complete bipartite graphs, we have the following; the upper bound shows that Conjecture 1.4 holds for complete bipartite graphs. If a and b are positive integers ($a \leq b$), then we denote the interval of integers from a to b by $[a, b] = \{a, a + 1, \ldots, b\}$.

Theorem 4.2. *If $a < b$ ($a, b \in \mathbb{N}$), then*

$$1 + \left\lceil \frac{b}{a} \right\rceil \leq \check{s}\left(K_{a,b}\right) \leq 1 + \frac{b}{\gcd(a,b)}.$$

Proof: The lower bound follows from Lemma 4.1.

We set $d = \gcd(a, b)$ and now show that $\check{s}(K_{a,b}) \leq 1 + \frac{b}{d}$. Let

$$V(K_{a,b}) = \{u_1, \ldots, u_a, v_1, \ldots, v_b\} \text{ and } E(K_{a,b}) = \{u_i v_j : 1 \leq i \leq a, 1 \leq j \leq b\}.$$

Also, let G be a subgraph of $K_{a,b}$ induced by vertices $\{u_1, \ldots, u_d, v_1, \ldots, v_d\}$; so G is isomorphic to the graph $K_{d,d}$.

We define an edge coloring α of G as follows: for $1 \leq i \leq d$ and $1 \leq j \leq d$, let

$$\alpha(u_i v_j) = \begin{cases} i + j - 1 \pmod{d}, & \text{if } i + j \neq d + 1, \\ d, & \text{if } i + j = d + 1. \end{cases}$$

The coloring α is a proper edge coloring of G and $S_G(u_i, \alpha) = S_G(v_i, \alpha) = [1, d]$ for $1 \leq i \leq d$.

Next we construct a proper b-edge coloring of $K_{a,b}$. Before we give the explicit definition of the coloring, we need two auxiliary functions f and h. For $i \in \mathbb{N}$, we define $f(i) = 1 + (i - 1) \pmod{d}$ and for $i, j \in \mathbb{N}$, we define

$$h(i, j) = \left(\left\lfloor \frac{i-1}{d} \right\rfloor + \left\lfloor \frac{j-1}{d} \right\rfloor \right) \pmod{\frac{b}{d}}.$$

Now we define an edge coloring β of $K_{a,b}$ by, for $1 \leq i \leq a$ and $1 \leq j \leq b$, setting

$$\beta(u_i v_j) = \alpha\left(u_{f(i)} v_{f(j)}\right) + dh(i, j).$$

Let us verify that β is a proper b-edge coloring of $K_{a,b}$ with exactly $1 + \frac{b}{a}$ palettes. By the definition of β and taking into account that $S_G(u_i, \alpha) = S_G(v_i, \alpha) = [1, d]$ for $1 \leq i \leq d$, we have

$$S(u_i, \beta) = [1, b] \text{ for } 1 \leq i \leq a,$$

and

$$S\left(v_{(j-1)d+1}, \beta\right) = S\left(v_{(j-1)d+2}, \beta\right) = \cdots = S\left(v_{jd}, \beta\right) = \bigcup_{i=0}^{\frac{a}{d}-1} \{a_i d + 1, \ldots, a_i d + d\},$$

for $1 \leq j \leq \frac{b}{d}$, and where $a_i = \left(i + \left\lceil \frac{j-1}{d} \right\rceil\right) \pmod{\frac{b}{d}}$. This implies that β is a proper b-edge coloring of $K_{a,b}$ with $1 + \frac{b}{d}$ distinct palettes. □

From the preceding theorem, we deduce the following, which was first obtained in Horňák and Hudák (2018).

Corollary 4.3. *If* $\gcd(a, b) = a$ *(*$a < b$*), then* $\check{s}(K_{a,b}) = 1 + \frac{b}{a}$.

In Horňák and Hudák (2018) the palette index of the complete bipartite graphs $K_{2,2r}$ was determined; the following generalization follows from Theorem 2.2. Here, and in the following, we assume r to be a positive integer.

Corollary 4.4. *If* G *is a* $(2, 2r)$*-biregular graph, then* $\check{s}(G) = r + 1$.

