

The LexCycle on $\overline{P_2 \cup P_3}$ -free Cocomparability Graphs*

Xiao-Lu Gao

Shou-Jun Xu[†]

School of Mathematics and Statistics, Lanzhou University, Gansu, China

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A graph G is a cocomparability graph if there exists an acyclic transitive orientation of the edges of its complement graph \overline{G} . LBFS⁺ is a variant of the generic Lexicographic Breadth First Search (LBFS), which uses a specific tie-breaking mechanism. Starting with some ordering σ_0 of G , let $\{\sigma_i\}_{i \geq 1}$ be the sequence of orderings such that $\sigma_i = \text{LBFS}^+(G, \sigma_{i-1})$. The $\text{LexCycle}(G)$ is defined as the maximum length of a cycle of vertex orderings of G obtained via such a sequence of LBFS⁺ sweeps. Dusart and Habib conjectured in 2017 that $\text{LexCycle}(G)=2$ if G is a cocomparability graph and proved it holds for interval graphs. In this paper, we show that $\text{LexCycle}(G)=2$ if G is a $\overline{P_2 \cup P_3}$ -free cocomparability graph, where a $\overline{P_2 \cup P_3}$ is the graph whose complement is the disjoint union of P_2 and P_3 . As corollaries, it's applicable for diamond-free cocomparability graphs, cocomparability graphs with girth at least 4, as well as interval graphs.

Keywords: cocomparability graph, LBFS⁺, LexCycle, $\overline{P_2 \cup P_3}$ -free, diamond-free, girth 4

1 Introduction

Lexicographic Breadth First Search (LBFS) is a graph search paradigm which was developed by Rose, Tarjan and Lueker Rose et al. (1976) for providing a simple linear time algorithm to recognize chordal graphs, namely, graphs containing no induced cycle of length greater than three. Since then, researchers have done plenty of studies on the properties and applications of LBFS Brandstädt et al. (1997); Habib et al. (2000). At each step of an LBFS procedure, a vertex is visited only if it has the lexicographically largest label. If there exists more than one such eligible vertex at some step, these vertices are said to be *tied* at this step.

A multi-sweep algorithm is an algorithm that produces a sequence of orderings $\{\sigma_i\}_{i \geq 0}$ where each ordering $\sigma_i (i \geq 1)$ breaks ties using specified tie-breaking rules by referring to the previous ordering σ_{i-1} . In particular, LBFS⁺ is one of the most widely used variants of LBFS, which is a multi-sweep algorithm that chooses the rightmost tied vertex in the previous sweep, and therefore produces a unique vertex ordering. It was first investigated in Ma (2000); Simon (1991), and has been used to recognize

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[†]Corresponding author.

several well-known classes of graphs, such as unit interval graphs Corneil (2004), interval graphs Corneil et al. (2010); Li and Wu (2014) and cocomparability graphs Dusart and Habib (2017). Here we present a description of the generic LBFS procedure in Algorithm 1 which starts with a distinguished vertex and then allows arbitrary tie-breaking; following the LBFS procedure we impose the specific tie-breaking mechanism LBFS⁺ in Algorithm 2.

Algorithm 1 LBFS (G, v)

Require: a graph $G(V, E)$ and a distinguished vertex v of G

Ensure: an ordering σ_v of vertices of G

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1: label( $v$ )  $\leftarrow |V|$ 
2: assign the label  $\epsilon$  to all the vertices of  $V - \{v\}$ 
3: for  $i \leftarrow 1$  to  $|V|$  do
4:   pick any unnumbered vertex  $u$  with the lexicographically largest label (§)
5:    $\sigma_i \leftarrow u$ 
6:   for each unnumbered vertex  $w \in N(u)$  do
7:     append  $(n - i)$  to label( $w$ )
8:   end for
9: end for
10: return  $\sigma_v$ 

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Algorithm 2 LBFS⁺ (G, π)

Require: a graph $G(V, E)$ and an ordering π of vertices of G

Ensure: an ordering σ of vertices of G

We run LBFS($G, \pi(|V|)$). In step (§) of the LBFS procedure, let L be the set of unnumbered vertices with the lexicographically largest label. Choose u to be the vertex in L that appears rightmost in π .

The LexCycle of a graph G is the size of the longest cycle resulting from a series of LBFS⁺'s on G . Since a finite graph has a finite number of vertex orderings, this series will converge to a number of fixed orderings that produce a cycle, the largest size of which is captured by this LexCycle parameter. Charbit et al. first introduced this new graph parameter Charbit et al. (2017). They believed that a small LexCycle often leads to a linear structure that has been exploited algorithmically on a number of graph classes.

Definition 1.1. Charbit et al. (2017) Let G be a graph, the *LexCycle*(G) is defined as the maximum length of a cycle of vertex orderings of G obtained via a sequence of LBFS⁺ sweeps starting with an arbitrary vertex ordering of G .

