

Number of Orbits of Discrete Interval Exchanges

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A new recursive function on discrete interval exchange associated to a composition of length r , and the permutation $\sigma(i) = r - i + 1$ is defined. Acting on a composition c , this recursive function counts the number of orbits of the discrete interval exchange associated to the composition c . Moreover, minimal discrete interval exchanges, *i.e.* the ones having only one orbit, are reduced to the composition $(1, 1)$ which label the root of the Raney tree. Therefore, we describe a generalization of the Raney tree using our recursive function.

Keywords: Interval exchanges, Rauzy induction, Raney tree, orbits

1 Introduction

An interval exchange is a map which cuts an interval into r pieces and reorders them. These maps were introduced by Oseledec in 1966 [11] to generalize circle rotations following some ideas of Arnold [1, 2]. The discrete analogue maps were introduced by Ferenczi and Zamboni in 2013 [7] by replacing the interval by a discrete interval *i.e.* a set of integers between a and b . A discrete interval exchange is called minimal if it has only one orbit. In the same paper, Ferenczi and Zamboni give a bijection between minimal discrete interval exchange and a special subset of words called π -clustering words (see [7] for more details on π -clustering words). A natural question arises from that bijection: Can we identify compositions associated with minimal discrete interval exchanges?

Some partial answers can be found in the literature when the pieces are reordered in reverse order. It is well-known that a discrete interval exchange with 2 pieces is minimal if and only if its composition has coprime parts. Moreover, a discrete interval exchange with 3 pieces $(\lambda_1, \lambda_2, \lambda_3)$ is minimal if and only if $\gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3) = 1$ as shown by Pak and Redlich [12]. We propose a new recursive function on compositions which counts the number of orbits of a discrete interval exchange and agrees with the greatest common divisor recursion when $r = 2$. Other functions that coincide with the greatest common divisor when $r = 2$ are known. For example, Rauzy induction is a generalization of Euclid's algorithm which renormalized intervals exchange and was introduced by Rauzy in 1979 [14]. Other examples are given by the multidimensional continued fraction algorithms (see [9, 16]).

This paper is organized as follows. In Section 2, a formal definition of symmetric discrete interval exchanges is presented. The presentation of the recursive function counting the number of orbits is the main focus of the Section 3. In Section 4, a generalization of the Raney tree where each circular composition

appears is shown. Finally, we describe the number of orbits of any symmetric discrete interval exchange with 3 pieces extending the result given by Pak and Redlich [12]. Moreover, we give the cyclic type of these discrete interval exchanges.

Note that the recursive function counting the number of orbits presented in Theorem 1, Corollary 3 giving the cyclic type of a discrete interval exchange of length 3 and a slightly modified version of the tree of circular compositions were conjectured by Christophe Reutenauer [15].

After submission, we discovered that Karnauhova and Liebscher [8] (2017) have given a similar algorithm as ours (Theorem 1). They work on a geometric generalization of symmetric discrete interval exchanges, called bi-rainbow meanders. Their main contribution is a formula which counts the number of connected components of bi-rainbow meanders; this is the same as counting the number of orbits of symmetric discrete interval exchanges. They do not study the construction, nor the enumeration of connected bi-rainbow meanders, which corresponds in our language to minimal symmetric discrete interval exchanges.

2 Discrete interval exchange

An *integer interval*, denoted by $\llbracket a, b \rrbracket$, is the set $[a, b] \cap \mathbb{Z}$ for some integers a and b . A *composition* of n is a finite sequence $(\lambda_1, \dots, \lambda_r)$ of positive integers whose sum is n . A *permutation* σ of S_r can be written as the *word* $\sigma(1)\sigma(2)\dots\sigma(r)$. Recall that the *cycle notation* of a permutation σ is written as the product of its orbits. For example, the permutation written as the word 451326, is written in cycle notation as $(1, 4, 3)(2, 5)(6)$. The number of orbits of σ is denoted by $\gamma(\sigma)$. A permutation is *circular* if it has exactly one orbit.

We consider a particular case of *discrete interval exchanges*, as they are defined in [7] (as they are defined in [7]: for $\pi \in S_r$, let $\pi(k) = r - k + 1$ for all $k \in \llbracket 1, r \rrbracket$).

Definition 1. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a composition of n . The associated symmetric discrete interval exchange T_λ is defined as follows: let B_1, \dots, B_r be the subintervals of $\llbracket 1, n \rrbracket$ defined by $B_i = \llbracket 1 + \sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \rrbracket$; let

$$s_i = \sum_{j > i} \lambda_j - \sum_{j < i} \lambda_j.$$

Then for all $x \in B_i$, $T_\lambda(x) = x + s_i$. We call s_i the i -th translation parameter and (s_1, \dots, s_r) the translation vector.

The symmetric discrete interval exchange T_λ is a permutation on the set $\llbracket 1, n \rrbracket$. For example, the symmetric discrete interval exchange $T_{(3,5,4,2)}$ is $(1, 12, 6, 9, 3, 14, 2, 13)(4, 7, 10)(5, 8, 11)$ as shown in Figure 1. The subintervals of $\llbracket 1, 14 \rrbracket$ are $B_1 = \{1, 2, 3\}$, $B_2 = \{4, 5, 6, 7, 8\}$, $B_3 = \{9, 10, 11, 12\}$, $B_4 = \{13, 14\}$ and the translation vector of $\llbracket 1, 14 \rrbracket$ is $s = (5 + 4 + 2, 4 + 2 - 3, 2 - 3 - 5, -3 - 5 - 4) = (11, 3, -6, -12)$. A discrete interval exchange is *minimal* if it has only one orbit.

The translation vector of a symmetric discrete interval exchange is strictly decreasing. Moreover, the inverse of a symmetric discrete interval exchange $T_{(\lambda_1, \dots, \lambda_r)}$ is $T_{(\lambda_r, \dots, \lambda_1)}$. Evidently, they both have the same number of orbits.

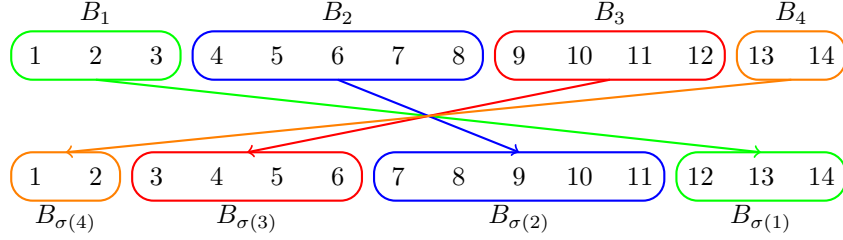


Fig. 1: The symmetric discrete interval exchange $T_{(3,5,4,2)}$

3 Counting Orbits

In this section, we define a recursive function on compositions that counts the number of orbits of a symmetric discrete interval exchange. In fact, the function subtracts the i -th translation parameter to λ_i . To ensure that the result is positive, we first prove that each composition has a part which is greater than its translation parameter.