Proof: The upper bound follows from Theorem 2.2. The lower bound follows from the fact that assuming that at most $r - 1$ palettes occur at vertices of degree 2 implies that G has a proper edge coloring with $2r - 2$ colors. □

Similarly, we have the following:

Corollary 4.5. *If G is a $(2r-2, 2r)$-biregular graph, then $\check{s}(G) \le r + 1$.*

This upper bound is sharp e.g. for complete bipartite graphs of small order, since $\check{s}(K_{2,4}) = 3$ and $\check{s}(K_{4,6}) = 4$.

We remark that the two previous corollaries do not only hold for biregular graphs, but for any bipartite graph where the vertex degrees lie in the set $\{2, 2r\}$ and $\{2r-2, 2r\}$, respectively.

Our next result on biregular graphs is an easy consequence of a result on interval colorings. In Hanson et al. (1998); Kamalian and Mirumian (1997), it was proved that every $(2, 2r+1)$-biregular graph has an interval coloring using $2r + 2$ colors.

Proposition 4.6. *If G is a $(2, 2r+1)$-biregular graph, then $r + 2 \le \check{s}(G) \le 2r + 2$.*

Proof: Let f be an interval coloring of G using exactly $2r+2$ colors. By taking all colors modulo $2r+1$, we obtain a cyclic interval $(2r+1)$-coloring of G; such a coloring yields at most $2r+2$ distinct palettes in G.

The lower bound can be proved as in the proof of Corollary 4.4. $\square$

We note that the upper bound in the preceding proposition is sharp, since $\check{s}(K_{2,3}) = 4$; in fact it is not hard to see that the upper bound in Proposition 4.6 is sharp for all $(2, 3)$-biregular graphs.

Next, we shall establish that Conjecture 1.4 holds for some families of biregular graphs with small vertex degrees. In fact, we shall deduce these results from more general propositions.

Corollary 2.3 implies that Conjecture 1.1 holds for all $(3, 3r)$-biregular and $(3r - 3, 3r)$-biregular graphs; the upper bound from Corollary 2.3 can be slightly improved as follows.

Proposition 4.7. *Let G be a bipartite graph.*

  (i) If G is $(3, 3r)$-biregular $(r \ge 2)$, then $r + 1 \le \check{s}(G) \le r^2 + 1$.

  (ii) If G is $(3r - 3, 3r)$-biregular graph $(r \ge 2)$, then $\check{s}(G) \le r^2 + 1$.

Proof: Let us first note that the the lower bound in (i) follows from Lemma 4.1.

We shall prove the upper bound in (i); the proof of the upper bound in (ii) is similar. Consequently, let G be a $(3, 3r)$-biregular bipartite graph with bipartition (X, Y), and let us show that $\check{s}(G) \le r^2 + 1$.

Define a new graph H from G by replacing each vertex $y \in Y$ by r vertices $y^{(1)}, y^{(2)}, \dots, y^{(r)}$ of degree 3, where each $y^{(i)}$ is adjacent to three neighbors of y in G, and $y^{(i)}$ and $y^{(j)}$ have disjoint neighborhoods if $i \ne j$. Clearly, H is a cubic bipartite graph, and so by Hall's matching theorem, H contains a perfect matching M.

In the graph G, M induces a subgraph F in which each vertex $y \in Y$ has degree r and each vertex $x \in X$ has degree 1. Let us consider the graph $G' = G - E(F)$. Since G' is a $(2, 2r)$-biregular graph, by proceeding as in the proof of Theorem 2.2 it can be shown that G' has a proper $2r$-edge coloring α such that for each $y \in Y$, $S(y, \alpha) = [1, 2r]$, and for each $x \in X$, $S(x, \alpha) = \{2i - 1, 2i\}$ for some i $(1 \le i \le r)$. Let us now define an edge coloring β of F as follows: for each vertex $y \in Y$, we color the edges of F incident with y with colors $2r + 1, 2r + 2, \dots, 3r$.

Finally, we define an edge coloring γ of G as follows:

1) for every $e \in E(G')$, let $\gamma(e) = \alpha(e)$;

2) for every $e \in E(F)$, let $\gamma(e) = \beta(e)$.