Comparability graphs are the graphs that admit an acyclic transitive orientation of the edges; that is, there is an orientation of the edges such that for every three vertices x, y, z , if the edges xy, yz are oriented $x \rightarrow y \rightarrow z$, then $xz \in E$ and $x \rightarrow z$. Cocomparability graphs are the complement graphs of comparability graphs and have been widely studied Kratsch and Stewart (1993); Köhler and Mouatadid (2016); Dusart et al. (2016); Mertzios and Corneil (2012); Corneil et al. (2016, 2013). The well-studied interval graphs, co-bipartite graphs, permutation graphs and trapezoid graphs are subclasses of cocomparability

graphs; and both comparability graphs and cocomparability graphs are well-known subclasses of perfect graphs Golumbic (2004).

Charbit et al. Charbit et al. (2017) reintroduced the conjecture that $\text{LexCycle}(G)=2$ if G is a cocomparability graph which was firstly raised in Dusart and Habib (2017). In particular, they showed that $\text{LexCycle}(G)=2$ for some subclasses of cocomparability graphs (proper interval, interval, co-bipartite, domino-free cocomparability graphs) as well as trees. They mentioned that to prove the conjecture, a good way is to start by proving that it holds for k -ladder-free cocomparability graphs for any positive integer k . Further, they conjectured that $\text{LexCycle}(G)=2$ even for AT-free graphs, which strictly contain cocomparability graphs. The k -ladder and asteroidal triple (AT) will be introduced in section 2.

In this paper, we show that $\text{LexCycle}(G)=2$ for $\overline{P_2 \cup P_3}$ -free cocomparability graphs, i.e., Theorem 3.2. These graphs strictly contain interval graphs and are unrelated under inclusion to domino-free cocomparability graphs, where the LexCycle of any of these two graphs is proved to be 2 in Charbit et al. (2017). The rest of this paper is organized as follows. We present in section 2 some preliminary definitions, notations and known results. In section 3, we present the main results. In the final section we present concluding remarks.

2 Preliminaries

In this paper, we consider simple finite undirected graphs $G = (V, E)$ on $n = |V|$ vertices. An *ordering* σ of G is a bijection σ from V to $\{1, 2, \dots, n\}$. We write $u \prec_\sigma v$ if and only if $\sigma(u) < \sigma(v)$ and u is said to be to the *left* of v in σ if $u \prec_\sigma v$. Given a sequence of orderings $\{\sigma_i\}_{i \geq 0}$, we write $u \prec_i v$ if $u \prec_{\sigma_i} v$; and $u \prec_{i,j} v$ if both $u \prec_{\sigma_i} v$ and $u \prec_{\sigma_j} v$. For $S \subseteq V$, the *induced subgraph* $G[S]$ of G is the graph whose vertex set is S and whose edge set consists of all the edges in E with both end-vertices in S ; we write $\sigma[S]$ to denote the ordering of σ restricted to the vertices of S . G is called *H-free* if G does not contain H as an induced subgraph. P_n and C_n denote a path and cycle respectively on n vertices. A *domino* is a pair of C_4 's sharing an edge. The *girth* $g(G)$ of G is the minimum length of a cycle in G ($g(G) = \infty$ if G does not contain a cycle). A *k-ladder* is a graph G with $V(G) = \{a, a_1, a_2, \dots, a_k, b, b_1, b_2, \dots, b_k\}$ and edge set $E(G) = \{ab, aa_1, bb_1\} \cup \{a_j b_j | 1 \leq j \leq k\} \cup \{a_j a_{j+1}, b_j b_{j+1} | 1 \leq j \leq k-1\}$, as shown in Fig. 1.

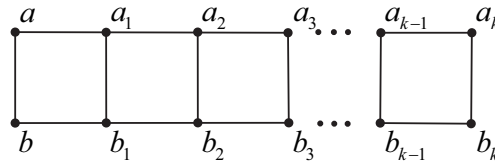


Fig. 1: A k -ladder.

Theorem 2.1. Corneil and Krueger (2008) *A graph $G = (V, E)$ is a cocomparability graph if and only if there exists a vertex ordering σ such that if $x \prec_\sigma y \prec_\sigma z$ and $xz \in E$, then either $xy \in E$ or $yz \in E$ or both.*

Such an ordering in Theorem 2.1 is called a *cocomparability ordering*, or an *umbrella-free ordering*. G is an *interval graph* if its vertices can be put in one-to-one correspondence with intervals on the real line

such that two vertices are adjacent in G if and only if the corresponding intervals intersect. An *asteroidal triple* (AT) is an independent triple of vertices u, v, w such that every pair of the triple is connected when removing the closed neighbourhood of the third vertex from the graph.

There is a nice vertex ordering characterization of LBFS as shown in Lemma 2.1, known as the *4-Point Condition*, which plays a key role in the proof of the correctness of our result.

Lemma 2.1. Corneil and Krueger (2008) (**4-Point Condition**) *A vertex ordering σ of a graph G with vertex set V is an LBFS ordering if and only if for any triple $x \prec_\sigma y \prec_\sigma z$, where $xz \in E$ and $xy \notin E$, there exists a vertex $w \prec_\sigma x$ such that $wy \in E$ and $wz \notin E$.*

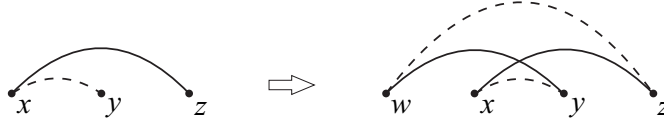


Fig. 2: The 4-Point Condition.

