Colouring the Square of the Cartesian Product of Trees
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We prove upper and lower bounds on the chromatic number of the square of the cartesian product of trees. The bounds are equal if each tree has even maximum degree.

Keywords: cartesian product, colouring, square graph

1 Introduction

This paper studies colourings of the square of cartesian products of trees. For simplicity we assume that a tree has at least one edge.

For our purposes, a colouring of a graph $G$ is a function $c : V(G) \rightarrow \mathbb{Z}$ such that $c(v) \neq c(w)$ for every edge $vw$ of $G$. The square graph $G^2$ of $G$ has vertex set $V(G)$, where two vertices are adjacent in $G^2$ whenever they are adjacent in $G$ or have a common neighbour in $G$. Thus, a colouring of $G^2$ corresponds to a colouring of $G$, such that in addition, vertices with a common neighbour in $G$ are assigned distinct colours.

Let $[a, b] := \{a, a + 1, \ldots, b\}$. The cartesian product of graphs $G_1, \ldots, G_d$ is the graph $G_1 \square \cdots \square G_d$ with vertex set $\{(v_1, \ldots, v_d) : v_i \in V(G_i)\}$, where vertices $v = (v_1, \ldots, v_d)$ and $w = (w_1, \ldots, w_d)$ are adjacent whenever $v_i w_i \in E(G_i)$ for some $i \in [1, d]$, and $v_j = w_j$ for all $j \neq i$. In this case, $vw$ is in dimension $i$. Let $\Delta(G)$ be the maximum degree of $G$.

**Theorem 1** Let $T_1, \ldots, T_d$ be trees. Let $G := T_1 \square T_2 \square \cdots \square T_d$. Then

$$1 + \sum_{i=1}^{d} \Delta(T_i) \leq \chi(G^2) \leq 1 + 2 \sum_{i=1}^{d} \left\lceil \frac{1}{2} \Delta(T_i) \right\rceil.$$  

This upper bound improves upon a similar bound by [Jamison et al., 2006], who proved $\chi(G^2) \leq 1 + 2 \sum_{i=1}^{d} (\Delta(T_i) - 1)$, assuming that each $\Delta(T_i) \geq 2$. Theorem 1 implies:

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Corollary 1 Let $T_1, \ldots, T_d$ be trees, such that $\Delta(T_i)$ is even for all $i \in [1, d]$. Let $G := T_1 \square T_2 \square \cdots \square T_d$. Then

$$\chi(G^2) = 1 + \sum_{i=1}^d \Delta(T_i).$$

This corollary generalises a result of [Fertin et al., 2003], who proved it when each $T_i$ is a path, and thus $G$ is a $d$-dimensional grid. See [Sopena and Wu, 2010; Fertin et al., 2004; Por and Wood, 2009; Jamison et al., 2006; Chiang and Yan, 2008] for more related results.

2 The Proof

For a colouring $c$ of a graph $G$, the span of an edge $vw$ of $G$ is $|c(v) - c(w)|$. The following lemma is well known; see [Por and Wood, 2009] for example.

Lemma 1 Let $G$ be a graph. If $G^2$ has a colouring in which every edge of $G$ has span at most $s$, then $G^2$ is $(2s + 1)$-colourable.

Proof: Let $c : V(G^2) \to \mathbb{Z}$ be the given colouring of $G^2$. Since every edge of $G$ has span at most $s$, every edge of $G^2$ has span at most $2s$. Let $c'(v) := c(v) \mod (2s + 1)$ for each vertex $v$. Then $c'(v) \neq c'(w)$ for each edge $vw$ of $G^2$. Thus $G^2$ is $(2s + 1)$-colourable.

Lemma 2 For every tree $T$ and non-negative integer $s$, $T^2$ has a colouring such that every edge of $T$ has span in $[s + 1, s + \lfloor \frac{1}{2} \Delta(T) \rfloor]$.

Proof: We proceed by induction on $|V(T)|$. If $|V(T)| = 2$ the result is trivial. Now assume that $|V(T)| \geq 3$. Let $v$ be a leaf vertex of $T$. Let $w$ be the neighbour of $v$. By induction, $(T - v)^2$ has a colouring $c$ such that every edge of $T - v$ has span in $[s + 1, s + \lfloor \frac{1}{2} \Delta(T) \rfloor]$. Let

$$X := \{x \in \mathbb{Z} : x \in [s + 1, s + \lfloor \frac{1}{2} \Delta(T) \rfloor] \}.$$

Each neighbour of $w$ in $T - v$ is coloured $c(w) + x$ for some $x \in X$. Since $|X| \geq \Delta(T)$ and $w$ has degree less than $\Delta(T)$ in $T - v$, for some $x \in X$, no neighbour of $w$ is coloured $c(w) + x$. Set $c(v) := c(w) + x$. Thus $|c(v) - c(w)| = |x| \in [s + 1, s + \lfloor \frac{1}{2} \Delta(T) \rfloor]$. No two neighbours of $w$ receive the same colour. Hence $c$ is the desired colouring of $T$.

Proof of Theorem[1] The lower bound is well known [Jamison et al., 2006]. In particular, for $i \in [1, d]$, let $v_i$ be a vertex of maximum degree in $T_i$. Then $(v_1, \ldots, v_d)$ has degree $\sum_i \Delta(T_i)$ in $G$. This vertex and its neighbours in $G$ receive distinct colours in any colouring of $G^2$. Thus $\chi(G^2) \geq 1 + \sum_i \Delta(T_i)$.

Now we prove the upper bound. Let $s_1 := 0$ and $s_i := \sum_{j=1}^{i-1} \lfloor \frac{1}{2} \Delta(T_j) \rfloor$. By Lemma 2, $T^2_d$ has a colouring $c_i$ such that every edge of $T_i$ has span in $[s_i + 1, s_i + \lfloor \frac{1}{2} \Delta(T_i) \rfloor]$. Thus the spans of edges in distinct trees are distinct.

Colour each vertex $v = (v_1, \ldots, v_d)$ of $G$ by $c(v) := \sum_{i=1}^d c_i(v_i)$.

Suppose on the contrary that $c(v) = c(w)$ for some edge $vw$ of $G$. Say $vw$ is in dimension $i$. Thus $v_j = w_j$ for all $j \neq i$. Hence $c_i(v_i) = c_i(w_i)$, and $c_i$ is not a colouring of $G$. This contradiction proves that $c$ is a colouring of $G$. 

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Suppose on the contrary that \( c(x) = c(y) \) for two vertices \( x \) and \( y \) with a common neighbour \( v \) in \( G \). Say \( vx \) is in dimension \( i \), and \( vy \) is in dimension \( j \). Thus \( v_\ell = x_\ell \) for all \( \ell \neq i \), and \( v_\ell = y_\ell \) for all \( \ell \neq j \). Now \( c_i(x) - c_i(v) = c(x) - c(v) = c(y) - c(v) = c_j(y) - c_j(v_j) \). Thus the edges \( x, v_i \) and \( y, v_j \) have the same span. Since the spans of edges in distinct trees are distinct, \( i = j \). Hence \( c_i(x_i) = c_i(y_i) \). However, \( v_i \) is a common neighbour of \( x_i \) and \( y_i \) in \( T_i \), implying \( c_i \) is not a colouring of \( T_i^2 \). This contradiction proves that \( c \) is a colouring of \( G^2 \).

Each edge of \( G \) has span at most \( \sum_{i=1}^{d} \lceil \frac{1}{2} \Delta(T_i) \rceil \). The result follows from Lemma 1. \( \square \)

References