Clearly, γ is a proper edge coloring of G with colors $1, 2, \ldots, 3r$ such that for each $y \in Y$, $S(y, \gamma) = [1, 3r]$, and for each $x \in X$, $S(x, \gamma) = \{2i - 1, 2i, 2r + j\}$ for some $i, j \in [1, r]$. This implies that $\check{s}(G) \leq r^2 + 1$. $\qquad\square$

We remark that the lower bound in part (i) of Proposition 4.7 is sharp by Corollary 4.3. Hence, this also holds for parts (i) and (ii) of the following consequence of Proposition 4.7.

Corollary 4.8. *Let G be a bipartite graph.*

(i) *If G is $(3, 6)$-biregular, then $3 \leq \check{s}(G) \leq 5$.*

(ii) *If G is $(3, 9)$-biregular, then $4 \leq \check{s}(G) \leq 10$.*

(iii) *If G is $(6, 9)$-biregular, then $\check{s}(G) \leq 10$.*

The preceding result shows that Conjecture 1.4 holds for some biregular graphs with vertex degrees divisible by three. Let us now turn to biregular graphs with vertex degrees divisible by four. In Section 3, we deduced that Conjecture 1.4 holds for $(4, 6)$-biregular graphs. If G is a $(4, 4r)$-biregular or $(4r-4, 4r)$-biregular graph, then Theorem 2.2 implies that $\check{s}(G) \leq 1 + r(2r-1)$. Our next proposition yields a slightly better bound.

Proposition 4.9. *Let G be a bipartite graph.*

(i) *If G is $(4, 4r)$-biregular ($r \geq 2$), then $r + 1 \leq \check{s}(G) \leq r^2 + 1$.*

(ii) *If G is $(4r - 4, 4r)$-biregular ($r \geq 2$), then $\check{s}(G) \leq r^2 + 1$.*

Proof: As in the proof of the preceding proposition, the lower bound in (i) follows from Lemma 4.1. Let us prove the upper bound in part (i); part (ii) can be proved similarly. Consequently, let G be a $(4, 4r)$-biregular bipartite graph with bipartition (X, Y) and let us show that $\check{s}(G) \leq r^2 + 1$.

Without loss of generality, we may assume that G is connected (otherwise, we color every component of G as below). Since G is bipartite and all vertex degrees in G are even, G has a closed Eulerian trail C with an even number of edges. We color the edges of G with colors "Red" and "Blue" by traversing the edges of G along the trail C; we color an odd-indexed edge in C with color Red, and an even-indexed edge in C with color Blue. Let E_R and E_B be the sets of all Red and Blue edges in G, respectively; then $E(G) = E_R \cup E_B$ and $E_R \cap E_B = \emptyset$. Define the subgraphs G_R and G_B of G as follows:

$$V(G_R) = V(G_B) = V(G) \text{ and } E(G_R) = E_R, E(G_B) = E_B.$$

Since G is $(4, 4r)$-biregular, each of the subgraphs G_R and G_B of G is a $(2, 2r)$-biregular bipartite graph with bipartition (X, Y). Hence, by proceeding as in the proof of the preceding proposition, we deduce that G_R has a proper $2r$-edge coloring α such that for each $y \in Y$ $S(y, \alpha) = [1, 2r]$, and for each $x \in X$ $S(x, \alpha) = \{2i - 1, 2i\}$ for some $i \in [1, r]$. Similarly, G_B has a proper $2r$-edge coloring β such that for each $y \in Y$ $S(y, \beta) = [1, 2r]$, and for each $x \in X$ $S(x, \beta) = \{2j - 1, 2j\}$ for some $j \in [1, r]$. We define a new edge coloring β' of G_B from β as follows: for every $e \in E(G_B)$, let $\beta'(e) = \beta(e) + 2r$; then β' is a proper edge coloring of G_B with colors $2r + 1, 2r + 2, \ldots, 4r$. Moreover, for each $y \in Y$,

$S(y, \beta') = [2r + 1, 4r]$, and for each $x \in X$, $S(x, \beta') = \{2(r+j) - 1, 2(r+j)\}$ for some $j \in [1, r]$.

Finally, we define an edge coloring γ of G as follows:

1) for every $e \in E(G_R)$, let $\gamma(e) = \alpha(e)$;

2) for every $e \in E(G_B)$, let $\gamma(e) = \beta'(e)$.