Given a pair of vertices y and z , we call a vertex w where $wy \in E$ and $wz \notin E$ a *private neighbour* of y with respect to z . A triple (x, y, z) satisfying $x \prec_\sigma y \prec_\sigma z$ where $xz \in E$ and $xy \notin E$ is called a *bad triple* with respect to σ , where σ is an ordering of G . In this paper, we always choose the vertex w in Lemma 2.1 as the *leftmost* private neighbour of y with respect to z in σ , and write it as $w = \text{LMPN}(y|_\sigma z)$.

It follows directly from Theorem 2.1 and Lemma 2.1 that an LBFS cocomparability ordering satisfies the following property.

Theorem 2.2. Charbit et al. (2017) (**LBFS C_4 Property**) *Let $G = (V, E)$ be a cocomparability graph and σ an LBFS cocomparability ordering of G . Then for every triple $x \prec_\sigma y \prec_\sigma z$ with $xz \in E$ and $xy \notin E$, there exists a vertex $w \prec_\sigma x$ such that $\{w, x, y, z\}$ induces a cycle where $wx, wy, yz \in E$.*

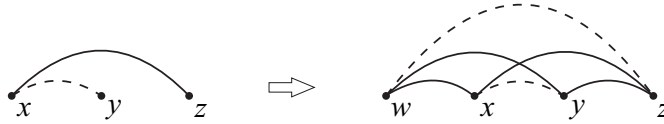


Fig. 3: The LBFS C_4 Property.

Theorem 2.3. Corneil et al. (2016) *Let G be a cocomparability graph, and π a cocomparability ordering of G . Then the LBFS ordering $\sigma = \text{LBFS}^+(\pi)$ is also a cocomparability ordering of G .*

Dusart and Habib presented a simple multi-sweep algorithm called *Repeated LBFS⁺*, where the algorithm *Repeated LBFS⁺* starts with an arbitrary LBFS ordering σ_1 and produces $n = |V(G)|$ consecutive LBFS orderings $\sigma_i (1 \leq i \leq n)$ such that $\sigma_i = \text{LBFS}^+(\sigma_{i-1})$ for $2 \leq i \leq n$. The authors proved in Dusart and Habib (2017) that G is a cocomparability graph if and only if the *Repeated LBFS⁺* algorithm computes a cocomparability ordering. They further conjectured that this series always falls into a cycle of length 2. We state these results below.

Lemma 2.2. Dusart and Habib (2017) G is a cocomparability graph if and only if $\mathcal{O}(n)$ LBFS⁺ sweeps compute a cocomparability ordering.

Conjecture 2.1. Dusart and Habib (2017) If G is a cocomparability graph, then $\text{LexCycle}(G) = 2$.

The conjecture is formulated based on the easy but very important tool called the *Flipping Lemma* about LBFS on cocomparability graphs.

Lemma 2.3. Corneil et al. (2016) (**The Flipping Lemma**) Let $G = (V, E)$ be a cocomparability graph, σ a cocomparability ordering of G and $\tau = \text{LBFS}^+(\sigma)$. Then for every non-edge $uv \notin E$, $u \prec_\sigma v \Leftrightarrow v \prec_\tau u$.

3 Main results

This paper presents a proof of a subcase of Conjecture 2.1. In the following we will show that $\text{LexCycle}(G)=2$ where G is a $\overline{P_2 \cup P_3}$ -free cocomparability graph. The graph $\overline{P_2 \cup P_3}$ is shown in Fig. 4.

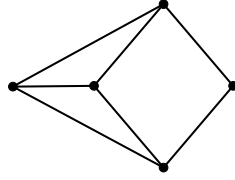


Fig. 4: the graph $\overline{P_2 \cup P_3}$.

Theorem 3.1. Let G be a $\overline{P_2 \cup P_3}$ -free cocomparability graph, π an arbitrary cocomparability ordering of G , and $\{\sigma_i\}_{i \geq 0}$ a sequence of LBFS⁺ orderings where $\sigma_{i+1} = \text{LBFS}^+(\sigma_i)$ and $\sigma_0 = \text{LBFS}^+(\pi)$. Then $\sigma_1 = \sigma_3$.

Proof: We prove this theorem by contradiction. We will show an infinite structure of G , which is a contradiction to the finiteness of G .

Since π is a cocomparability ordering of G , it follows from Theorem 2.3 that each ordering σ_i ($i \geq 0$) is an LBFS cocomparability ordering of G . Suppose to the contrary that $\sigma_1 \neq \sigma_3$. Let $\sigma_1 = u_1, u_2, \dots, u_n$ and $\sigma_3 = v_1, v_2, \dots, v_n$. Denote k the index of the leftmost vertex where σ_1 and σ_3 differ. Let a_1 (resp. b_1) denote the k^{th} vertex of σ_1 (resp. σ_3). Then $u_i = v_i$ for any $i < k$ and $u_k = a_1, v_k = b_1$. Thus $a_1 \prec_1 b_1$ and $b_1 \prec_3 a_1$. The following claim presents the infinite structure of G .