Lemma 1. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a composition. There exists an integer t in the set $\llbracket 1, r \rrbracket$ such that $\lambda_t \geq |s_t|$.*

Proof: There exists an integer t such that

$$\sum_{i < t} \lambda_i \leq \sum_{i \geq t} \lambda_i \text{ and } \sum_{i \leq t} \lambda_i \geq \sum_{i > t} \lambda_i,$$

since all λ_i are positive integers. By isolating λ_t in both inequalities, we obtain

$$\sum_{i < t} \lambda_i - \sum_{i > t} \lambda_i \leq \lambda_t \text{ and } \lambda_t \geq \sum_{i > t} \lambda_i - \sum_{i < t} \lambda_i,$$

We combine both inequalities to obtain

$$\lambda_t \geq \left| \sum_{i > t} \lambda_i - \sum_{i < t} \lambda_i \right| = |s_t|,$$

by Definition 1. □

Remark: The value t where $\lambda_t \geq |s_t|$ is not unique if $\lambda_t = |s_t|$. For example, in the composition $(2, 1, 2, 1)$ we have that $\lambda_2 \geq |s_2|$ and $\lambda_3 \geq |s_3|$. However, one can show that there are at most two possible values. These properties are proven in Section 4.

Let t be the smallest integer belonging to the set $\llbracket 1, r \rrbracket$ such that $\lambda_t \geq |s_t|$ where (s_1, \dots, s_r) is the translation vector of T_λ . Define a function f mapping from the set of compositions to the natural numbers

λ	(3, 5, 4, 2)	(3, 2, 4, 2)	(3, 2, 1, 2)	(3, 1, 2)	(1, 2)	(1, 1)	(1)
t	2	3	2	1	2	1	
$ s_t $	3	3	0	3	1	1	

Tab. 1: The first row gives the successive steps in the computation of $f(3, 5, 4, 2)$, the second row gives the indices where the recursion for f is applied on the composition and the third row gives the absolute value of the t -th translation parameter.

as

$$f(\lambda_1, \dots, \lambda_r) = \begin{cases} \lambda_1 & \text{if } r = 1, \\ \lambda_t + f(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r) & \text{if } |s_t| = 0, \\ f(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r) & \text{if } \lambda_t = |s_t|, \text{ and} \\ f(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r) & \text{otherwise.} \end{cases} \quad (1)$$

We show that f counts the number of orbits of any symmetric discrete interval exchange i.e. $f(\lambda) = \gamma(T_\lambda)$.

Example 1. The symmetric discrete interval exchange $T_{(3,5,4,2)}$ has 3 cycles as shown previously and

$$\begin{aligned} f(3, 5, 4, 2) &= f(3, 2, 4, 3) = f(3, 2, 1, 2) \\ &= 2 + f(3, 1, 2) = 2 + f(1, 2) = 2 + f(1, 1) = 2 + f(1) \\ &= 3. \end{aligned}$$

The values of t and the corresponding translation parameter at each step are in Table 1.

We say that $(\lambda_1, \dots, \lambda_r) \leq (\beta_1, \dots, \beta_s)$ if $r < s$ or $r = s$ and $(\lambda_1, \dots, \lambda_r) \leq_{lex} (\beta_1, \dots, \beta_s)$, where \leq_{lex} is the lexicographic order i.e. $\lambda_1 < \beta_1$ or $\lambda_1 = \beta_1$ and $(\lambda_2, \dots, \lambda_r) \leq_{lex} (\beta_2, \dots, \beta_s)$. One can check that the composition in the recursive step of f is smaller than the composition.

The next few lemmas show that the operations used in the recursive step of the function f are valid.

Lemma 2. If the value of $|s_t|$ is zero, then the symmetric discrete interval exchange T_λ has

$$\lambda_t + \gamma(T_{(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)})$$

orbits.

For example, $T_{(3,2,4,5)}$ has 5 orbits and its third translation parameter is 0. Moreover, one can check that $T_{(3,2,5)}$ has one orbit, hence $\gamma(T_{(3,2,4,5)}) = 5 = 4 + \gamma(T_{(3,2,5)})$ as predicted.

Proof: Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a composition and t an integer in $\llbracket 1, r \rrbracket$ such that $s_t = 0$. Let x be an integer in the set $\llbracket 1, n \rrbracket$. By definition, the symmetric discrete interval exchange T_λ is

$$T_\lambda(x) = x + s_i \quad (2)$$

where x belongs to B_i with $B_i = \llbracket 1 + \sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \rrbracket$ and $s_i = \sum_{j > i} \lambda_j - \sum_{j < i} \lambda_j$ as in Definition 1.

If $x \in B_t$, then Equation (2) becomes $T_\lambda(x) = x$. Thus, each integer in B_t is a fixed point of T_λ and its orbit contains only itself. Thus $T_\lambda|_{B_t}$ is the identity and T_λ stabilizes $H = \llbracket 1, n \rrbracket \setminus B_t$.

We show that $T_\lambda|H$ is conjugated to $T_{\lambda'}$; that is, $T_\lambda|H = \omega^{-1} \circ T_{\lambda'} \circ \omega$, where ω is the unique decreasing bijection $H \rightarrow \llbracket 1, n - \lambda_t \rrbracket$. This implies that the number of orbits of $T_\lambda|H$ is equal to $\gamma(T_{\lambda'})$.

Suppose that $x \notin B_t$. Let $\lambda' = (\lambda'_1, \dots, \lambda'_{r-1})$ where $\lambda'_i = \lambda_i$ if $i < t$ and $\lambda'_i = \lambda_{i+1}$ otherwise. Note that $T_{\lambda'}$ is the symmetric discrete interval exchange where B'_1, \dots, B'_{r-1} are subintervals of $\llbracket 1, n - \lambda_t \rrbracket$ defined by $B'_i = \llbracket 1 + \sum_{j < i} \lambda'_j, \sum_{j \leq i} \lambda'_j \rrbracket$. Its translation parameters are defined by $s'_i = \sum_{j > i} \lambda'_j - \sum_{j < i} \lambda'_j$. Then for all $x \in B'_i$, $T_{\lambda'}(x) = x + s'_i$. Note that $T_{\lambda'} = T_{(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)}$.

First, we describe the sets B_i using the sets B'_i . We have two cases to consider.

If $i < t$, then

$$B_i = \left[\left[1 + \sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \right] \right] = \left[\left[1 + \sum_{j < i} \lambda'_j, \sum_{j \leq i} \lambda'_j \right] \right] = B'_i$$

since $j \leq i < t$, we know that $\lambda_j = \lambda'_j$.

Otherwise, if $i > t$, then

$$B_i = \left[\left[1 + \sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \right] \right] = \left[\left[1 + \lambda_t + \sum_{j < i} \lambda'_j, \lambda_t + \sum_{j \leq i} \lambda'_j \right] \right] = \{x + \lambda_t \mid x \in B'_i\},$$

since $i > t$ and $\sum_{j < i} \lambda_j = \sum_{j < t} \lambda_j + \sum_{t < j < i} \lambda_j + \lambda_t = \lambda_t + \sum_{i < j} \lambda'_j$.