Clearly, γ is a proper edge coloring of G with colors $1, 2, \ldots, 4r$ such that for each $y \in Y$, $S(y, \gamma) = [1, 4r]$, and for each $x \in X$, $S(x, \gamma) = \{2i - 1, 2i, 2(r+j) - 1, 2(r+j)\}$ for some i and j ($i, j \in [1, r]$). This implies that $\check{s}(G) \leq r^2 + 1$. $\qquad\square$

Once again, we remark that the lower bound in part (i) of Proposition 4.9 is sharp by Corollary 4.3, so this also holds for parts (i)-(iii) of the following consequence of Proposition 4.9.

Corollary 4.10. *Let G be a bipartite graph.*

(i) *If G is $(4, 8)$-biregular, then $3 \leq \check{s}(G) \leq 5$.*

(ii) *If G is $(4, 12)$-biregular, then $4 \leq \check{s}(G) \leq 10$.*

(iii) *If G is $(4, 16)$-biregular, then $5 \leq \check{s}(G) \leq 17$.*

(iv) *If G is $(8, 12)$-biregular, then $\check{s}(G) \leq 10$.*

(v) *If G is $(12, 16)$-biregular, then $\check{s}(G) \leq 17$.*

Our next result establishes an upper bound on the palette index of $(5, 5r)$-biregular graphs.

Proposition 4.11. *If G is a $(5, 5r)$-biregular ($r \geq 2$) bipartite graph, then $r + 1 \leq \check{s}(G) \leq r^3 + 1$.*

Proof: The lower bound follows from Lemma 4.1, so let us prove the upper bound.

Let G be a $(5, 5r)$-biregular bipartite graph with bipartition (X, Y), and let us show that $\check{s}(G) \leq r^3 + 1$. As in the proof of Proposition 4.7, we define a new graph H from G by replacing each vertex $y \in Y$ by r vertices $y^{(1)}, y^{(2)}, \ldots, y^{(r)}$ of degree 5, where each $y^{(i)}$ is adjacent to five neighbors of y in G, and $y^{(i)}$ and $y^{(j)}$ have disjoint neighborhoods if $i \neq j$. Clearly, H is a 5-regular bipartite graph, and by Hall's matching theorem, H contains a perfect matching M.

In the graph G, M induces a subgraph F in which each vertex $y \in Y$ has degree r and each vertex $x \in X$ has degree 1. Let us consider the graph $G' = G - E(F)$. Since G' is $(4, 4r)$-biregular, by proceeding as in the proof of Proposition 4.9, we can construct a proper $4r$-edge coloring α of G' such that for each $y \in Y$, $S(y, \alpha) = [1, 4r]$ and for each $x \in X$, $S(x, \alpha) = \{2i - 1, 2i, 2(r+j) - 1, 2(r+j)\}$ for some $i, j \in [1, r]$. Let us now define an edge-coloring β of F as follows: for each vertex $y \in Y$, we color the edges of F incident with y with colors $4r + 1, 4r + 2, \ldots, 5r$.

Finally, we define an edge coloring γ of G as follows:

1) for every $e \in E(G')$, let $\gamma(e) = \alpha(e)$;

2) for every $e \in E(F)$, let $\gamma(e) = \beta(e)$.

Clearly, γ is a proper edge coloring of G with colors $1, 2, \ldots, 5r$ such that for each $y \in Y$, $S(y, \gamma) = [1, 5r]$, and for each $x \in X$,

$$S(x, \gamma) = \{2i - 1, 2i, 2(r + j) - 1, 2(r + j), 4r + k\}$$

for some $i, j, k \in [1, r]$. This implies that $\check{s}(G) \leq r^3 + 1$. $\qquad\square$

We remark that it is possible to prove a similar upper bound for $(5r - 5, 5r)$-biregular graphs. From the preceding proposition we deduce the following.

Corollary 4.12. *If G is a $(5, 10)$-biregular graph, then $3 \leq \check{s}(G) \leq 9$.*

Again, by Corollary 4.3, the lower bound in the preceding corollary (and in Proposition 4.11) is sharp. This also applies to the next proposition which concerns $(r, 2r)$-biregular graphs.