Claim 1. Assume that a_1, b_1 were given as defined previously. Then, for any integer $t \geq 2$, there always exists a $(t-1)$ -ladder with vertex set $\{a_1, a_2, \dots, a_t, b_1, b_2, \dots, b_t\}$, satisfying that:

- (1) $a_{j+1} = \text{LMPN}(a_j |_{\sigma_2} b_j)$, $b_{j+1} = \text{LMPN}(b_j |_{\sigma_0} a_j)$, $\forall 1 \leq j \leq t-1$.
- (2) $E(G[\{a_1, a_2, \dots, a_t, b_1, b_2, \dots, b_t\}]) = \{a_j b_j | 1 \leq j \leq t\} \cup \{a_j a_{j+1}, b_j b_{j+1} | 1 \leq j \leq t-1\}$.
- (3) $b_t \prec_0 a_t \prec_0 b_{t-1} \prec_0 a_{t-1} \prec_0 \dots \prec_0 b_1 \prec_0 a_1$; $a_1 \prec_1 b_1 \prec_1 a_2 \prec_1 b_2 \prec_1 \dots \prec_1 a_t \prec_1 b_t$;
 $a_t \prec_2 b_t \prec_2 a_{t-1} \prec_2 b_{t-1} \prec_2 \dots \prec_2 a_1 \prec_2 b_1$.

We prove the claim by induction on t . We first show it holds for the base case $t = 2$.

Let $S = \{u_1, u_2, \dots, u_{k-1}\} = \{v_1, v_2, \dots, v_{k-1}\}$ (S might be empty), then $\sigma_1[S] = \sigma_3[S]$. Since at the time a_1 was chosen in σ_1 after the ordering of S , b_1 was simultaneously chosen in σ_3 , it follows that

label(a)=label(b) at iteration k in both σ_1 and σ_3 , i.e., $S \cap N(a_1) = S \cap N(b_1)$. Therefore when a_1 was chosen in σ_1 , the “+” rule was applied to break ties between a_1 and b_1 and so $b_1 \prec_0 a_1$. Similarly, we have $a_1 \prec_2 b_1$.

Since $a_1 \prec_{1,2} b_1$, we know that $a_1 b_1 \in E$ (by the Flipping Lemma) and there exists a vertex left of a_1 in σ_2 which is a private neighbour of a_1 with respect to b_1 . We choose a_2 as $a_2 = \text{LMPN}(a_1 |_{\sigma_2} b_1)$. Using the Flipping Lemma on the non-edge $a_2 b_1$, we place a_2 in the remaining orderings and obtain that $a_2 \prec_0 b_1, b_1 \prec_1 a_2$. This gives rise to a bad triple in σ_0 where $a_2 \prec_0 b_1 \prec_0 a_1$ and $a_2 a_1 \in E, a_2 b_1 \notin E$.

By the LBFS C_4 Property, we choose vertex b_2 as $b_2 = \text{LMPN}(b_1 |_{\sigma_0} a_1)$ and thus $b_2 a_2 \in E$. We again use the Flipping Lemma on $b_2 a_1 \notin E$ to place b_2 in the remaining orderings, and obtain that $a_1 \prec_1 b_2, b_2 \prec_2 a_1$.

Consider the position of b_2 in σ_2 . If $b_2 \prec_2 a_2$, then (b_2, a_1, b_1) is a bad triple, contradicting to the choice of a_2 as $a_2 = \text{LMPN}(a_1 |_{\sigma_2} b_1)$. Therefore $a_2 \prec_2 b_2 \prec_2 a_1$, as shown in Fig. 5.

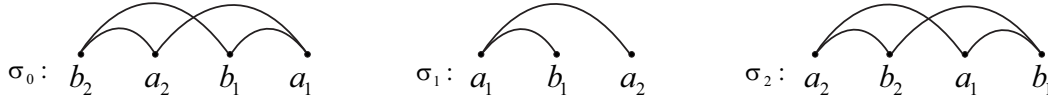


Fig. 5: initial positions of a_1, b_1 in σ_0, σ_1 and σ_2 , respectively.

Now we consider the position of b_2 in σ_1 . We know that $a_1 \prec_1 b_2$. This gives rise to three cases: (i) $a_1 \prec_1 b_2 \prec_1 b_1$, or (ii) $b_1 \prec_1 b_2 \prec_1 a_2$, or (iii) $a_2 \prec_1 b_2$.

(i) If $a_1 \prec_1 b_2 \prec_1 b_1$, then (a_1, b_2, b_1) is a bad triple in σ_1 . Thus there exists a vertex $c \prec_1 a_1$ (thus $c \in S$) such that $cb_2, ca_1 \in E$ and $cb_1 \notin E$, contradicting that $S \cap N(a_1) = S \cap N(b_1)$.

(ii) If $b_1 \prec_1 b_2 \prec_1 a_2$, then (a_1, b_2, a_2) is a bad triple in σ_1 . Thus there exists a vertex $c \prec_1 a_1$ (thus $c \in S$) such that $cb_2, ca_1 \in E$ and $ca_2 \notin E$. Since $S \cap N(a_1) = S \cap N(b_1)$, $cb_1 \in E$. Then $\{c, a_1, b_1, b_2, a_2\}$ induces a $\overline{P_2} \cup \overline{P_3}$, where P_2 is the path $a_1 - b_2$ and P_3 is the path $c - a_2 - b_1$, a contradiction.

