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On packing of two copies of a hypergraph

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A 2-packing of a hypergraph $H$ is a permutation $\sigma$ on $V(H)$ such that if an edge $e$ belongs to $E(H)$, then $\sigma(e)$ does not belong to $E(H)$. Let $H$ be a hypergraph of order $n$ which contains edges of cardinality at least 2 and at most $n - 2$. We prove that if $H$ has at most $n - 2$ edges then it is 2-packable.

Keywords: packing, hypergraphs

1 Introduction

Let $H = (V, E)$ be a hypergraph, where $V$ is the vertex set and $E \subset 2^V$ is the edge set. We allow empty edges for technical reasons, hence a complete simple hypergraph of order $n$ has $2^n$ edges. We consider only finite hypergraphs. The edge of cardinality $t$ is called $t$-edge, and 1-edge is called a singleton. A vertex is isolated if it does not belong to any edge. The number $d(v)$ of edges containing a vertex $v$ is called the degree of $v \in V$. A hypergraph is t-uniform if $|e| = t$ for all $e \in E$. Let $H$ be a hypergraph of order $n$. A packing of two copies of $H$ (2-packing of $H$) is a permutation $\sigma$ on $V(H)$ such that, if an edge $e = \{x_1, ..., x_k\}$ belongs to $E(H)$, then the edge $\sigma(e) = \{\sigma(x_1), ..., \sigma(x_k)\}$ does not belong to $E(H)$. Such a permutation (a packing permutation) is also called an embedding of $H$ into its complement. Consider a hypergraph $H$ and a permutation $\sigma$ on $V$. We have $\sigma(V) = V$ and $\sigma(\emptyset) = \emptyset$. So, if $V \in E$ or $\emptyset \in E$, then $H$ cannot be packable.

We proved the following result in [4].

**Theorem 1** If a hypergraph $H$ of order $n$ and size at most $\frac{1}{2}n$ has neither the empty edge nor its complement, then $H$ is 2-packable.

Observe that this bound is sharp. Namely, if $H$ is a hypergraph of order $n$, and it has more than $\frac{1}{2}n$ edges, and each edge is a singleton, then evidently $H$ is not packable.

The aim of this paper is to show that if empty edges and singletons (and their complements, i.e. $n$-edges and $(n - 1)$-edges) are excluded, then the bound on the size can be improved. We call a hypergraph $H$ of order $n$ admissible if $2 \leq |H| \leq n - 2$ holds for all edges $H \in H$.

We shall prove the following theorem.

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Theorem 2  An admissible hypergraph $\mathcal{H}$ of order $n$ and size at most $n - 2$ is 2-packable.

Recall that a 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H. P. Yap ([2],[8]), or survey papers by H. P. Yap or M. Woźniak ([9],[6],[7] and [5])). One of the first results in this area was the following theorem (see [3]).

Theorem 3  A graph $G$ of order $n$ and size at most $n - 2$ is 2-packable.

This bound is tight. Namely, if $G$ is a star (of order $n$ and size $n - 1$), then $G$ is not packable.

Let $\mathcal{H}$ be an admissible hypergraph of order $n$. First, denote by $\mathcal{H}_k$ a $k$-uniform hypergraph of order $n$, which is induced by all $k$-edges in $\mathcal{H}$, and let $m_k$ be the size of $\mathcal{H}_k$. Let $m$ be the size of $\mathcal{H}$. Thus

$$n - 2 \geq m = m_2 + m_3 + \ldots + m_{n-2}.$$ 

Let $\mathcal{H} = (V, E)$ be a hypergraph. Consider the hypergraph $\tilde{\mathcal{H}} = (V, \tilde{E})$ with the same vertex set $V$ and the edge set $\tilde{E}$, obtained from $E$ in the following way: if $e \in E$ has at most $\frac{n}{2}$ vertices then $e$ belongs to $\tilde{E}$ and if $e$ has more than $\frac{n}{2}$ vertices, then $e$ is replaced by $V \setminus e$, with the convention that a double edge conceivably created in this way is replaced by a single one.

Remark 4  Let $\mathcal{H}$ be an admissible hypergraph of order $n$. If the hypergraph $\tilde{\mathcal{H}}$ is 2-packable, then also $\mathcal{H}$ is 2-packable. Therefore, we shall assume that $\mathcal{H}$ of order $n$ is restricted to have edges of size at most $n/2$ only.