Proposition 4.13. *If G is an $(r, 2r)$-biregular ($r \geq 2$) bipartite graph, then $3 \leq \check{s}(G) \leq 2^{\lceil \frac{r}{2} \rceil} + 1$.*

Proof: As in the proofs of the preceding propositions, the lower bound follows from Lemma 4.1. Let G be an $(r, 2r)$-biregular bipartite graph with bipartition (X, Y), and let us show that $\check{s}(G) \leq 2^{\lceil \frac{r}{2} \rceil} + 1$. We consider two cases.

Case 1. *r is even:* Let $r = 2k$ ($k \in \mathbb{N}$). Since G is $(2k, 4k)$-biregular, it has a decomposition into k $(2, 4)$-biregular graphs $G_1, \ldots, G_k$; this follows by splitting vertices of degree $2k$ into two vertices of degree k, vertices of degree $4k$ into four vertices of degree k, and taking perfect matchings in the resulting k-regular bipartite graph. As in the proof of Theorem 2.2 it can be shown that each graph G_i has a proper 4-edge coloring α_i such that for each $y \in Y$, $S(y, \alpha_i) = [4i - 3, 4i]$, and for each $x \in X$, either $S(x, \alpha_i) = \{4i - 3, 4i - 2\}$ or $S(x, \alpha_i) = \{4i - 1, 4i\}$ ($1 \leq i \leq k$). Let us now define an edge-coloring β of G as follows: for $1 \leq i \leq k$ and for every $e \in E(G_i)$, let $\beta(e) = \alpha_i(e)$.

Clearly, β is a proper edge coloring of G with colors $1, 2, \ldots, 4k$ such that for each $y \in Y$, $S(y, \beta) = [1, 4k]$, and for each $x \in X$, $S(x, \beta)$ is one of 2^k possible palettes. This implies that $\check{s}(G) \leq 2^k + 1$.

Case 2. *r is odd:* Let $r = 2k + 1$ ($k \in \mathbb{N}$). Since G is $(2k + 1, 4k + 2)$-biregular, it has a $(1, 2)$-biregular subgraph F; this follows by splitting vertices of degree $4k + 2$ into two vertices of degree $2k + 1$, and taking a perfect matching in the resulting $(2k + 1)$-regular bipartite graph. Let us consider the graph $G' = G - E(F)$. Since G' is a $(2k, 4k)$-biregular graph, it follows from the proof in Case 1 that G' has a proper $4k$-edge coloring α such that for each $y \in Y$, $S(y, \alpha) = [1, 4k]$ and for each $x \in X$, $S(x, \alpha)$ is one of 2^k possible palettes. Let us now define an edge coloring β of F as follows: for each vertex $y \in Y$, we color the edges of F incident with y with colors $4k + 1$ and $4k + 2$.

Finally, we define an edge coloring γ of G as follows:

1) for every $e \in E(G')$, let $\gamma(e) = \alpha(e)$;

2) for every $e \in E(F)$, let $\gamma(e) = \beta(e)$.

Clearly, γ is a proper edge coloring of G with colors $1, 2, \ldots, 4k + 2$ such that for each $y \in Y$, $S(y, \gamma) = [1, 4k + 2]$, and for each $x \in X$,

$$\text{either } S(x, \gamma) = S(x, \alpha) \cup \{4k + 1\} \text{ or } S(x, \gamma) = S(x, \alpha) \cup \{4k + 2\};$$

thus there are at most 2^{k+1} possible choices for the palette $S(x, \gamma)$. This implies that $\check{s}(G) \leq 2^{k+1} + 1$. □

Corollary 4.14. *Let G be a bipartite graph.*

  (i) If G is $(6, 12)$-biregular, then $3 \leq \check{s}(G) \leq 9$.

  (ii) If G is a $(8, 16)$-biregular, then $3 \leq \check{s}(G) \leq 17$.

Our final result for biregular graphs shows that a slightly weaker form of Conjecture 1.4 holds for $(3, 5)$-biregular graphs.