Secondly, we want to show that $T_\lambda(x) = T_{\lambda'}(x) \pm \lambda_t$. Recall that $T_\lambda(x) = x + \sum_{j > i} \lambda_j - \sum_{j < i} \lambda_j$. We factorize λ_t to obtain

$$T_\lambda(x) = \begin{cases} x + \sum_{j > t} \lambda_j + \sum_{t > j > i} \lambda_j - \sum_{j < i} \lambda_j + \lambda_j & \text{if } i < t \\ x + \sum_{j > i} \lambda_j - \sum_{j < t} \lambda_j - \sum_{t < j < i} \lambda_j - \lambda_j & \text{if } i > t. \end{cases}$$

Replacing λ_j by λ'_j if $j < t$ and by λ'_{j-1} otherwise gives

$$\begin{aligned} T_\lambda(x) &= \begin{cases} x + \sum_{j > i} \lambda'_j - \sum_{j < i} \lambda'_j + \lambda_t & \text{if } i < t \\ x + \sum_{j > i} \lambda'_j - \sum_{j < i} \lambda'_j + \lambda_t & \text{if } i > t, \end{cases} \\ &= \begin{cases} x + s_i + \lambda_t & \text{if } i < t \\ x + s_i - \lambda_t & \text{if } i > t. \end{cases} \end{aligned}$$

Hence, $T_\lambda|H$ is conjugated to $T_{\lambda'}$ and the number of orbits of T_λ is $\lambda_t + \gamma(T_{(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)})$ as claimed. \square

Lemma 3. *Let $(\lambda_1, \dots, \lambda_r)$ be a composition of n . The symmetric discrete interval exchange $T_{(\lambda_1, \dots, \lambda_r)}$ and $T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)}$ have the same number of orbits.*

We begin by understanding how the orbits of $T_{(\lambda_1, \dots, \lambda_r)}$ can be transformed into the orbits of $T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)}$. Through the cycle notation, a symmetric discrete interval exchange can be viewed as a set of circular words on the alphabet $\llbracket 1, n \rrbracket$. Therefore, a substitution on $T_{(\lambda_1, \dots, \lambda_r)}$ describes this operation:

$$\psi_t(x) = \begin{cases} x & \text{if } x \in \llbracket 1, k \rrbracket \\ x \cdot (x + |s_t|) & \text{if } x \in \llbracket k + 1, k + |s_t| \rrbracket \text{ and } s_t > 0 \\ (x + |s_t|) \cdot x & \text{if } x \in \llbracket k + 1, k + |s_t| \rrbracket \text{ and } s_t < 0 \\ x + |s_t| & \text{otherwise} \end{cases} \quad (3)$$

where $x \in \llbracket 1, n \rrbracket$ is a letter, \cdot denotes concatenation and

$$k = \begin{cases} \sum_{j \leq t} \lambda_j & \text{if } s_t \geq 0 \\ \sum_{j < t} \lambda_j - |s_t| & \text{if } s_t < 0. \end{cases}$$

For example, the symmetric discrete interval exchange of $T_{(5,1,2)}$ is $(1, 4, 7)(2, 5, 8)(3, 6)$ and the symmetric discrete interval exchange of $T_{(5,4,2)}$ is $(1, 7, 4, 10)(2, 8, 5, 11)(3, 9, 6)$. Let us apply ψ_2 on $T_{(5,1,2)}$. We have that $t = 2$, $s_t = -3 < 0$ and $k = 2$. For $T_{(5,1,2)}$, the substitution is

$$\psi_2(x) = \begin{cases} x & \text{if } x \in \llbracket 1, 2 \rrbracket \\ (x+3) \cdot x & \text{if } x \in \llbracket 3, 5 \rrbracket \\ x+3 & \text{otherwise.} \end{cases} \quad (4)$$

Hence $\psi_2(T_{(5,1,2)}) = \psi_2((1, 4, 7)(2, 5, 8)(3, 6)) = (1, 7, 4, 10)(2, 8, 5, 11)(6, 3, 9)$ as predicted.

Lemma 4. *Let t be an integer in $\llbracket 1, r \rrbracket$. The identity*

$$\psi_t(T_{(\lambda_1, \dots, \lambda_r)}) = T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)}$$

holds.

Here ψ_t acts as a substitution on the orbits of $T_{(\lambda_1, \dots, \lambda_r)}$. Recall that the circular factors of length 2 of a circular word (a_1, \dots, a_n) are the words $a_i a_{(i+1) \bmod n}$, with $i \in \llbracket 1, n \rrbracket$. We use the following obvious but useful result in the next proof: the orbits of σ and α have the same circular factors of length 2.

Proof: First, suppose that $s_t = 0$. Thus, $\psi_t(T_\lambda) = T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)}$, since ψ_t is the identity morphism when $s_t = 0$.

Let t be an integer in the set $\llbracket 1, r \rrbracket$ and T_λ a symmetric discrete interval exchange as in Definition 1. Let $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ be a composition where $\lambda'_t = \lambda_t + |s_t|$ and $\lambda'_i = \lambda_i$ if $t \neq i$. Recall that $T_{\lambda'}$ is the symmetric discrete interval exchange where B'_1, \dots, B'_r are subintervals of $\llbracket 1, n + |s_t| \rrbracket$ defined by $B'_i = \llbracket 1 + \sum_{j < i} \lambda'_j, \sum_{j \leq i} \lambda'_j \rrbracket$ and its translation parameters are $s'_i = \sum_{j > i} \lambda'_j - \sum_{j < i} \lambda'_j$. Then for all $x \in B'_i$, $T_{\lambda'}(x) = x + s'_i$. Note that the previous definition implies $T_{\lambda'_t} = T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)}$.

We begin by describing the circular factors of length 2 of $T_{\lambda'}$ using λ and B_i instead of λ' and B'_i .

To describe B'_i using the sets B_i , we have three cases to consider. First, if $i < t$, then

$$B'_i = \left[\left[1 + \sum_{j < i} \lambda'_j, \sum_{j \leq i} \lambda'_j \right] \right] = \left[\left[1 + \sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \right] \right] = B_i$$

since $j \leq i < t$, we know that $\lambda'_j = \lambda_j$. Secondly, if $i = t$, then

$$B'_t = \left[\left[1 + \sum_{j < t} \lambda'_j, \sum_{j \leq t} \lambda'_j \right] \right] = \left[\left[1 + \sum_{j < t} \lambda_j, |s_t| + \sum_{j \leq t} \lambda_j \right] \right] = B_t \cup \left[\left[1 + \sum_{j \leq t} \lambda_j, |s_t| + \sum_{j \leq t} \lambda_j \right] \right]$$

as $\lambda'_t = \lambda_t + |s_t|$ and $\lambda'_i = \lambda_i$. Finally, if $i > t$, then

$$B'_i = \left[\left[1 + \sum_{j < i} \lambda'_j, \sum_{j \leq i} \lambda'_j \right] \right] = \left[\left[1 + |s_t| + \sum_{j < i} \lambda_j, |s_t| + \sum_{j \leq i} \lambda_j \right] \right] = \{x + |s_t| \mid x \in B_i\}.$$

We have that $T_{\lambda'}(x) = x + \sum_{j>i} \lambda'_j - \sum_{j<i} \lambda'_j$. We factorize λ'_t to obtain

$$T_{\lambda'}(x) = \begin{cases} x + \sum_{t>j>i} \lambda'_j + \sum_{j>t} \lambda'_j - \sum_{j<i} \lambda'_j + \lambda'_t & \text{if } i < t \\ x + \sum_{j>i} \lambda'_j - \sum_{j<i} \lambda'_j & \text{if } i = t \\ x + \sum_{j>i} \lambda'_j - \sum_{j<t} \lambda'_j - \sum_{t<j<i} \lambda'_j - \lambda'_t & \text{if } i > t. \end{cases}$$

