On packing of two copies of a hypergraph

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A *2-packing* of a hypergraph $\mathcal{H}$ is a permutation σ on $V(\mathcal{H})$ such that if an edge e belongs to $\mathcal{E}(\mathcal{H})$, then $\sigma(e)$ does not belong to $\mathcal{E}(\mathcal{H})$. Let $\mathcal{H}$ be a hypergraph of order n which contains edges of cardinality at least 2 and at most $n-2$. We prove that if $\mathcal{H}$ has at most $n-2$ edges then it is 2-packable.

Keywords: packing, hypergraphs

1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, where V is the *vertex set* and $\mathcal{E} \subset 2^V$ is the *edge set*. We allow empty edges for technical reasons, hence a complete simple hypergraph of order n has 2^n edges. We consider only finite hypergraphs. The edge of cardinality t is called t-edge, and 1-edge is called a *singleton*. A vertex is *isolated* if it does not belong to any edge. The number $d(v)$ of edges containing a vertex v is called the *degree of* $v \in V$. A hypergraph is *t-uniform* if $|e| = t$ for all $e \in \mathcal{E}$. Let $\mathcal{H}$ be a hypergraph of order n. A *packing* of two copies of $\mathcal{H}$ (2-*packing* of $\mathcal{H}$) is a permutation σ on $V(\mathcal{H})$ such that, if an edge $e = \{x_1, ..., x_k\}$ belongs to $\mathcal{E}(\mathcal{H})$, then the edge $\sigma(e) = \{\sigma(x_1), ..., \sigma(x_k)\}$ does not belong to $\mathcal{E}(\mathcal{H})$. Such a permutation (a *packing permutation*) is also called an *embedding* of $\mathcal{H}$ into its complement. Consider a hypergraph $\mathcal{H}$ and a permutation σ on V. We have $\sigma(V) = V$ and $\sigma(\emptyset) = \emptyset$. So, if $V \in \mathcal{E}$ or $\emptyset \in \mathcal{E}$, then $\mathcal{H}$ cannot be packable.

We proved the following result in [4].

Theorem 1 *If a hypergraph $\mathcal{H}$ of order n and size at most $\frac{1}{2}n$ has neither the empty edge nor its complement, then $\mathcal{H}$ is 2-packable.*

Observe that this bound is sharp. Namely, if $\mathcal{H}$ is a hypergraph of order n, and it has more than $\frac{1}{2}n$ edges, and each edge is a singleton, then evidently $\mathcal{H}$ is not packable.

The aim of this paper is to show that if empty edges and singletons (and their complements, i.e. n-edges and $(n-1)$-edges) are excluded, then the bound on the size can be improved. We call a hypergraph $\mathcal{H}$ of order n *admissible* if $2 \leq |H| \leq n-2$ holds for all edges $H \in \mathcal{H}$.

We shall prove the following theorem.

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Theorem 2 *An admissible hypergraph $\mathcal{H}$ of order n and size at most $n-2$ is 2-packable.*

Recall that a 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H. P. Yap ([2],[8]), or survey papers by H. P. Yap or M. Woźniak ([9], [6], [7] and [5])). One of the first results in this area was the following theorem (see [3]).

Theorem 3 *A graph G of order n and size at most $n-2$ is 2-packable.*

This bound is tight. Namely, if G is a star (of order n and size $n-1$), then G is not packable.

Let $\mathcal{H}$ be an admissible hypergraph of order n. First, denote by $\mathcal{H}_k$ a k-uniform hypergraph of order n, which is induced by all k-edges in $\mathcal{H}$, and let m_k be the size of $\mathcal{H}_k$. Let m be the size of $\mathcal{H}$. Thus

$$n-2 \geq m = m_2 + m_3 + ... + m_{n-2}.$$

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Consider the hypergraph $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ with the same vertex set V and the edge set $\tilde{\mathcal{E}}$, obtained from $\mathcal{E}$ in the following way: if $e \in \mathcal{E}$ has at most $\frac{n}{2}$ vertices then e belongs to $\tilde{\mathcal{E}}$ and if e has more than $\frac{n}{2}$ vertices, then e is replaced by $V \setminus e$, with the convention that a double edge conceivably created in this way is replaced by a single one.