Proposition 4.15. *If G is a $(3, 5)$-biregular bipartite graph, then $5 \leq \check{s}(G) \leq 7$.*

Proof: Let G be a $(3, 5)$-biregular bipartite graph with parts X and Y. Since G is $(3, 5)$-biregular, we have that $|X| = 5k$ and $|Y| = 3k$ for some positive integer k.

By Lemma 4.1, we obtain that $\check{s}(G) \geq 3$. Moreover, if $\check{s}(G) = 3$, then in a proper edge coloring attaining this value, vertices in X in G must have two distinct palettes. If φ is such a coloring, then the vertices of degree five all have the same palette under φ. This implies that φ is a proper 5-edge coloring, and so there is some color appearing at all vertices of degree three in G. However, this contradicts that φ is a proper 5-edge coloring. Hence, $\check{s}(G) \geq 4$.

Now assume that $\check{s}(G) = 4$. Using similar counting arguments as before, it follows that vertices in X must have at least two distinct palettes. Vertices in Y must also have at least two distinct palettes, because suppose there is only one palette $\{1, 2, 3, 4, 5\}$ of size 5 and three palettes of size 3; then, since there are three palettes of size 3 and no color can appear in all these three palettes, there is exactly one color, say 1, that appears in exactly one palette of size 3, say $\{1, 2, 3\}$; the remaining palettes of size three are then $\{2, 4, 5\}$ and $\{3, 4, 5\}$. Now, since $|X| = 5k$ and $|Y| = 3k$, we have that the number of vertices in X with the palette $\{1, 2, 3\}$ is $3k$. But then colors 4 and 5 appear at all $3k$ vertices of Y but only at $2k$ vertices in X, a contradiction. Hence, the vertices in Y have at least two distinct palettes, and so, there are exactly two palettes of vertices in X and two palettes of vertices in Y.

Now, if the two distinct palettes of vertices in X are not disjoint, then at most 5 colors are used in a proper edge coloring of G with a minimum number of palettes, which contradicts that two distinct palettes appear at vertices in Y. Thus there is a proper edge coloring φ with 4 distinct palettes, and where the two palettes of vertices in X are disjoint, say $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Now, since exactly 6 colors are used in φ, and since only two distinct palettes appear at vertices of Y, some color appears at all vertices of Y, say color 1. This implies that the number of vertices in X with the palette $\{1, 2, 3\}$ is $|Y| = 3k$. However, some color in $\{4, 5, 6\}$ must also appear at all vertices in Y, which implies that the number of vertices in X with the palette $\{4, 5, 6\}$ is $3k$, a contradiction because $|X| = 5k$. Hence $\check{s}(G) \geq 5$.

Let us now show that $\check{s}(G) \leq 7$. By Hall's matching theorem, G has a matching M that saturates all the vertices of degree 5. The graph $G' = G - M$ is a bipartite graph with $\Delta(G') = 4$. As in the proof of Corollary 3.2, G' has a proper edge coloring α with colors $1, 2, 3, 4$ such that the vertices of degree 2 in G' have 2 possible palettes and the vertices of degree 3 in G' have 4 possible palettes.

We now define a proper edge coloring β of G as follows:

  1) for every $e \in E(G')$, let $\beta(e) = \alpha(e)$;

2) for every $e \in M$, let $\beta(e) = 5$.

In the coloring β the vertices of degree 5 in G all have the same palette, the vertices of degree 3 in G that are covered by M have again 2 possible palettes and the rest of the vertices of degree 3 in G have 4 possible palettes. This implies that $\check{s}(G) \leq 7$. □

We remark that the lower bound in the preceding proposition is sharp since $\check{s}(K_{3,5}) = 5$, as proved by Horňák and Hudák (2018).

5   Graphs with large palette index

For every graph G we clearly have $\check{s}(G) \leq |V(G)|$. In this section we shall characterize the graphs G with largest possible palette index in the sense that G satisfies $\check{s}(G) = |V(G)|$. Throughout this section we only consider graphs with no multiple edges.

Denote by $\hat{K}_3^j$ the graph obtained from K_3 and $K_{1,j}$ by identifying the central vertex of $K_{1,j}$ with a vertex of K_3. Moreover, we denote by $\hat{K}_3^{j+}$ the graph obtained from K_3 and $K_{1,j}$ by adding an edge between the central vertex of $K_{1,j}$ and some vertex of K_3.