Let $\mathcal{H} = (V, E)$ be an admissible hypergraph, and let $x$ be a vertex of $\mathcal{H}$. We define the hypergraph $\mathcal{H}' = (V', E') := \mathcal{H} - x$ as follows: $V' = V \setminus \{x\}$, and the set of edges is obtained from $E$ by deleting 2-edges containing $x$, and replacing all remaining edges containing $x$ by new edges with $x$ deleted. It should be noted that it may happen that the assumption of Remark 4 does not apply to the hypergraph $\tilde{\mathcal{H}}$. So, if necessary, we use $\tilde{\mathcal{H}}'$ instead of $\mathcal{H}'$.

2 Lemmas

In the proof of Theorem 2 we shall use the following lemmas.

Lemma 5  Let $\mathcal{H}$ be an admissible hypergraph of order $n \geq 7$. Let $x$ be an isolated vertex in $\mathcal{H}_2$, and let $y$ be a vertex of degree at least two in $\mathcal{H}_2$. Suppose that there does not exist any 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. If $\mathcal{H}' = \mathcal{H} - x - y$ is 2-packable, then $\mathcal{H}$ is also 2-packable. Moreover, $\mathcal{H}'$ is an admissible hypergraph.

Proof:  Let $x$ and $y$ be two vertices satisfying the assumptions. It is easy to see that $\mathcal{H}'$ is an admissible hypergraph, since, by assumptions, there is no singleton in $\mathcal{H}'$, because there is no 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. On the other hand, since $n \geq 7$, there is no $(n' - 1)$-edge in $\mathcal{H}'$ (where $n' = n - 2$).

Let $\sigma'$ be a packing permutation of $\mathcal{H}'$. By the choice of $x$ and $y$ and the property of $\sigma'$, it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$, where $(xy)$ denotes a transposition, is a packing permutation of $\mathcal{H}$.

The proof of Lemma 6 is analogous to that of Lemma 5.
Lemma 6 Let $\mathcal{H}$ be an admissible hypergraph of order $n \geq 7$. Let $x$ and $y$ be two not adjacent vertices of degree one in $\mathcal{H}_2$ such that the neighbors $x'$ of $x$ and $y'$ of $y$ are distinct. Suppose that there does not exist any 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. If $\mathcal{H}' = \mathcal{H} - x - y$ is 2-packable, then $\mathcal{H}$ is also 2-packable. Moreover, $\mathcal{H}'$ is an admissible hypergraph.

Lemma 7 Let $\mathcal{H}$ be an admissible hypergraph of order $n$ and size at most $n - 2$. If $m_2 \leq \frac{n}{2}$, then $\mathcal{H}$ is 2-packable.

Proof: Using a probabilistic argument we shall show that a packing permutation exists for $\mathcal{H}$.

Let $e$ and $f$ be two edges of $\mathcal{H}$ of the same cardinality and let $\sigma$ be a random permutation on $V$. We say that edge $e$ covers edge $f$ (with respect to $\sigma$), if $\sigma(e) = f$. We denote this fact by $(e \, \bowtie \, f)$.

Let $e$ and $f$ be two $k$-edges. The event $A$ such that $e$ covers $f$ (denoted by $A(e \, \bowtie \, f)$) has probability equal to

$$Pr(A(e \, \bowtie \, f)) = \frac{k!(n-k)!}{n!} = \left(\frac{n}{k}\right)^{-1}.$$ 

Observe, that there are $m_2^2$ ways to choose a pair $e$, $f$ of $k$-edges such that $e$ covers $f$. So, we have

$$Pr\left(\bigcup_{e,f \in \mathcal{H}} A(e \, \bowtie \, f)\right) \leq \sum_{e,f \in \mathcal{H}} Pr(A(e \, \bowtie \, f)) = m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + m_3^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \ldots + m_{\left\lceil \frac{n}{2} \right\rceil}^2 \left(\begin{array}{c} n \\end{array}\right)^{-1}. $$

Since $k \leq \frac{n}{2}$, the sequence $\left(\begin{array}{c} n \\end{array}\right)^{-1}, \left(\begin{array}{c} n \\end{array}\right)^{-1}, \ldots$ is decreasing, and we have

$$m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + m_3^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \ldots + m_{\left\lceil \frac{n}{2} \right\rceil}^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} \leq m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + m_3^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \ldots + m_{\left\lceil \frac{n}{2} \right\rceil}^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} \leq m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-2-m_2)^2.$$ 

If $m_2 = 0$, then $n \geq 5$, and

$$m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-2-m_2)^2 = \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-2)^2.$$ 