Remark 4 *Let $\mathcal{H}$ be an admissible hypergraph of order n. If the hypergraph $\tilde{\mathcal{H}}$ is 2-packable, then also $\mathcal{H}$ is 2-packable. Therefore, we shall assume that $\mathcal{H}$ of order n is restricted to have edges of size at most $n/2$ only.*

Let $\mathcal{H} = (V, \mathcal{E})$ be an admissible hypergraph, and let x be a vertex of $\mathcal{H}$. We define the hypergraph $\mathcal{H}' = (V', \mathcal{E}') := \mathcal{H} - x$ as follows: $V' = V \setminus \{x\}$, and the set of edges is obtained from $\mathcal{E}$ by deleting 2-edges containing x, and replacing all remaining edges containing x by new edges with x deleted. It should be noted that it may happen that the assumption of Remark 4 does not apply to the hypergraph $\tilde{\mathcal{H}}$. So, if necessary, we use $\tilde{\mathcal{H}}'$ instead of $\mathcal{H}'$.

2 Lemmas

In the proof of Theorem 2, we shall use the following lemmas.

Lemma 5 *Let $\mathcal{H}$ be an admissible hypergraph of order $n \geq 7$. Let x be an isolated vertex in $\mathcal{H}_2$, and let y be a vertex of degree at least two in $\mathcal{H}_2$. Suppose that there does not exist any 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. If $\mathcal{H}' = \mathcal{H} - x - y$ is 2-packable, then $\mathcal{H}$ is also 2-packable. Moreover, $\mathcal{H}'$ is an admissible hypergraph.*

Proof: Let x and y be two vertices satisfying the assumptions. It is easy to see that $\mathcal{H}'$ is an admissible hypergraph, since, by assumptions, there is no singleton in $\mathcal{H}'$, because there is no 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. On the other hand, since $n \geq 7$, there is no $(n'-1)$-edge in $\mathcal{H}'$ (where $n' = n-2$).

Let σ' be a packing permutation of $\mathcal{H}'$. By the choice of x and y and the property of σ', it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$, where (xy) denotes a transposition, is a packing permutation of $\mathcal{H}$.

□

The proof of Lemma 6 is analogous to that of Lemma 5.

Lemma 6 *Let $\mathcal{H}$ be an admissible hypergraph of order $n \geq 7$. Let x and y be two not adjacent vertices of degree one in $\mathcal{H}_2$ such that the neighbors x' of x and y' of y are distinct. Suppose that there does not exist any 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. If $\mathcal{H}' = \mathcal{H} - x - y$ is 2-packable, then $\mathcal{H}$ is also 2-packable. Moreover, $\mathcal{H}'$ is an admissible hypergraph.*

Lemma 7 *Let $\mathcal{H}$ be an admissible hypergraph of order n and size at most $n-2$. If $m_2 \leq \frac{n}{2}$, then $\mathcal{H}$ is 2-packable.*

Proof: Using a probabilistic argument we shall show that a packing permutation exists for $\mathcal{H}$.

Let e and f be two edges of $\mathcal{H}$ of the same cardinality and let σ be a random permutation on V. We say that edge e *covers* edge f (with respect to σ), if $\sigma(e) = f$. We denote this fact by $(e \curvearrowright f)$.

Let e and f be two k-edges. The event A such that e covers f (denoted by $A(e \curvearrowright f)$) has probability equal to

$$Pr(A(e \curvearrowright f)) = \frac{k!(n-k)!}{n!} = \binom{n}{k}^{-1}.$$

Observe, that there are m_k^2 ways to choose a pair e, f of k-edges such that e covers f. So, we have

$$Pr\left(\bigcup_{e,f\in\mathcal{H}} A(e \curvearrowright f)\right) \leq \sum_{e,f\in\mathcal{H}} Pr\left(A(e \curvearrowright f)\right) = m_2^2\binom{n}{2}^{-1} + m_3^2\binom{n}{3}^{-1} + \dots + m_{\lfloor\frac{n}{2}\rfloor}^2\binom{n}{\lfloor\frac{n}{2}\rfloor}^{-1}.$$

Since $k \leq \frac{n}{2}$, the sequence $\left(\binom{n}{2}^{-1}, \binom{n}{3}^{-1}, \dots\right)$ is decreasing, and we have

$$m_2^2\binom{n}{2}^{-1} + m_3^2\binom{n}{3}^{-1} + \dots + m_{\lfloor\frac{n}{2}\rfloor}^2\binom{n}{\lfloor\frac{n}{2}\rfloor}^{-1} \leq m_2^2\binom{n}{2}^{-1} + \binom{n}{3}^{-1}\left(m_3^2 + \dots + m_{\lfloor\frac{n}{2}\rfloor}^2\right) \leq$$

$$\leq m_2^2\binom{n}{2}^{-1} + \binom{n}{3}^{-1}(n-2-m_2)^2.$$

If $m_2 = 0$, then $n \geq 5$, and

$$m_2^2\binom{n}{2}^{-1} + \binom{n}{3}^{-1}(n-2-m_2)^2 = \binom{n}{3}^{-1}(n-2)^2.$$