Theorem 5.1. *If G is a graph with no isolated vertices, then $\check{s}(G) = |V(G)|$ if and only if G is isomorphic to K_3, $K_{1,j}$ with $j \geq 2$, $\hat{K}_3^j$ with $j \geq 1$, or one of $\hat{K}_3^{j+}$ and $K_3 \cup K_{1,j}$ with $j \geq 3$.*

Proof: Sufficiency is straightforward, so let us prove necessity. Let G be a graph with $\check{s}(G) = |V(G)|$.

By the pigeonhole principle, there are at least two vertices in G that have equal degrees; let us first prove that any such pair of vertices have vertex degrees 1 or 2. Suppose that G contains two vertices u and v of equal degree greater than 2. It is straightforward to verify that there is a partial edge coloring of G such that an edge of G is colored if and only if it is incident with u or v, and such that u and v have the same palettes. However, any proper extension of such a partial edge coloring of G (not necessarily using a minimum number of colors) produces at most $|V(G)| - 1$ distinct palettes. Thus if two vertices in G have equal degree, then they both have degree 1 or 2.

Let us first assume that there are two vertices u and v of degree 2 in G. Unless u and v are contained in a cycle of length 3, there is a similar partial edge coloring as in the preceding paragraph. Moreover, if three vertices of G have degree 2, and these vertices are not contained in a component isomorphic to K_3, then $\check{s}(G) < |V(G)|$. Hence, either G contains a component isomorphic to K_3 or G contains two vertices of degree 2 that lie on a cycle of length three, and no other vertex of G has degree 2. Let F be the component of G containing u and v. We shall prove that if $F \not\cong K_3$, then $F \cong \hat{K}_3^j$ or $F \cong \hat{K}_3^{j+}$.

Suppose first that F does not contain any vertices of degree 1. Let w be a vertex of maximum degree in F and assume that $\Delta(F) \geq 3$. Now, since F has no more than one vertex of degree d for each $d \in \{3, \ldots, \Delta(F) - 1\}$, the degree of w is at most $2 + |\{3, \ldots, \Delta(F) - 1\}| = \Delta(F) - 1$, a contradiction. We conclude that if $F \not\cong K_3$, then F must contain some vertex of degree 1.

Assume, consequently, that F contains some vertex of degree 1. If two vertices of degree 1 in F have distinct neighbors, then there is a proper edge coloring of F where these two vertices have the same palette, contradicting that $\check{s}(G) = |V(G)|$. Hence, all vertices of degree 1 in F are adjacent to a fixed vertex y of F. Since F contains vertices of degree 1, and u and v both have degree 2, $\Delta(F) \geq 3$. Moreover, since all vertices of degree greater than 3 in F have distinct degrees, there is a unique vertex w of maximum degree $\Delta(F)$ in F. Furthermore, it follows from the same argument that w is adjacent to

some vertex of degree 1 in F. Thus, all vertices of degree 1 in F are adjacent to w. If all vertex degrees in F are in the set $\{1, 2, \Delta(F)\}$, then $G \cong \hat{K}_3^j$.

Suppose that there is some vertex x of degree k, $3 < k < \Delta(F)$ in F. Without loss of generality, we assume that x has second largest degree in F. Since all vertices of degree 1 in F are adjacent to w, x must be adjacent to u, v, w and exactly $k - 3 \geq 1$ vertices of distinct degrees in the set $\{3, \ldots, k - 1\}$. Therefore, x is adjacent to a vertex y of degree $k - 1$. However, y can be adjacent only to vertices of degrees in the set $\{3, \ldots, k - 2\} \cup \{k, \Delta(F)\}$, so that its degree is at most $k - 2$, a contradiction. We conclude that there is no vertex of degree greater than 3 in F except for w. Moreover, if all vertex degrees of F are in the set $\{1, 2, 3, \Delta(F)\}$, where $\Delta(F) > 3$, then $F \cong \hat{K}_3^{j+}$, because all vertices of degree 1 in F are adjacent to w.

We conclude that u and v must lie in a component F of G that is isomorphic to K_3, $\hat{K}_3^j$ or $\hat{K}_3^{j+}$.