Replacing λ'_i by λ_i if $i \neq t$ and λ'_t by $\lambda_t + |s_t|$ give that

$$\begin{aligned} T_{\lambda'}(x) &= \begin{cases} x + \sum_{t>j>i} \lambda_j + \sum_{j>t} \lambda_j - \sum_{j<i} \lambda_j + \lambda_t + |s_t| & \text{if } i < t \\ x + \sum_{j>i} \lambda_j - \sum_{j<i} \lambda_j & \text{if } i = t \\ x + \sum_{j>i} \lambda_j - \sum_{j<t} \lambda_j - \sum_{t<j<i} \lambda_j + \lambda_t - |s_t| & \text{if } i > t, \end{cases} \\ &= \begin{cases} x + s_i + |s_t| & \text{if } i < t \\ x + s_i & \text{if } i = t \\ x + s_i - |s_t| & \text{if } i > t. \end{cases} \end{aligned}$$

Therefore, the circular factors of length 2 of $T_{\lambda'}$ are $x \cdot (x + s_i + |s_t|)$ if $x \in B_i$ and $i < t$, $x \cdot (x + s_i)$ if $x \in B_t$ and $x \cdot (x + s_i - |s_t|)$, otherwise.

Now, it remains to find the circular factors of length 2 of $\psi(T_\lambda)$. There are two cases to analyze depending on the translation parameter s_t .

Let x be an integer in $\llbracket 1, n \rrbracket$. There exist an integer i such that $x \in B_i$.

Suppose that s_t is positive. Thus, we have that $k = \sum_{j \leq t} \lambda_j$.

If $i < t$, we have that $x \in \llbracket 1, \sum_{j < t} \lambda_j \rrbracket$, then $x < k$. Also, we have that $T_{(\lambda_1, \dots, \lambda_r)}(x) > k + |s_t|$, since $\sum_{j > t} \lambda_j = s_t + \sum_{j < t} \lambda_j$. Therefore, we obtain that $\psi_t(x \cdot T_\lambda(x)) = x \cdot (T_\lambda(x) + |s_t|) = x \cdot (x + s_i + |s_t|)$. Since $i < t$, we know that $x \in B_i$ implies that $x \in B'_i$ and $T_{\lambda'}(x) = x + s_i + |s_t|$, the element x is sent to the same element in both symmetric discrete interval exchanges, as desired.

If $i = t$, we have that $x \in \llbracket 1 + \sum_{j < t} \lambda_j, \sum_{j \leq t} \lambda_j \rrbracket$, then $x < k$. Moreover, we have $T_{(\lambda_1, \dots, \lambda_r)}(x) \in \llbracket \sum_{j < t} \lambda_j + 1, k + |s_t| \rrbracket$, since $\sum_{j > t} \lambda_j = s_t + \sum_{j < t} \lambda_j$. Therefore, we obtain that

$$\psi_t(x \cdot T_\lambda(x)) = \begin{cases} x \cdot (x + s_i) & \text{if } x < k \\ x \cdot (x + s_i) \cdot (x + s_i + |s_t|) & \text{otherwise.} \end{cases}$$

Which means that all elements of B_t are sent to $(x + s_i)$, as desired.

If $i > t$, we have that $x \in \llbracket 1 + \sum_{j \leq t} \lambda_j, n \rrbracket$, then $x > k$. Moreover, we have that $T_\lambda(x) < k + |s_t|$, since $\sum_{j \leq t} \lambda_j > |s_t| + \sum_{j > t} \lambda_j$. Therefore, we obtain that

$$\psi_t(x \cdot T_\lambda(x)) = \begin{cases} x \cdot (x + |s_t|) & \text{if } x \leq k + |s_t| \text{ and } \sigma(x) \leq k, \\ x \cdot (x + |s_t|) \cdot T_\lambda(x) \cdot (T_\lambda(x) + |s_t|) & \text{if } x \leq k + |s_t| \text{ and } \sigma(x) > k, \\ (x + |s_t|) \cdot T_\lambda(x) & \text{if } x > k + |s_t| \text{ and } T_\lambda(x) \leq k, \\ (x + |s_t|) \cdot T_\lambda(x) \cdot (T_\lambda(x) + |s_t|) & \text{if } x > k + |s_t| \text{ and } T_\lambda(x) > k. \end{cases} \quad (5)$$

Hence, we have that every element between $k+1$ and $k+|s_t|$ is sent to $x + s_i$. Thus, all elements of B'_t are sent to the correct value. Moreover, every element of B'_i with $i > t$ are of the form $x + |s_t|$ corresponding to

the last two cases in Equation (5). The circular factors of length 2 are $(x+|s_t|) \cdot T_\lambda(x) = (x+|s_t|) \cdot (x+s_i)$, which can be rewritten to $y \cdot (y + s_i - |s_t|)$ if we substitute y to $x + |s_t|$, as desired.

If s_t is negative, a similar case-by-case analysis shows the equality. \square

Now, we are ready to prove Lemma 3.

Proof of Lemma 3: By Lemma 4, we have that

$$\psi_t(T_{(\lambda_1, \dots, \lambda_r)}) = T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)},$$

for all $t \in \llbracket 1, r \rrbracket$. As a substitution on a set of circular words preserves the number of circular words, we have that

$$\gamma(T_{(\lambda_1, \dots, \lambda_r)}) = \gamma(\psi_t(\lambda_1, \dots, \lambda_r)) = \gamma(T_{(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)})$$

as claimed. \square

Theorem 1. *The function f counts the orbits of the symmetric discrete interval exchange T_λ .*

Proof: By Lemma 1, we know there exists an integer $t \in \llbracket 1, r \rrbracket$ such that $\lambda_t \geq |s_t|$ for any composition λ . Also, $(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r)$ and $(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)$ are both compositions, since $\lambda_t - |s_t| \geq 0$. Hence, the function f is well-defined.

We show case by case that the recursive step of the function f preserves the number of orbits. First, $\gamma(\lambda_1) = \lambda_1$, since $T_{\lambda_1}(x) = x$ for all $x \in \llbracket 1, \lambda_1 \rrbracket$. Secondly, if $|s_t| = 0$, we know that $\gamma(\lambda_1, \dots, \lambda_r) = \lambda_t + \gamma(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)$ by Lemma 2. Thirdly, if $|s_t| = \lambda_t$, we know that $\gamma(\lambda_1, \dots, \lambda_r) = \gamma(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r)$ by Lemma 3. However, $\lambda_t = |s_t|$ implies $(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r) = (\lambda_1, \dots, \lambda_{t-1}, 0, \lambda_{t+1}, \dots, \lambda_r) = \gamma(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)$ as desired. Finally, if $\lambda_t > |s_t|$, the equality $\gamma(\lambda_1, \dots, \lambda_r) = \gamma(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r)$ follows directly from Lemma 3. Thus, the function f counts the number of orbits of T_λ . \square

Corollary 1. *A symmetric discrete interval exchange is minimal if and only if $f(\lambda_1, \dots, \lambda_r) = 1$.*

4 Tree of circular compositions

How can we use the function f to enumerate the compositions defining minimal symmetric discrete interval exchange? For minimal symmetric discrete 2-interval exchanges, solutions are given by the Raney tree [3, 13] (also called Calkin-Wilf tree [5]) or the Stern-Brocot tree [4, 17]. A composition λ is called *circular* if the symmetric discrete interval exchange T_λ is minimal. We propose a generalization of the Raney tree to enumerate circular compositions of any lengths.