If $m_2 = 1$, then $n \geq 3$, and

$$m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-2-m_2)^2 = \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-2)^2 + \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-3)^2.$$ 

If $m_2 \geq 2$, then $n \geq 4$, and

$$m_2^2 \left(\begin{array}{c} n \\end{array}\right)^{-1} + \left(\begin{array}{c} n \\end{array}\right)^{-1} (n-2-m_2)^2 \leq \frac{2n^2}{4n(n-1)} + \frac{6(n-4)^2}{n(n-1)(n-2)}.$$
It is easy to check that in each case
\[
Pr \left( \bigcup_{e,f \in H} A(e \succ f) \right) < 1.
\]
Consequently, a 2-packing of an admissible hypergraph \( H \) of order \( n \) and size at most \( n - 2 \) exists, if \( m_2 \leq \frac{n}{2} \).

3 Proof of Theorem 2

By Remark 4, we consider only hypergraphs with edges of cardinality at most \( \frac{n}{2} \). It is easy to see that for \( n \leq 6 \), either \( H \) has only 2-edges, and we can apply Theorem 3, or the number of 2-edges is less than or equal to \( n/2 \), and we can apply Lemma 7. So, let \( n \geq 7 \).

Observe that, by Lemma 7, our claim holds if \( H_2 \) is empty. Therefore, the proof will be divided into two main cases corresponding to the structure of \( H_2 \) which is supposed to be non-empty.

The proof goes by induction on \( n \). Let \( x, y \) be two vertices satisfying the assumptions of Lemma 5 or of Lemma 6. A 3-edge containing both of them will be called a blocking edge. Observe that if there is no blocking edge in \( H \), then the induction hypothesis can be applied. Below, we shall very often estimate the number of blocking edges in order to get a contradiction with the size of \( H \).

Case 1. There is no vertex of degree one in \( H_2 \).

The hypergraph \( H_2 \) has at most \( n - 2 \) edges, so it has at least two isolated vertices. Denote by \( w \) the number of non-isolated vertices in \( H_2 \). Observe that \( w \geq 3 \) and \( w \leq m_2 \). Let \( y \) be a vertex of degree at least 2 in \( H_2 \). If we can choose an isolated vertex \( x \) in \( H_2 \) such that there is no 3-edge containing both \( x \) and \( y \), then we are done. So, suppose that a 3-edge containing both \( x \) and \( y \) exists in \( H \) for every isolated vertex \( x \) in \( H_2 \) and for any \( y \). Observe that one 3-edge can cover at most two pairs of vertices \( x, y \) satisfying the assumptions of Lemma 5. Hence,

\[
m_3 \geq \frac{1}{2} w(n - w) \geq \frac{1}{2} w(n - m_2),
\]

\[
2m_3 + wm_2 \geq wn.
\]

Hence,

\[
w(m_2 + m_3) \geq wn,
\]

but \( m_2 + m_3 \leq n - 2 \), a contradiction.

Case 2. There is a vertex of degree one in \( H_2 \).

Let \( b = m_3 + \ldots + m_{\lfloor \frac{n}{2} \rfloor} \). If \( b = 0 \), then \( H \) is a graph, and the claim is true. Hence, let \( b > 0 \). Then \( m_2 = n - 2 - b \). Denote by \( t \) the number of tree components in \( H_2 \). So, \( t \geq b + 2 \). Next, denote by \( i \) the number of isolated vertices in \( H_2 \), by \( j \) the number of isolated edges, by \( k \) the number of stars with at least two leaves, and by \( l \) the number of trees with diameter greater than two. Thus, \( t = i + j + k + l \). We shall consider four subcases.
Case 2A. There are at least two vertices of degree at least two in \( H \), and \( j + k + l \geq 2 \).

As above, we shall count, how many blocking edges have to be in \( H \). Denote by \( n_2 \) the number of vertices of degree at least two in \( H_2 \). By assumption, \( n_2 \geq 2 \). So, if we are not able to apply Lemma 5, we should have at least \( \left( \frac{1}{2} in_2 \right) \) 3-edges in \( H \). By assumption, \( n_2 \geq 2 \). So, if we are not able to apply Lemma 5, we should have at least \( \left( \frac{1}{2} in_2 \right) \) 3-edges in \( H \). Observe that one 3-edge can cover at most three pairs of vertices \( x, y \) which satisfy the assumptions of Lemma 6. Moreover, between every two tree components with at least two leaves, there are at least four such pairs. There are \( \binom{j+k+l}{2} \) such pairs. Observe that all 3-edges mentioned above have to be distinct. Hence, we have

\[
\frac{1}{2} in_2 + \frac{4}{3} \cdot \binom{j+k+l}{2} \leq b \leq t - 2 = i + j + k + l - 2.
\]

Observe that

\[
\frac{1}{2} in_2 \geq i,
\]

and

\[
\frac{4}{3} \cdot \frac{1}{2} \cdot (j + k + l)(j + k + l - 1) \geq 1 \cdot 1 \cdot (j + k + l - 1).
\]

Again, we obtain a contradiction.