Suppose that G has more than one component, and let H be a component of $G - V(F)$. Now, since the palette index of G is $|V(G)|$, H does not contain any vertex of degree 2. Thus any two vertices of equal degree in H have degree 1. Moreover, by the pigeonhole principle at least two vertices of H have equal degree, and it is easy to see that if two vertices x and y of degree 1 in H are not adjacent to the same vertex, then $\check{s}(G) < |V(G)|$. We conclude that there are at least two vertices of degree one in H that are adjacent to the same vertex in H (unless H consists of a single edge). Since all other vertex degrees in H are different, all vertices of degree 1 in H are adjacent to the vertex of maximum degree in H. Moreover, it follows, as in the preceding paragraph, that the only vertex degrees in H are $\Delta(H)$ and 1; and so, H is isomorphic to a star.

Now, if there are vertices of degree 1 in different components of G, then clearly $\check{s}(G) < |V(G)|$. Thus, if $G - V(F)$ is non-empty, then $F \cong K_3$ and $G - V(F)$ is a star.

We conclude that G is isomorphic to K_3, $\hat{K}_3^j$, $\hat{K}_3^{j+}$ or to the disjoint union of K_3 and a star.

The case when there are no two vertices of degree 2 in G, can be dealt with similarly by first deducing that two vertices in G have degree 1, and, as before, all such vertices of G are adjacent to the vertex of maximum degree in G. By proceeding as above it is now easy to prove that the only vertex degrees in G are 1 and $\Delta(G)$, and that G must be connected. Hence, G is isomorphic to a star. $\qquad\square$

Although the preceding theorem only holds for graphs with no isolated vertices, we note that if G is a graph with no isolated vertices and $\check{s}(G) = |V(G)|$, then $\check{s}(G \cup K_1) = |V(G \cup K_1)|$.

Consider a non-regular graph G which is the union of two regular edge-disjoint Class 1 graphs H_1 and H_2 satisfying that $V(H_1) \subseteq V(H_2)$. Since both H_1 and H_2 are Class 1 and G is non-regular, we have that $\check{s}(G) = 2$. It is not difficult to see that the converse holds as well. Indeed, assume that G is a graph with $\check{s}(G) = 2$, and let ϕ be a proper edge coloring of G attaining this minimum.

It follows from a result of Horňák et al. (2014) that G is not regular, and thus exactly two different vertex degrees appear in G; d_1 and d_2, say, where $d_1 > d_2$. For $i = 1, 2$, let $C_i(\phi)$ be the set of all colors appearing on edges incident with vertices of degree d_i under ϕ. If $C_2(\phi) \nsubseteq C_1(\phi)$, then there is some color $j \in C_2(\phi)$ which does not appear at any vertex of degree d_1 in G, and since $|C_1(\phi)| > |C_2(\phi)|$, there is some color k which does not appear on any edge incident with a vertex of degree d_2. Hence, by recoloring all edges with color j by color k, we obtain, from ϕ, a proper edge coloring ϕ' of G with two distinct palettes, and such that

$$|C_2(\phi') \setminus C_1(\phi')| < |C_2(\phi) \setminus C_1(\phi)|.$$

We conclude that we may assume that $C_2(\phi) \subseteq C_1(\phi)$. Now, let H_1 be the edge-induced subgraph of G induced by all edges with colors in $C_1(\phi) \setminus C_2(\phi)$, and let H_2 be the edge-induced subgraph of G induced by all edges with colors in $C_2(\phi)$. The graph H_1 is a regular Class 1 graph and the graph H_2 is a regular Class 1 graph. Moreover, since $C_2(\phi) \subseteq C_1(\phi)$, $V(H_1) \subseteq V(H_2)$. We have thus proved the following.

Proposition 5.2. *If G is a graph, then $\check{s}(G) = 2$ if and only if G is a non-regular graph which is the union of two regular edge-disjoint Class 1 graphs H_1 and H_2, satisfying that $V(H_1) \subseteq V(H_2)$.*

A partial characterization of graphs with palette index 3 was obtained in Bonvicini and Mazzuoccolo (2016). We would like to pose the following question.

Problem 5.3. *Is it possible to characterize graphs G satisfying that $\check{s}(G) = |V(G)| - 1$?*

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