The *Raney tree* is a complete infinite binary tree whose nodes are reduced fractions, described recursively by the following rules :

- the root of the tree is the fraction $\frac{1}{1}$;
- each vertex $\frac{i}{j}$ has two children: the left child is the fraction $\frac{i}{i+j}$ and the right child is the $\frac{i+j}{j}$.

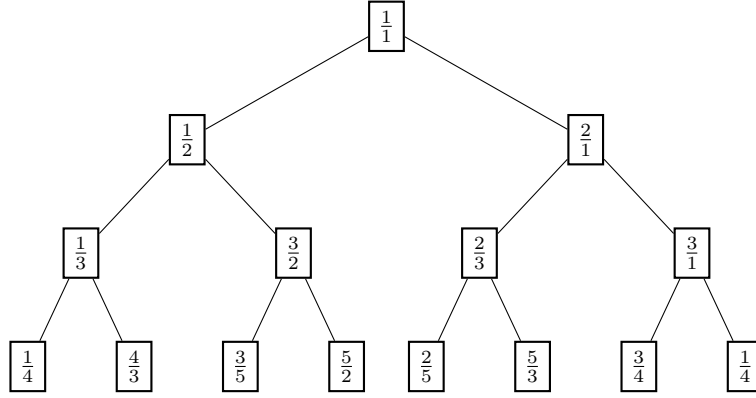


Fig. 2: Raney tree

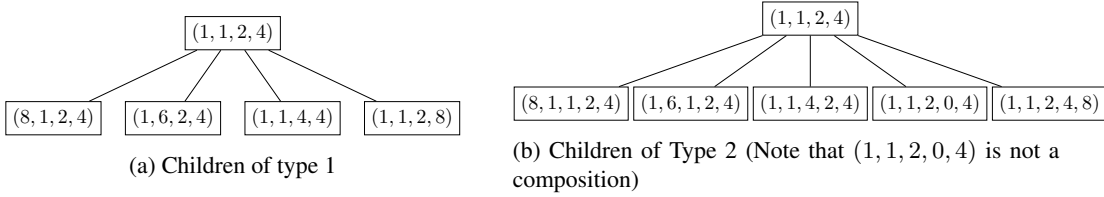


Fig. 3: Children of Type 1 and 2 of the composition $(1, 1, 2, 4)$ whose translation vector is $(7, 5, 2, -4)$ and $\delta = (8, 6, 4, 0, -8)$

Figure 2 shows the first few levels of the Raney tree. This tree contains each reduced positive rational number exactly once as shown by Berstel and de Luca [3], and by Calkin and Wilf [5]. Thus it contains all pairs of coprime natural numbers. Therefore, the Raney tree contains every circular composition of length 2 exactly once.

Observe that applying the function f to any vertex in the Raney tree gives the path from that vertex to the root $1/1$ in the Raney tree. Therefore, we can use the function f to generalize the Raney tree. The root of this generalized tree is still $(1, 1)$, since it is an extension of the Raney tree. A circular composition $(\lambda_1, \dots, \lambda_r)$ has children of the form $(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)$ for all $t \in \llbracket 1, r \rrbracket$ and $(\lambda_1, \dots, \lambda_t, |\delta_t|, \lambda_{t+1}, \dots, \lambda_r)$ for all $t \in \llbracket 0, r \rrbracket$, where $\delta_t = n - 2 \sum_{j \leq t} \lambda_j$. Figure 3 shows all the possible children of the composition $(1, 1, 2, 4)$. Note that $|s_t|$ cannot be zero, otherwise the composition is not circular by Lemma 2. Moreover, the children of $(\lambda_1, \dots, \lambda_r)$ of the form $(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)$ are circular by Lemma 3. There are two problems with the children of $(\lambda_1, \dots, \lambda_r)$ of the form $(\lambda_1, \dots, \lambda_t, |\delta_t|, \lambda_{t+1}, \dots, \lambda_r)$. First, if $\delta_t = 0$ (e.g. δ_3 in Figure 3), the result is the same symmetric discrete interval exchange. We want to avoid this case as it does not add any new information. Secondly, the resulting composition has two parents. For example, the parents of the circular composition $(1, 1, 2, 4)$ are the compositions $(1, 1, 4)$ ($\delta = (6, 4, 2, -6)$) and $(1, 1, 2)$ ($\delta = (4, 2, 0, -4)$) (as shown in Figure 4).

Adding δ_t preserves the number of orbits of the symmetric discrete interval exchange, since the $(t +$

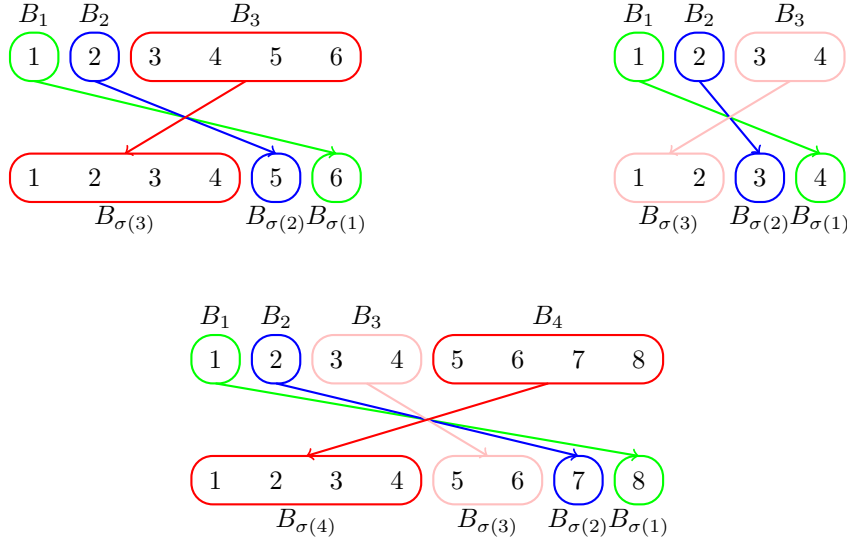


Fig. 4: The top right discrete interval is $(1, 1, 4)$ and the top left one is $(1, 1, 2)$ both can be the parent of $(1, 1, 2, 4)$. The δ of $(1, 1, 4)$ is $(6, 4, 2, -6)$ and the δ of $(1, 1, 2)$ is $(4, 2, 0, -4)$. Hence, $(1, 1, |\delta_2|, 2) = (1, 1, 2, 4) = (1, 1, 2, |\delta_4|)$

1)-th translation parameter of the composition $(\lambda_1, \dots, \lambda_t, |\delta_t|, \lambda_{t+1}, \dots, \lambda_r)$ is by definition $s_{t+1} = \sum_{j>t} \lambda_j - \sum_{j \leq t} \lambda_j = n - 2 \sum_{j \leq t} \lambda_j = \delta_t$. By Lemma 3, these children of $(\lambda_1, \dots, \lambda_r)$ are also circular. Compositions constructed by the two rules have two parents, since adding δ_t or λ_t if $\delta_t > 0$ (resp. λ_{t+1} if $\delta_t < 0$) produces the same composition (see Figure 4). To ensure that every child is a composition and that each composition has a single parent, we add a condition on δ .