Therefore, b_2 must be placed as $a_2 \prec_1 b_2$, and thus we have completely determined the positions of vertices of $\{a_1, a_2, b_1, b_2\}$ in σ_0, σ_1 and σ_2 , respectively, as shown in Fig. 6. Therefore it holds for the base case.

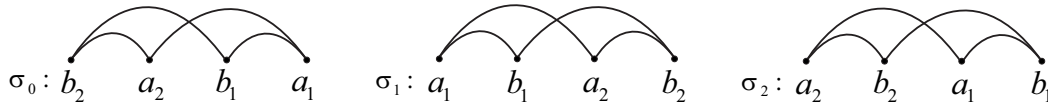


Fig. 6: positions of a_2, b_2 in σ_0, σ_1 and σ_2 , respectively.

From now on we suppose that it is true for $t = i$ and will prove the case when $t = i + 1$. By the inductive hypothesis, there exists a sequence of vertices $a_1, a_2, \dots, a_i, b_1, b_2, \dots, b_i$ satisfying the three conditions in Claim 1.

Since $a_i \prec_{1,2} b_i$, there exists a vertex left of a_i in σ_2 which is adjacent to a_i but not to b_i . Choose a_{i+1} as $a_{i+1} = \text{LMPN}(a_i |_{\sigma_2} b_i)$. Using the Flipping Lemma on the non-edge $b_i a_{i+1}$, we have that $a_{i+1} \prec_0 b_i, b_i \prec_1 a_{i+1}$. This gives rise to a bad triple (a_{i+1}, b_i, a_i) in σ_0 where $a_{i+1} a_i \in E$ and $a_{i+1} b_i \notin E$.

Choose b_{i+1} as $b_{i+1} = \text{LMPN}(b_i |_{\sigma_0} a_i)$. Using the Flipping Lemma on the non-edge $b_{i+1}a_i$, we obtain that $b_{i+1} \prec_2 a_i$, $a_i \prec_1 b_{i+1}$. If $b_{i+1} \prec_2 a_{i+1}$, then (b_{i+1}, a_i, b_i) is a bad triple, contradicting that $a_{i+1} = \text{LMPN}(a_i |_{\sigma_2} b_i)$. Thus, we have that $a_{i+1} \prec_2 b_{i+1} \prec_2 a_i$, as shown in Fig. 7.

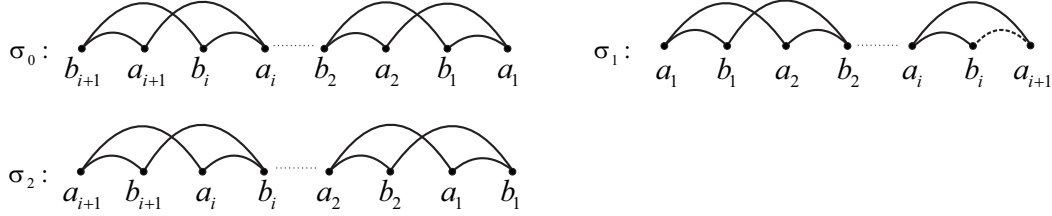


Fig. 7: initial positions of a_{i+1}, b_{i+1} in σ_0, σ_1 and σ_2 , respectively.

Now we show that b_{i+1} is adjacent to none of the vertices $a_1, a_2, \dots, a_{i-1}, b_1, b_2, \dots, b_{i-1}$, and similarly, we show that a_{i+1} is adjacent to none of these vertices. We have shown that $b_{i+1}a_i \notin E$. Since also that $a_i b_j \notin E$ and $b_{i+1} \prec_0 a_i \prec_0 b_j$ for any $1 \leq j \leq i-1$, it follows that $b_{i+1}b_j \notin E$ (by the definition a cocomparability ordering) for any $1 \leq j \leq i-1$. Similarly, $a_{i+1}a_j \notin E$ for any $1 \leq j \leq i-1$. It holds that $b_{i+1}a_{i-1} \notin E$, since otherwise, $(b_{i+1}, b_{i-1}, a_{i-1})$ is a bad triple in σ_0 , contradicting that $b_i = \text{LMPN}(b_{i-1} |_{\sigma_0} a_{i-1})$. On the other hand, since $b_{i+1} \prec_0 a_i \prec_0 a_j$ and $b_{i+1}a_i, a_i a_j \notin E$ for any $1 \leq j \leq i-2$, we have that $b_{i+1}a_j \notin E$ for any $1 \leq j \leq i-2$. Therefore $b_{i+1}a_j \notin E$, for any $1 \leq j \leq i-1$. Similarly, we deal with a_{i+1} in σ_2 and obtain that $a_{i+1}b_j \notin E$, for any $1 \leq j \leq i-1$. So far, we have proved the correctness of conditions (1) and (2).

What remains to be shown is the position of b_{i+1} in the ordering σ_1 . We know that $a_i \prec_1 b_{i+1}$. This gives rise to three cases: (i) $a_i \prec_1 b_{i+1} \prec_1 b_i$, or (ii) $b_i \prec_1 b_{i+1} \prec_1 a_{i+1}$, or (iii) $a_{i+1} \prec_1 b_{i+1}$. We will show that b_{i+1} must be placed as in (iii).