Case 2B. There are at least two vertices of degree at least two in \( H_2 \), and \( j + k + l < 2 \).

Thus, we have \( l \leq 1 \) and \( n_2 \geq 2 \). Analogously as in Case 2A, we consider blocking edges in \( H \). If \( l = 0 \), we obtain two cases:

1) if \( j + k = 0 \), then

\[
i \leq \frac{1}{2} in_2 \leq b \leq t - 2 = i - 2;
\]

2) if \( j + k = 1 \), then

\[
i \leq \frac{1}{2} in_2 \leq b \leq t - 2 = i - 1.
\]

If \( l = 1 \) we have at least one blocking edge more. Then,

\[
i + 1 \leq \frac{1}{2} in_2 + 1 \leq b \leq t - 2 = i + l - 2 = i - 1.
\]

In all cases we get a contradiction.

Case 2C. There is at most one vertex of degree at least two in \( H_2 \), and \( j + k + l < 2 \).

By definition, \( l = 0 \). Therefore, we have three subcases to consider. If \( k = j = 0 \) or \( k = 0 \) and \( j = 1 \), then by Lemma 7 our claim is true. Thus, let \( j = 0 \) and \( k = 1 \). So, \( H_2 \) consists of a star \( K_{1,p} \) and \( i \) isolated vertices. Observe that if \( p \leq \frac{2}{3} \), then we are done by Lemma 7.

Hence, let \( p > \frac{2}{3} \). Then, \( n = i + p + 1 \). Let \( y \) be the center of the star, and let \( x \) be an isolated vertex in \( H_2 \). If for any vertex \( z \), the set \( \{x, y, z\} \) is not an edge of \( H \), then we are done by Lemma 5.
If the vertex $y$ belongs to two edges of the form $\{x, y, z\} \in \mathcal{E}(\mathcal{H})$ for any isolated vertex $x$, then we have the inequality
\[ p + 2 \cdot \frac{i}{2} \leq n - 2. \]

Since $p + 2 \cdot \frac{i}{2} = n - 1$, we obtain a contradiction.

Therefore, there exists an isolated vertex $x$ such that $\mathcal{H}$ contains exactly one 3-edge $\{x, y, z\}$. Now, we construct a hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ such that $V' = V - \{x, y\}$ and the set of edges is obtained from $\mathcal{E}$ as follows: we delete all 2-edges as well as the edge $\{x, y, z\}$, and we replace all remaining edges containing $x$ or $y$ (or $x$ and $y$) by new edges with these vertices deleted. Then $\mathcal{H}'$ has two vertices less, and at least $p + 1$ edges less than $\mathcal{H}$.

We shall show that there exists a packing permutation $\sigma'$ of $\mathcal{H}'$ without fixed points.

By the choice of $x$ and $y$ and the property of $\sigma'$, it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$, where $(xy)$ denotes a transposition, will be a packing permutation of $\mathcal{H}$.

An edge of the form $\{x, s, t\} \in \mathcal{H}$ (where $s \neq y$ and $t \neq y$) will be called an $x$-edge. Analogously, an edge of the form $\{y, s, t\} \in \mathcal{H}$ (where $s \neq x$ and $t \neq x$) will be called a $y$-edge.

First, we consider the case where $\mathcal{H}$ has either $x$-edges or $y$-edges. We construct the hypergraph $\mathcal{H}'' = (V'', \mathcal{E}'')$ as follows: $V'' = V'$, and the set of edges is obtained from $\mathcal{E}'$ by deleting all $x$-edges and $y$-edges. So $m''_x = 0$ in $\mathcal{H}''$. Now, we use a probabilistic argument as in the proof of Lemma 7.

\[ \Pr \left( \bigcup_{e,f \in \mathcal{H}''} A(e \cap f) \right) \leq \left( \frac{n}{3} \right)^{-1} (n - 2 - p - 1)^2 \leq \frac{6(n - 6)^2}{4(n - 2)(n - 3)(n - 4)} < \frac{1}{e} - \frac{1}{n!}. \]

It is easy to observe that the last inequality holds for $n \geq 6$. (Recall that the probability that a random permutation has no fixed point is greater than or equal to $\frac{1}{e} - \frac{1}{n!}$.)