The *tree of circular compositions* is an infinite tree described by the following recursive rules:

- the root of the tree is the composition $(1, 1)$;
- each vertex $(\lambda_1, \dots, \lambda_r)$ has children of the form:

Type 1: $(\lambda_1, \dots, \lambda_t + |s_t|, \dots, \lambda_r)$ for all $t \in \llbracket 1, r \rrbracket$;

Type 2: if $\delta_t > \lambda_{t+1}$ or $-\delta_t > \lambda_t$, $(\lambda_1, \dots, \lambda_t, |\delta_t|, \lambda_{t+1}, \dots, \lambda_r)$ for all $t \in \llbracket 0, r \rrbracket$;

where $\delta_t = n - \sum_{j \leq t} \lambda_j$.

Figure 5 shows the first few levels of the tree of circular compositions. Note that $\delta_0 > \lambda_1$ and $-\delta_r > \lambda_r$, hence the condition is always satisfied in those cases. Thus, the parent of $(1, 1, 2, 4)$ is $(1, 1, 2)$, since $-\delta_3 = 4 > \lambda_3$ and $(1, 1, 4)$ is not its parent since $\delta_3 = 2 < \lambda_4 = 4$ and $-\delta_3 = -2 < \lambda_3 = 2$. Now we prove that each circular composition is a child of type 1 or 2 but not both and has a unique parent.

Lemma 5. *Let $(\lambda_1, \dots, \lambda_r)$ be a circular composition and t an integer in $\llbracket 1, r \rrbracket$. If $\lambda_t > |s_t|$, then $\lambda_i < |s_i|$ for all $i \in \llbracket 1, r \rrbracket, i \neq t$.*

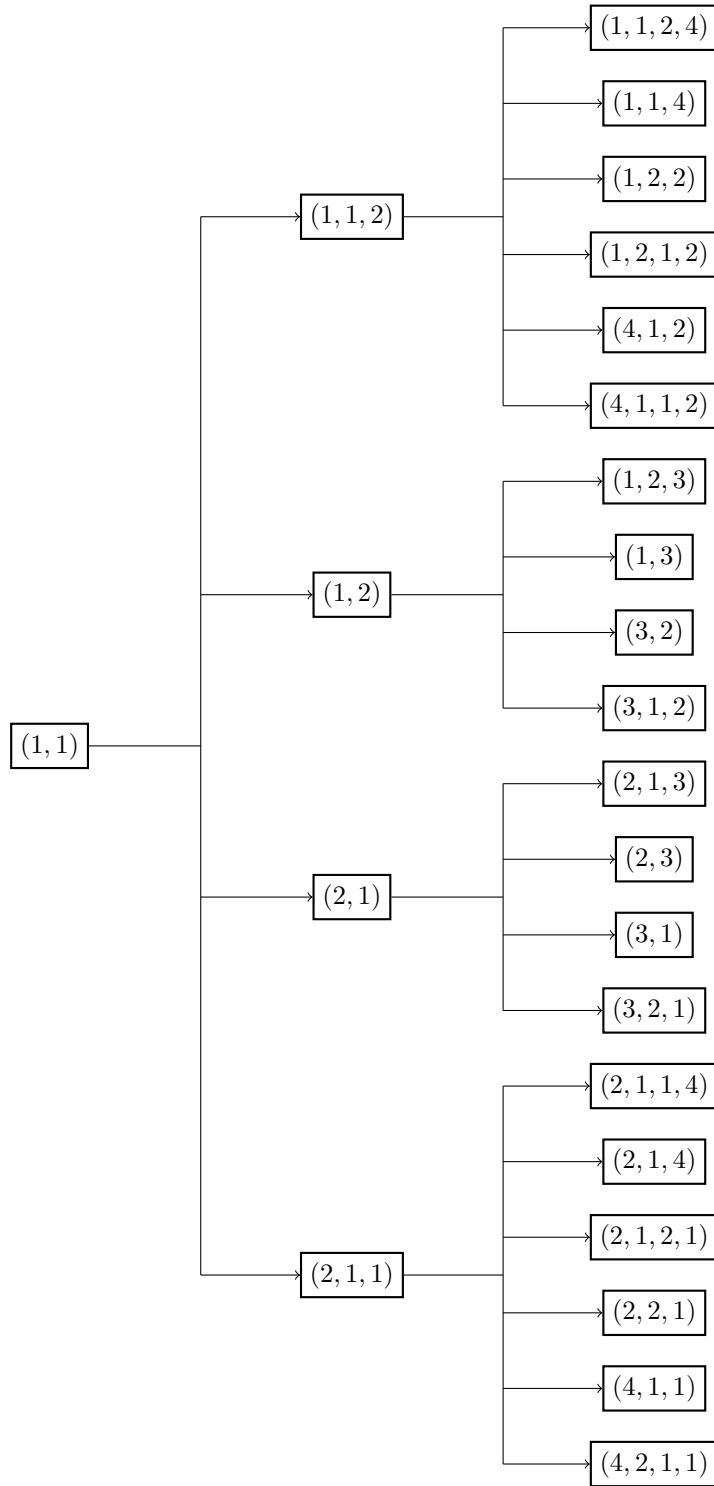


Fig. 5: The tree of circular compositions

Proof: Let t be an integer in $\llbracket 1, r \rrbracket$ such that $\lambda_t > |s_t|$. First, we show that $s_i > 0$ if $i < t$ (resp. $s_i < 0$ if $i > t$). Let i be an integer in the set $\llbracket 1, t-1 \rrbracket$.

If $s_t > 0$, by the definition of s_t the following inequality is satisfied:

$$\sum_{k>t} \lambda_k > \sum_{k<t} \lambda_k.$$

Since $i < t$ and all λ_k 's are positive, we have that

$$\sum_{k>i} \lambda_k > \sum_{k>t} \lambda_k > \sum_{k<t} \lambda_k > \sum_{k<i} \lambda_k.$$

Hence $s_i = \sum_{k>i} \lambda_k - \sum_{k<i} \lambda_k > 0$.

If $s_t < 0$, by the definition of s_t , we know that

$$\lambda_t > |s_t| = -s_t = \sum_{k<t} \lambda_k - \sum_{k>t} \lambda_k.$$

Moreover, adding $\sum_{k>t} \lambda_k$ on both sides, we have that

$$\lambda_t + \sum_{k>t} \lambda_k > \sum_{k<t} \lambda_k.$$

Since $i < t$ and all λ_k 's are positive, the previous inequality implies that

$$\sum_{k>i} \lambda_k \geq \lambda_t + \sum_{k>t} \lambda_k > \sum_{k<t} \lambda_k > \sum_{k<i} \lambda_k.$$

Hence, $s_i = \sum_{k>i} \lambda_k - \sum_{k<i} \lambda_k > 0$ as claimed.