Case 1: $a_i \prec_1 b_{i+1} \prec_1 b_i$.

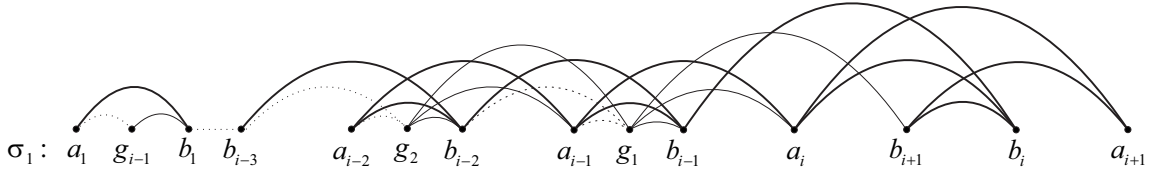


Fig. 8: Case 1. $a_i \prec_1 b_{i+1} \prec_1 b_i$.

In this case, (b_{i-1}, b_{i+1}, b_i) is a bad triple in σ_1 , so we choose g_1 as $g_1 = \text{LMPN}(b_{i+1} |_{\sigma_1} b_i)$ and so $g_1 b_{i-1} \in E$. The ladder structure implies that g_1 can't be any of the vertices $\{a_j, b_j\}_{1 \leq j \leq i}$. Because of the fact that $g_1 \prec_1 a_i \prec_1 b_{i+1}$ and $a_i b_{i+1} \notin E$, we have $g_1 a_i \in E$. If $g_1 \prec_1 a_{i-1}$, then since $a_{i-1} b_{i+1} \notin E$, $g_1 a_{i-1} \in E$. Thus, $\{g_1, a_i, a_{i-1}, b_{i-1}, b_i\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_i - b_{i-1}$ and P_3 is the path $g_1 - b_i - a_{i-1}$, a contradiction. Therefore, $a_{i-1} \prec_1 g_1 \prec_1 b_{i-1}$ and $g_1 a_{i-1} \notin E$. Thus $g_1 b_{i-2} \notin E$. The triple (b_{i-2}, g_1, b_{i-1}) is a bad triple in σ_1 .

We choose g_2 as $g_2 = \text{LMPN}(g_1 |_{\sigma_1} b_{i-1})$ and thus $g_2 b_{i-2} \in E$. The non-edge $g_1 a_{i-1} \notin E$ implies that $g_2 a_{i-1} \in E$. It holds that $g_2 a_{i-2} \notin E$, since otherwise $\{g_2, a_{i-1}, a_{i-2}, b_{i-2}, b_{i-1}\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_{i-1} - b_{i-2}$ and P_3 is the path $g_2 - b_{i-1} - a_{i-2}$, a contradiction. We show (by contradiction) that $a_{i-2} \prec_1 g_2 \prec_1 b_{i-2}$. If $g_2 \prec_1 a_{i-2}$, then $g_2 a_{i-2} \notin E$ implies that $g_1 a_{i-2} \in E$. Note that $a_{i-1} \prec_1 g_1$ and $a_{i-1} g_1 \notin E$, thus $g_1 \prec_2 a_{i-1}$. Observe that $g_1 a_{i-2} \in E$ and $g_1 b_{i-2} \notin E$, contradicting that $a_{i-1} = \text{LMPN}(a_{i-2} |_{\sigma_2} b_{i-2})$. If $g_2 = a_{i-2}$, then we have $g_1 a_{i-2} \in E$ and $g_1 b_{i-2} \notin E$, which leads to the same contradiction. Therefore, $a_{i-2} \prec_1 g_2 \prec_1 b_{i-2}$. Since $g_2 a_{i-2} \notin E$, $g_2 b_{i-3} \notin E$. The triple (b_{i-3}, g_2, b_{i-2}) is a bad triple in σ_1 .

We choose g_3 as $g_3 = \text{LMPN}(g_2 |_{\sigma_1} b_{i-2})$. We deal with g_3 in the same way as with g_2 , and thus obtain a sequence of vertices $\{g_j\}_{2 \leq j \leq i-1}$ such that $g_j = \text{LMPN}(g_{j-1} |_{\sigma_1} b_{i-j+1})$ and $a_{i-j} \prec_1 g_j \prec_1 b_{i-j}$, satisfying that $g_j b_{i-j-1} \notin E$ (if $j \leq i-2$), $g_j a_{i-j} \notin E$, $g_j b_{i-j} \in E$ and $g_j a_{i-j+1} \in E$, as shown in Fig. 8. Especially, $a_1 \prec_1 g_{i-1} \prec_1 b_1$, and $g_{i-1} a_1 \notin E$. Then, (a_1, g_{i-1}, b_1) is a bad triple in σ_1 , resulting that there exists a vertex left of a_1 in σ_1 which is adjacent to a_1 and g_{i-1} but not to b_1 , contradicting that $S \cap N(a_1) = S \cap N(b_1)$.

Case 2: $b_i \prec_1 b_{i+1} \prec_1 a_{i+1}$.