If $m_2 = 1$, then $n \geq 3$, and

$$m_2^2\binom{n}{2}^{-1} + \binom{n}{3}^{-1}(n-2-m_2)^2 = \binom{n}{2}^{-1} + \binom{n}{3}^{-1}(n-3)^2.$$

If $m_2 \geq 2$, then $n \geq 4$, and

$$m_2^2\binom{n}{2}^{-1} + \binom{n}{3}^{-1}(n-2-m_2)^2 \leq \frac{2n^2}{4n(n-1)} + \frac{6(n-4)^2}{n(n-1)(n-2)}.$$

It is easy to check that in each case

$$Pr\left(\bigcup_{e,f\in\mathcal{H}} A(e\curvearrowright f)\right) < 1.$$

Consequently, a 2-packing of an admissible hypergraph $\mathcal{H}$ of order n and size at most $n-2$ exists, if $m_2 \leq \frac{n}{2}$.

□

3 Proof of Theorem 2

By Remark 4, we consider only hypergraphs with edges of cardinality at most $\frac{n}{2}$. It is easy to see that for $n \leq 6$, either $\mathcal{H}$ has only 2-edges, and we can apply Theorem 3, or the number of 2-edges is less than or equal to $n/2$, and we can apply Lemma 7. So, let $n \geq 7$.

Observe that, by Lemma 7, our claim holds if $\mathcal{H}_2$ is empty. Therefore, the proof will be divided into two main cases corresponding to the structure of $\mathcal{H}_2$ which is supposed to be non-empty.

The proof goes by induction on n. Let x, y be two vertices satisfying the assumptions of Lemma 5 or of Lemma 6. A 3-edge containing both of them will be called *a blocking edge*. Observe that if there is no blocking edge in $\mathcal{H}$, then the induction hypothesis can be applied. Below, we shall very often estimate the number of blocking edges in order to get a contradiction with the size of $\mathcal{H}$.

Case 1. There is no vertex of degree one in $\mathcal{H}_2$.

The hypergraph $\mathcal{H}_2$ has at most $n-2$ edges, so it has at least two isolated vertices. Denote by w the number of non-isolated vertices in $\mathcal{H}_2$. Observe that $w \geq 3$ and $w \leq m_2$. Let y be a vertex of degree at least 2 in $\mathcal{H}_2$. If we can choose an isolated vertex x in $\mathcal{H}_2$ such that there is no 3-edge containing both x and y, then we are done. So, suppose that a 3-edge containing both x and y exists in $\mathcal{H}$ for every isolated vertex x in $\mathcal{H}_2$ and for any y. Observe that one 3-edge can cover at most two pairs of vertices x, y satisfying the assumptions of Lemma 5. Hence,

$$m_3 \geq \frac{1}{2}w(n-w) \geq \frac{1}{2}w(n-m_2),$$

$$2m_3 + wm_2 \geq wn.$$

Hence,

$$w(m_2+m_3) \geq wn,$$

but $m_2 + m_3 \leq n-2$, a contradiction.

Case 2. There is a vertex of degree one in $\mathcal{H}_2$.

Let $b = m_3 + ... + m_{\lfloor \frac{n}{2} \rfloor}$. If $b = 0$, then $\mathcal{H}$ is a graph, and the claim is true. Hence, let $b > 0$. Then $m_2 = n-2-b$. Denote by t the number of tree components in $\mathcal{H}_2$. So, $t \geq b+2$. Next, denote by i the number of isolated vertices in $\mathcal{H}_2$, by j the number of isolated edges, by k the number of stars with at least two leaves, and by l the number of trees with diameter greater than two. Thus, $t = i+j+k+l$. We shall consider four subcases.

Case 2A. There are at least two vertices of degree at least two in $\mathcal{H}_2$, and $j+k+l \geq 2$.