Now, suppose that there are $\xi$ $x$-edges and $\eta$ $y$-edges in $\mathcal{H}$. Observe that we have at least $p + 3$ edges in $\mathcal{H}$ (there are $p$ edges of the star, the edge $\{x, y, z\}$, at least one $x$-edge and at least one $y$-edge).

Then, $p + 3 \leq n - 2$. But $p > \frac{n}{2}$, hence $n \geq 11$. In general, we have at least $(\xi + \eta + 1 + p)$ edges in $\mathcal{H}$. Therefore $\xi + \eta \leq \frac{n}{2} - 3$. Then a product $\xi \eta$ is maximal if $\xi = \eta = \frac{1}{2} \left( \frac{n}{2} - 3 \right)$. Analogously as above, we use a probabilistic argument to show that there is a packing permutation $\sigma'$ without fixed points of $\mathcal{H}'$. Observe that there are $\xi + \eta$ edges in $\mathcal{H}'_2$, and an $x$-edge cannot be mapped by $\sigma'$ onto a $y$-edge (and vice versa). We have
\[ \Pr \left( \bigcup_{e,f \in \mathcal{H}'} A(e \cap f) \right) \leq \frac{2 \cdot 2 \xi \eta \cdot (n - 2)!}{n!} + \left( \frac{n}{3} \right)^{-1} (n - 2 - p - 3)^2 \leq \frac{(n - 6)^2}{4n(n - 1)} + \frac{3(n - 10)^2}{2(n - 2)(n - 3)(n - 4)} \leq \frac{1}{e} - \frac{1}{n!}. \]

It is easy to check that the last inequality is satisfied for $n \geq 11$, and consequently, there exists a packing permutation of $\mathcal{H}'$ without fixed points.
Case 2D. There is at most one vertex of degree at least two in $H_2$, and $j + k + l \geq 2$.

Then, $H_2$ has only tree components, $l = 0$ and $k \leq 1$.

If $k = 0$, then $j \geq 2$ and $j \leq \frac{n}{2}$ (because $j$ is the number of isolated edges in $H_2$). Then, by Lemma 7, the conclusion holds.

Thus, let $k = 1$ and $j \geq 1$. Denote by $K_{1,p}$ the star in $H_2$. If $p + j \leq \frac{n}{2}$, we are done by Lemma 7.

Hence $p + j > \frac{n}{2}$ and $n = i + 2j + p + 1$. If $j = 1$, then a 3-edge can block at most two possibilities for the choice of two leaves in $H_2$ if one leaf is in the star. So, if we are not able to apply Lemma 6, we have to have at least $\frac{2p}{2}$ blocking edges in $H$. If we are not able to apply Lemma 5, we have to have at least $\frac{1}{2}$ blocking edges in $H$. Observe that in both cases the blocking edges are distinct. Hence, taking into account all 2-edges we get

$$n - 2 \geq |E| \geq p + 1 + p + \frac{i}{2},$$

and

$$n - 3 \geq 2p + \frac{i}{2}.$$

On the other hand, $n - 3 = i + p$. Therefore, $\frac{i}{2} \geq p$. So, $n - 3 \geq 3p$. It follows that $p < \frac{n}{3}$, a contradiction.

Now, let $j \geq 2$. Observe that the number of 3-edges in $H$ is at least $\frac{i}{2}$ (because of Lemma 5), and at least $\frac{2p}{2}$ (because of Lemma 6). (In the latter case, we may assume that one of the leaves comes from the star.)

We have

$$n - 2 \geq |E| \geq j + p + pj + \frac{i}{2}.$$

But $j + p > \frac{n}{2}$, so

$$n - 2 \geq \frac{n}{2} + pj + \frac{i}{2}.$$

Hence

$$\frac{n}{2} - \frac{i}{2} - 2 \geq pj.$$

We know from a structure of the hypergraph that $n = i + 2j + p + 1$, so it follows from the above inequality that

$$\frac{2j + p + 1 - 4}{2} \geq pj.$$

This inequality together with the fact that $2pj \geq 2p + 2j$ for $p, j \geq 2$, implies

$$2j + p - 3 \geq 2p \geq 2p + 2j,$$

a contradiction.

This ends the proof of the theorem.
References