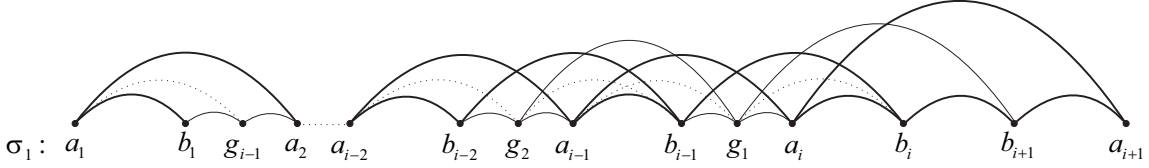


Fig. 9: Case 2. $b_i \prec_1 b_{i+1} \prec_1 a_{i+1}$.

In this case, (a_i, b_{i+1}, a_{i+1}) is a bad triple in σ_1 , so we choose g_1 as $g_1 = \text{LMPN}(b_{i+1} |_{\sigma_1} a_{i+1})$ and thus $g_1 a_i \in E$. It follows from the ladder structure that g_1 can't be any of the vertices $\{a_j, b_j\}_{1 \leq j \leq i}$. It holds that $g_1 b_i \notin E$, since otherwise $\{g_1, a_i, b_i, b_{i+1}, a_{i+1}\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_i - b_{i+1}$ and P_3 is the path $g_1 - a_{i+1} - b_i$, a contradiction.

Consider the position of g_1 in σ_1 . We know that $a_{i-1} b_{i+1}, b_{i-1} b_{i+1} \notin E$. If $g_1 \prec_1 a_{i-1}$, then $g_1 a_{i-1}, g_1 b_{i-1} \in E$. Thus, $\{g_1, a_i, a_{i-1}, b_{i-1}, b_i\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_i - b_{i-1}$ and P_3 is the path $g_1 - b_i - a_{i-1}$, a contradiction. If $a_{i-1} \prec_1 g_1 \prec_1 b_{i-1}$, then $g_1 b_{i-1} \in E$ as $b_{i-1} b_{i+1} \notin E$. Thus $g_1 a_{i-1} \notin E$ because of the same contradiction above, and thus $g_1 b_{i-2} \notin E$, resulting that (b_{i-2}, g_1, b_{i-1}) is a bad triple in σ_1 , which is the same as in Case 1. Therefore, we assume from now on that $b_{i-1} \prec_1 g_1 \prec_1 a_i$. Since $g_1 b_i \notin E$, $g_1 b_{i-1} \in E$. Similarly, we have $g_1 a_{i-1} \notin E$. The triple (a_{i-1}, g_1, a_i) is a bad triple in σ_1 .

We choose g_2 as $g_2 = \text{LMPN}(g_1 |_{\sigma_1} a_i)$ and thus $g_2 a_{i-1} \in E$. Since $b_{i-2} a_{i-1} \notin E$, $g_2 \neq b_{i-2}$. It holds that $g_2 b_{i-1} \notin E$, since otherwise $\{g_2, a_{i-1}, b_{i-1}, g_1, a_i\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_{i-1} - g_1$ and P_3 is the path $g_2 - a_i - b_{i-1}$, a contradiction.

In the following we consider the position of g_2 in σ_1 . If $g_2 \prec_1 a_{i-2}$, then $g_2 a_{i-2} \in E$. Since otherwise, if $g_2 a_{i-2} \notin E$, then $g_1 a_{i-2} \in E$. Note that $g_1 a_{i-1} \notin E$ and $b_{i-2} \prec_1 a_{i-1} \prec_1 g_1$ imply $g_1 b_{i-2} \notin E$. Since $a_{i-1} \prec_1 g_1$ and $a_{i-1} g_1 \notin E$, we have $g_1 \prec_2 a_{i-1}$, contradicting to the choice of a_{i-1} as $a_{i-1} = \text{LMPN}(a_{i-2} |_{\sigma_2} b_{i-2})$. Thus we have $g_2 a_{i-2} \in E$. Since $g_1 b_{i-2} \notin E$, $g_2 b_{i-2} \in E$.

Then $\{g_2, a_{i-1}, a_{i-2}, b_{i-2}, b_{i-1}\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_{i-1} - b_{i-2}$ and P_3 is the path $g_2 - b_{i-1} - a_{i-2}$, a contradiction. If $g_2 = a_{i-2}$, then immediately we have $g_1 a_{i-2} \in E$ and $g_1 b_{i-2} \notin E$, which still contradicts that $a_{i-1} = \text{LMPN}(a_{i-2} |_{\sigma_2} b_{i-2})$. If $a_{i-2} \prec_1 g_2 \prec_1 b_{i-2}$, then $g_2 a_{i-1} \in E$ (by the triple (g_2, a_{i-1}, g_1)) and $b_{i-2} a_{i-1} \notin E$ imply that $g_2 b_{i-2} \in E$ (otherwise, (g_2, b_{i-2}, a_{i-1}) would be a bad triple in σ_1). It holds that $g_2 a_{i-2} \notin E$, since otherwise $\{g_2, b_{i-2}, a_{i-2}, a_{i-1}, b_{i-1}\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $b_{i-2} - a_{i-1}$ and P_3 is the path $g_2 - b_{i-1} - a_{i-2}$, a contradiction. If $i > 3$, $b_{i-3} a_{i-2} \notin E$, $g_2 b_{i-3} \notin E$ implies (b_{i-3}, g_2, b_{i-2}) is a bad triple in σ_1 , which is the same as in Case 1. If $i = 3$, then $(a_1 = a_{i-2}, g_2, b_{i-2} = b_1)$ is a bad triple in σ_1 , contradicting that $S \cap N(a_1) = S \cap N(b_1)$ by using the LBFS C_4 Property. Therefore we assume from now on that $b_{i-2} \prec_1 g_2 \prec_1 a_{i-1}$.