As above, we shall count, how many blocking edges have to be in $\mathcal{H}$. Denote by n_2 the number of vertices of degree at least two in $\mathcal{H}_2$. By assumption, $n_2 \geq 2$. So, if we are not able to apply Lemma 5, we should have at least $(\frac{1}{2}in_2)$ 3-edges in $\mathcal{H}$. Similarly, if we are not able to apply Lemma 6, we should have at least $[\frac{1}{3} \cdot 4 \cdot \binom{j+k+l}{2}]$ 3-edges in $\mathcal{H}$. Observe that one 3-edge can cover at most three pairs of vertices x, y which satisfy the assumptions of Lemma 6. Moreover, between every two tree components with at least two leaves, there are at least four such pairs. There are $\binom{j+k+l}{2}$ such pairs. Observe that all 3-edges mentioned above have to be distinct. Hence, we have

$$\frac{1}{2}in_2 + \frac{4}{3} \cdot \binom{j+k+l}{2} \leq b \leq t-2 = i+j+k+l-2.$$

Observe that

$$\frac{1}{2}in_2 \geq i,$$

and

$$\frac{4}{3} \cdot \frac{1}{2} \cdot (j+k+l)(j+k+l-1) \geq 1 \cdot 1 \cdot (j+k+l-1).$$

Again, we obtain a contradiction.

Case 2B. There are at least two vertices of degree at least two in $\mathcal{H}_2$, and $j+k+l < 2$.

Thus, we have $l \leq 1$ and $n_2 \geq 2$. Analogously as in Case 2A, we consider blocking edges in $\mathcal{H}$. If $l = 0$, we obtain two cases:

1) if $j+k=0$, then

$$i \leq \frac{1}{2}in_2 \leq b \leq t-2 = i-2;$$

2) if $j+k=1$, then

$$i \leq \frac{1}{2}in_2 \leq b \leq t-2 = i-1.$$

If $l=1$ we have at least one blocking edge more. Then,

$$i+1 \leq \frac{1}{2}in_2 + 1 \leq b \leq t-2 = i+l-2 = i-1.$$

In all cases we get a contradiction.

Case 2C. There is at most one vertex of degree at least two in $\mathcal{H}_2$, and $j+k+l < 2$.

By definition, $l=0$. Therefore, we have three subcases to consider. If $k=j=0$ or $k=0$ and $j=1$, then by Lemma 7, our claim is true. Thus, let $j=0$ and $k=1$. So, $\mathcal{H}_2$ consists of a star $K_{1,p}$ and i isolated vertices. Observe that if $p \leq \frac{n}{2}$, then we are done by Lemma 7.

Hence, let $p > \frac{n}{2}$. Then, $n = i+p+1$. Let y be the center of the star, and let x be an isolated vertex in $\mathcal{H}_2$. If for any vertex z, the set $\{x, y, z\}$ is not an edge of $\mathcal{H}$, then we are done by Lemma 5.

If the vertex y belongs to two edges of the form $\{x, y, z\} \in \mathcal{E}(\mathcal{H})$ for any isolated vertex x, then we have the inequality

$$p + 2 \cdot \frac{i}{2} \leq n - 2.$$

Since $p + 2 \cdot \frac{i}{2} = n - 1$, we obtain a contradiction.

Therefore, there exists an isolated vertex x such that $\mathcal{H}$ contains exactly one 3-edge $\{x, y, z\}$. Now, we construct a hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ such that $V' = V - \{x, y\}$ and the set of edges is obtained from $\mathcal{E}$ as follows: we delete all 2-edges as well as the edge $\{x, y, z\}$, and we replace all remaining edges containing x or y (or x and y) by new edges with these vertices deleted. Then $\mathcal{H}'$ has two vertices less, and at least $p + 1$ edges less than $\mathcal{H}$.

We shall show that there exists a packing permutation σ' of $\mathcal{H}'$ without fixed points.

By the choice of x and y and the property of σ', it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$, where (xy) denotes a transposition, will be a packing permutation of $\mathcal{H}$.

An edge of the form $\{x, s, t\} \in \mathcal{H}$ (where $s \neq y$ and $t \neq y$) will be called an *x-edge*. Analogously, an edge of the form $\{y, s, t\} \in \mathcal{H}$ (where $s \neq x$ and $t \neq x$) will be called a *y-edge*.

First, we consider the case where $\mathcal{H}$ has either x-edges or y-edges. We construct the hypergraph $\mathcal{H}'' = (V'', \mathcal{E}'')$ as follows: $V'' = V'$, and the set of edges is obtained from $\mathcal{E}'$ by deleting all x-edges and y-edges. So $m_2'' = 0$ in $\mathcal{H}''$. Now, we use a probabilistic argument as in the proof of Lemma 7.

$$Pr\left(\bigcup_{e,f \in \mathcal{H}''} A(e \curvearrowright f)\right) \leq \binom{n}{3}^{-1} (n - 2 - p - 1)^2 \leq \frac{6(n-6)^2}{4(n-2)(n-3)(n-4)} < \frac{1}{e} - \frac{1}{n!}.$$

It is easy to observe that the last inequality holds for $n \geq 6$. (Recall that the probability that a random permutation has no fixed point is greater than or equal to $\frac{1}{e} - \frac{1}{n!}$.)