Similar arguments show that, if $i > t$, then $s_i < 0$.

Now we can show that $\lambda_i < |s_i|$ if $i \neq t$. There are two cases to consider.

Suppose that $s_t > 0$. If $i < t$, the translation parameter s_i can be expressed as a sum of s_t and some λ_k 's given below:

$$\begin{aligned} s_t + \lambda_t + 2 \sum_{i<j<t} \lambda_j + \lambda_i &= \sum_{j>t} \lambda_j - \sum_{j<t} \lambda_j + \lambda_t + 2 \sum_{i<j<t} \lambda_j + \lambda_i \\ &= \sum_{j>i} \lambda_j - \sum_{j<i} \lambda_j \\ &= s_i. \end{aligned}$$

Moreover, we have that $s_i = s_t + \lambda_t + 2 \sum_{i<j<t} \lambda_j + \lambda_i > \lambda_i$ since s_t is positive. Thus, $|s_i| = s_i > \lambda_i$ since $i < t$.

If $i > t$, the translation parameter s_i can be expressed by s_t and some λ_k as below:

$$\begin{aligned} s_t - \lambda_t - 2 \sum_{t<j<i} \lambda_j - \lambda_i &= \sum_{j>t} \lambda_j - \sum_{j<t} \lambda_j - \lambda_t - 2 \sum_{t<j<i} \lambda_j - \lambda_i \\ &= \sum_{j>i} \lambda_j - \sum_{j<i} \lambda_j \\ &= s_i. \end{aligned}$$

Since $i > t$, we have that $s_i < 0$, hence $|s_i| = -s_t + \lambda_t + 2 \sum_{t < j < i} \lambda_j + \lambda_i$ which is a sum of positive integers since $\lambda_t - s_t > 0$. Hence, $|s_i| > \lambda_i$ as claimed.

Similar arguments show that the lemma also holds in the case where s_t is negative. \square

Lemma 6. *Let $(\lambda_1, \dots, \lambda_r)$ be a circular composition and t be an integer in $\llbracket 1, r \rrbracket$. If $\lambda_t = |s_t|$, then there exists an integer $d \in \{t-1, t+1\}$ such that $\lambda_d = |s_d|$ and $\lambda_i < |s_i|$ for all $i \in \llbracket 1, r \rrbracket - \{t, d\}$. Moreover $\lambda_d \neq \lambda_t$.*

Proof: Let $(\lambda_1, \dots, \lambda_r)$ be a circular composition such that $\lambda_t = |s_t|$. First, we check that λ_d exists.

If $s_t > 0$, then

$$\lambda_t = \sum_{k>t} \lambda_k - \sum_{k<t} \lambda_k \quad \Rightarrow \quad \lambda_{t+1} = \sum_{k<t+1} \lambda_k - \sum_{k>t+1} \lambda_k = |s_{t+1}|.$$

Similarly, if $s_t < 0$, we have that $\lambda_{t-1} = |s_{t-1}|$.

Secondly, we have to show that $\lambda_t \neq \lambda_d$. Suppose that we are in the case $d = t+1$. Suppose further that $\lambda_t = \lambda_d$. Since $\lambda_t = |s_t|$, we have that $\gamma(\lambda_1, \dots, \lambda_r) = \gamma(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)$ by Theorem 1. Moreover, $\lambda_t = \sum_{k>t} \lambda_k - \sum_{k<t} \lambda_k$ and $\lambda_t = \lambda_{t+1}$ imply that $0 = \sum_{k>t+1} \lambda_k - \sum_{k<t} \lambda_k$. Hence, $\gamma(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r) = \lambda_{t+1} + \gamma(\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+2}, \dots, \lambda_r)$ by Lemma 2. Thus $\gamma(\lambda_1, \dots, \lambda_r)$ is greater than 1 since $\lambda_{t+1} > 0$, a contradiction. The case where $d = t-1$ is similar.

Finally, we have to check that $\lambda_i < |s_i|$ if $i \neq d$ and $i \neq t$. Suppose that $s_t > 0$, then $d = t+1$. If $i < t$, then we have that $\lambda_t = s_t = \sum_{k>t} \lambda_k - \sum_{k<t} \lambda_k$. We isolate λ_i in this equation to obtain $\lambda_i = -\sum_{j<i} \lambda_j - \sum_{i+1<j \leq t} \lambda_j + \sum_{j>t} \lambda_j$. Moreover, we have that

$$\begin{aligned} \lambda_i &= -\sum_{k<i} \lambda_k - \sum_{i+1<k \leq t} \lambda_k + \sum_{k>t} \lambda_k + s_i - s_i \\ &= s_i - 2 \sum_{i+1<k \leq t} \lambda_k \\ &< s_i \end{aligned}$$

since all λ_k 's are positive.

Similar arguments show that if $i > t$, then $\lambda_i < |s_i|$. To complete the proof, we use a similar argument to show that $\lambda_i < |s_i|$ if $s_t < 0$. \square

Corollary 2. *Let $(\lambda_1, \dots, \lambda_r)$ be a circular composition. If the composition is not $(1, 1)$, then it has a unique parent in the tree of circular composition.*

Proof: There exists a $t \in \llbracket 1, r \rrbracket$ such that $\lambda_t \geq |s_t|$ by Lemma 1 and s_t is positive, since the composition is circular by Lemma 2. Therefore, the composition $(\lambda_1, \dots, \lambda_r)$ is a child of type 1 or type 2 of the composition λ' .

Suppose that $(\lambda_1, \dots, \lambda_r)$ is a child of type 1. We know that $\lambda_i < |s_i|$ for all $i \in \llbracket 1, r \rrbracket - \{t\}$ by Lemma 5. Thus $(\lambda_1, \dots, \lambda_r)$ is not a child of type 2 and the only valid position to reduce the composition is t . So the parent of $(\lambda_1, \dots, \lambda_r)$ is $\lambda' = (\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r)$.

Suppose that $(\lambda_1, \dots, \lambda_r)$ is a child of type 2. We know that $\lambda_i < |s_i|$ for all $i \in \llbracket 1, r \rrbracket - \{t, d\}$. Thus $(\lambda_1, \dots, \lambda_r)$ is not a child of type 1, since any $\lambda_i \leq |s_i|$. If $\delta_t > \lambda_{t+1}$ or $-\delta_t > \lambda_t$, then the parent

of $(\lambda_1, \dots, \lambda_r)$ is $\lambda' = (\lambda_1, \dots, \lambda_{t-1}, \lambda_{t+1}, \dots, \lambda_r)$, otherwise it is $\lambda' = (\lambda_1, \dots, \lambda_t, \lambda_{t+2}, \dots, \lambda_r)$ if $\delta_t > 0$, or $\lambda' = (\lambda_1, \dots, \lambda_{t-2}, \lambda_t, \dots, \lambda_r)$ if $\delta_t < 0$. \square

Theorem 2. *Every circular composition appears exactly once on the tree of circular composition.*

Proof: First, we show that every composition in the tree is circular. The symmetric discrete interval exchange $T_{(1,1)}$ is the permutation $(1, 2)$ which is obviously minimal. Let $(\lambda_1, \dots, \lambda_r)$ be a composition that appears in the tree which is not associated with a circular permutation and such that all compositions appearing in previous levels of the tree are circular. There exists an integer $t \in \llbracket 1, r \rrbracket$ such that $\lambda_t \geq |s_t|$ and

$$\gamma(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r) = \gamma(\lambda_1, \dots, \lambda_r) > 1,$$

by Theorem 1. The composition $(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r)$ appears in a previous level of the tree, which is a contradiction. Thus, all the compositions in the tree are circular.