Since $g_2 b_{i-1} \notin E$, it follows that $g_2 b_{i-2} \in E$. If $g_2 a_{i-2} \in E$, then $\{g_2, a_{i-1}, a_{i-2}, b_{i-2}, b_{i-1}\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $a_{i-1} - b_{i-2}$ and P_3 is the path $g_2 - b_{i-1} - a_{i-2}$, a contradiction. Thus $g_2 a_{i-2} \notin E$. Then (a_{i-2}, g_2, a_{i-1}) is a bad triple in σ_1 . Choose g_3 as $g_3 = \text{LMPN}(g_2 |_{\sigma_1} a_{i-1})$. We deal with g_3 in the same way as with g_2 , and thus obtain a sequence of vertices $\{g_j\}_{2 \leq j \leq i-1}$, such that $g_j = \text{LMPN}(g_{j-1} |_{\sigma_1} a_{i-j+2})$ and $b_{i-j} \prec_1 g_j \prec_1 a_{i-j+1}$, satisfying $g_j a_{i-j+1} \in E$, $g_j b_{i-j+1} \notin E$, $g_j b_{i-j} \in E$ and $g_j a_{i-j} \notin E$, as shown in Fig. 9. Especially, $b_1 \prec_1 g_{i-1} \prec_1 a_2$, $g_{i-1} b_1 \in E$ and (a_1, g_{i-1}, a_2) is a bad triple in σ_1 . Thus there exists a vertex $c \prec_1 a_1$ (thus $c \in S$) such that $ca_1, cg_{i-1} \in E$ and $ca_2 \notin E$. Since $S \cap N(a_1) = S \cap N(b_1)$, it follows that $cb_1 \in E$. Then $\{c, g_{i-1}, b_1, a_1, a_2\}$ induces a $\overline{P_2 \cup P_3}$, where P_2 is the path $g_{i-1} - a_1$ and P_3 is the path $c - a_2 - b_1$, a contradiction.

Thus we obtain that b_{i+1} must be placed in σ_1 as $a_{i+1} \prec_1 b_{i+1}$, as required in condition (3). Therefore, we have completely proved the correctness of Claim 1.

Since we can always find such a sequence of vertices $a_1, a_2, \dots, a_t, b_1, b_2, \dots, b_t$ for any integer $t \geq 2$, we get a contradiction to G being finite. Thus $\sigma_1 = \sigma_3$, as required. \square

Combining Lemma 2.2 with Theorem 3.1, we immediately obtain our main result as following.

Theorem 3.2. *Let G be a $\overline{P_2 \cup P_3}$ -free cocomparability graph. Then $\text{LexCycle}(G) = 2$.*

Note that $\overline{P_2 \cup P_3}$ -free cocomparability graphs strictly contain both C_4 -free cocomparability graphs (i.e., interval graphs, which have been proved in Charbit et al. (2017)) and diamond-free cocomparability graphs, where a diamond consists of a complete graph K_4 minus one edge, we thus immediately obtain the following corollaries.

Corollary 3.1. Charbit et al. (2017) *Let G be an interval graph. Then $\text{LexCycle}(G) = 2$.*

Corollary 3.2. *Let G be a diamond-free cocomparability graph. Then $\text{LexCycle}(G) = 2$.*

Additionally, $\overline{P_2 \cup P_3}$ -free cocomparability graphs strictly contain triangle-free cocomparability graphs, we thus immediately obtain that this result also holds for cocomparability graphs with girth at least 4.

Corollary 3.3. *Let G be a cocomparability graph with girth $g(G) \geq 4$. Then $\text{LexCycle}(G) = 2$.*

4 Concluding remarks

In this paper we focus on the parameter called $\text{LexCycle}(G)$, recently introduced by Charbit et al. Charbit et al. (2017), and show that $\text{LexCycle}(G)=2$ if G is a $\overline{P_2 \cup P_3}$ -free cocomparability graph. As corollaries, it's applicable for diamond-free cocomparability graphs, cocomparability graphs with girth at least 4, as well as interval graphs. In the proof of Theorem 3.1, we have assumed that $b_{i+1} = \text{LMPN}(b_i |_{\sigma_0} a_i)$. In fact, using this requirement, we can get the strict ordering of $a_1, a_2, \dots, a_i, b_1, b_2, \dots, b_i$ in σ_3 as $b_1 \prec_3 a_1 \prec_3$

$$b_2 \prec_3 a_2 \prec_3 \dots \prec_3 b_i \prec_3 a_i.$$

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