Now, suppose that there are ξ x-edges and η y-edges in $\mathcal{H}$. Observe that we have at least $p+3$ edges in $\mathcal{H}$ (there are p edges of the star, the edge $\{x, y, z\}$, at least one x-edge and at least one y-edge). Then, $p + 3 \leq n - 2$. But $p > \frac{n}{2}$, hence $n \geq 11$. In general, we have at least $(\xi + \eta + 1 + p)$ edges in $\mathcal{H}$. Therefore $\xi + \eta \leq \frac{n}{2} - 3$. Then a product $\xi\eta$ is maximal if $\xi = \eta = \frac{1}{2}(\frac{n}{2} - 3)$. Analogously as above, we use a probabilistic argument to show that there is a packing permutation σ' without fixed points of $\mathcal{H}'$. Observe that there are $\xi + \eta$ edges in ${\mathcal{H}'}_2$, and an x-edge cannot be mapped by σ' onto a y-edge (and vice versa). We have

$$Pr\left(\bigcup_{e,f \in \mathcal{H}'} A(e \curvearrowright f)\right) \leq \frac{2 \cdot 2\xi\eta \cdot (n-2)!}{n!} + \binom{n}{3}^{-1} (n - 2 - p - 3)^2 \leq$$

$$\leq \frac{(n-6)^2}{4n(n-1)} + \frac{3(n-10)^2}{2(n-2)(n-3)(n-4)} < \frac{1}{e} - \frac{1}{n!}.$$

It is easy to check that the last inequality is satisfied for $n \geq 11$, and consequently, there exists a packing permutation of $\mathcal{H}'$ without fixed points.

Case 2D. There is at most one vertex of degree at least two in $\mathcal{H}_2$, and $j+k+l \geq 2$.

Then, $\mathcal{H}_2$ has only tree components, $l=0$ and $k \leq 1$.

If $k=0$, then $j \geq 2$ and $j \leq \frac{n}{2}$ (because j is the number of isolated edges in $\mathcal{H}_2$). Then, by Lemma 7, the conclusion holds.

Thus, let $k=1$ and $j \geq 1$. Denote by $K_{1,p}$ the star in $\mathcal{H}_2$. If $p+j \leq \frac{n}{2}$, we are done by Lemma 7.

Hence $p+j > \frac{n}{2}$ and $n = i+2j+p+1$. If $j=1$, then a 3-edge can block at most two possibilities for the choice of two leaves in $\mathcal{H}_2$ if one leaf is in the star. So, if we are not able to apply Lemma 6, we have to have at least $\frac{2p}{2}$ blocking edges in $\mathcal{H}$. If we are not able to apply Lemma 5, we have to have at least $\frac{i}{2}$ blocking edges in $\mathcal{H}$. Observe that in both cases the blocking edges are distinct. Hence, taking into account all 2-edges we get

$$n-2 \geq |\mathcal{E}| \geq p+1+p+\frac{i}{2},$$

and

$$n-3 \geq 2p+\frac{i}{2}.$$

On the other hand, $n-3 = i+p$. Therefore, $\frac{i}{2} \geq p$. So, $n-3 \geq 3p$. It follows that $p < \frac{n}{3}$, a contradiction.

Now, let $j \geq 2$. Observe that the number of 3-edges in $\mathcal{H}$ is at least $\frac{i}{2}$ (because of Lemma 5), and at least $\frac{2pj}{2}$ (because of Lemma 6). (In the latter case, we may assume that one of the leaves comes from the star.)

We have

$$n-2 \geq |\mathcal{E}| \geq j+p+pj+\frac{i}{2}.$$

But $j+p > \frac{n}{2}$, so

$$n-2 \geq \frac{n}{2}+pj+\frac{i}{2}.$$

Hence

$$\frac{n}{2}-\frac{i}{2}-2 \geq pj.$$

We know from a structure of the hypergraph that $n = i+2j+p+1$, so it follows from the above inequality that

$$\frac{2j+p+1-4}{2} \geq pj.$$

This inequality together with the fact that $2pj \geq 2p+2j$ for $p, j \geq 2$, implies

$$2j+p-3 \geq 2pj \geq 2p+2j,$$

a contradiction.

This ends the proof of the theorem.

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