Secondly, we prove that all circular compositions appear in the tree. The composition $(1, 1)$ is the root of the tree. Let S be the set of all circular compositions which are not in the tree and $(\lambda_1, \dots, \lambda_r)$ the smallest composition in S ordered them as in Section 3. There exists a natural number $t \in \llbracket 1, r \rrbracket$ such that $\lambda_t \geq |s_t|$ and

$$\gamma(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r) = \gamma(\lambda_1, \dots, \lambda_r)$$

by Lemma 1. Therefore the composition $(\lambda_1, \dots, \lambda_t - |s_t|, \dots, \lambda_r)$ cannot be an element of the tree, otherwise $(\lambda_1, \dots, \lambda_r)$ is also an element of the tree contradicting the minimality of $(\lambda_1, \dots, \lambda_r)$.

Thirdly, we show that no circular composition occurs more than once in the tree. Let $(\lambda_1, \dots, \lambda_r)$ be the smallest composition which appears more than once in the tree. By Corollary 2, we know that $(\lambda_1, \dots, \lambda_r)$ has a unique parent λ' . Then $(\lambda_1, \dots, \lambda_r)$ is the child of two distinct vertices both labelled by λ' , contradicting the minimality of $(\lambda_1, \dots, \lambda_r)$. Therefore, each circular composition appears only once in the tree. \square

5 Cyclic type

The *cyclic type* of a symmetric discrete interval exchange is a partition $\ell_1^{\alpha_1} \dots \ell_k^{\alpha_k}$ where $\ell_1 > \ell_2 > \dots > \ell_k$, ℓ_i is the length of an orbit of T_λ and α_i is the number of orbits of length ℓ_i . For example, the cyclic type of $T_{(3,5,4,2)}$ is the partition $8^1 3^2$. The cyclic type of symmetric discrete 2-interval exchanges is already known.

Lemma 7. (folklore) *Let (λ_1, λ_2) be a composition. The cyclic type of the symmetric discrete interval exchange $T_{(\lambda_1, \dots, \lambda_r)}$ is the partition*

$$\left(\frac{\lambda_1 + \lambda_2}{\gcd(\lambda_1, \lambda_2)} \right)^{\gcd(\lambda_1, \lambda_2)}.$$

The cyclic type of symmetric discrete 3-interval exchange can also be described by the gcd function.

Lemma 8. *Let $(\lambda_1, \lambda_2, \lambda_3)$ be a composition. The integers s_1 , s_2 and s_3 are multiples of $\gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)$.*

Proof: The result is straightforward for $s_1 = \lambda_2 + \lambda_3$ and $s_3 = -\lambda_1 - \lambda_2$. Moreover, subtracting $\lambda_2 + \lambda_3$ from $\lambda_1 + \lambda_2$, we obtain that $\gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3) = \gcd(\lambda_1 - \lambda_3, \lambda_2 + \lambda_3)$. Thus $s_2 = |\lambda_3 - \lambda_1|$ is a multiple of $\gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)$. \square

Now, we can describe the number of orbits of a symmetric discrete 3-interval exchange using the $\gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)$.

Theorem 3. *The orbit of $x \in \llbracket 1, n \rrbracket$ in the symmetric discrete interval exchange $T_{(\lambda_1, \lambda_2, \lambda_3)}$ is the set*

$$\left\{ x + kd \mid k \in \mathbb{N} \text{ and } \frac{1-x}{d} \leq k \leq \frac{n-x}{d} \right\},$$

where $d = \gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)$.

For example, the cyclic decomposition of the symmetric discrete interval exchange of $T_{(9,1,4)}$ is $(1, 6, 11)(2, 7, 12)(3, 8, 14)(4, 9, 14)$. Therefore, the orbit of 9 is $\{9 + 5k \mid -8/5 \leq k \leq 5/5\} = \{4, 9, 14\}$.

Proof: Let d be the $\gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)$ and x be an integer in $\llbracket 1, n \rrbracket$. First, we show by induction that $T_{(\lambda_1, \lambda_2, \lambda_3)}^{(k)}(x) = x + id$. We know that $T_{(\lambda_1, \lambda_2, \lambda_3)}^0(x) = x + 0 \cdot d$ and $T_{(\lambda_1, \lambda_2, \lambda_3)}^1(x) = x + s_i = x + id$ since s_i is a multiple of d as shown in Lemma 8. Let computes

$$T_{(\lambda_1, \lambda_2, \lambda_3)}^{(k+1)}(x) = T(T_{(\lambda_1, \lambda_2, \lambda_3)}^k(x)) = T(x + id) = x + id + s_i$$

by induction. Moreover, s_i is a multiple of d by Lemma 8. Hence, we have that $T_{(\lambda_1, \lambda_2, \lambda_3)}^{(k+1)}(x) = x + jd$.

By the definition of $T_{(\lambda_1, \lambda_2, \lambda_3)}$, we have that $1 \leq x + kd \leq n$ which gives that $(1-x)/d \leq k \leq (n-x)/d$. \square

Corollary 3. *Let $(\lambda_1, \lambda_2, \lambda_3)$ be a composition of $n = \lambda_1 + \lambda_2 + \lambda_3$. Take $d = \gcd(\lambda_1 + \lambda_2, \lambda_2 + \lambda_3)$. Let q and r be integers with $0 \leq r < d$ such that $n = qd + r$. The cyclic type of $T_{(\lambda_1, \lambda_2, \lambda_3)}$ is q^d if $r = 0$ and $(q+1)^r q^{d-r}$ otherwise. Moreover $d = \gamma(T_{(\lambda_1, \lambda_2, \lambda_3)})$.*

Proof: By Theorem 3, we have that the orbit of x is the set

$$\left\{ x + kd \mid k \in \mathbb{N} \text{ and } \frac{1-x}{d} \leq k \leq \frac{n-x}{d} \right\}.$$

Therefore, every element in the orbit of x is congruent to x modulo d . Hence, the number of orbit is the number of classes modulo d , which is exactly d . The cardinality of $\gamma(x)$ is $\lceil n/d \rceil$ if $x \bmod d < r$ or $\lfloor n/d \rfloor + 1$ if $x \bmod d \geq r$. \square

The symmetric discrete interval exchanges T_λ where λ is a composition of length 2 or 3 has simple cyclic type. For symmetric discrete interval exchanges of length 4 or more, no such description is known. As a matter of fact, it does not seem to have any relation between the length of the orbits in the cyclic type. However computer testing suggests that if $\ell_1^{\alpha_1} \dots \ell_k^{\alpha_k}$ is the cyclic type of $T_{(\lambda_1, \dots, \lambda_r)}$, then $k \leq \lceil \frac{r-1}{2} \rceil$.

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