On the Book Thickness of $k$-Trees

Vida Dujmović$^1$ and David R. Wood$^2$

$^1$School of Computer Science, Carleton University, Ottawa, Canada  
$^2$Department of Mathematics and Statistics, The University of Melbourne, Melbourne, Australia

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Every $k$-tree has book thickness at most $k + 1$, and this bound is best possible for all $k \geq 3$. Vandenbussche et al. [SIAM J. Discrete Math., 2009] proved that every $k$-tree that has a smooth degree-3 tree decomposition with width $k$ has book thickness at most $k$. We prove this result is best possible for $k \geq 4$, by constructing a $k$-tree with book thickness $k + 1$ that has a smooth degree-4 tree decomposition with width $k$. This solves an open problem of Vandenbussche et al.

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1 Introduction

Consider a drawing of a graph $G$ in which the vertices are represented by distinct points on a circle in the plane, and each edge is a chord of the circle between the corresponding points. Suppose that each edge is assigned one of $k$ colours such that crossing edges receive distinct colours. This structure is called a $k$-page book embedding of $G$: one can also think of the vertices as being ordered along the spine of a book, and the edges that receive the same colour being drawn on a single page of the book without crossings. The book thickness of $G$, denoted by $bt(G)$, is the minimum integer $k$ for which there is a $k$-page book embedding of $G$. Book embeddings, first defined by Olumann (1973), are ubiquitous structures with a variety of applications; see (Dujmović and Wood, 2004) for a survey with over 50 references. A book embedding is also called a stack layout, and book thickness is also called stacknumber, pagenumber and fixed outerthickness.

This paper focuses on the book thickness of $k$-trees. A vertex $v$ in a graph $G$ is $k$-simplicial if its neighbourhood, $N_G(v)$, is a $k$-clique. For $k \geq 1$, a $k$-tree is a graph $G$ such that either $G \simeq K_{k+1}$, or $G$ has a $k$-simplicial vertex $v$ and $G - v$ is a $k$-tree. In the latter case, we say that $G$ is obtained from $G - v$ by adding $v$ onto the $k$-clique $N_G(v)$.
What is the maximum book thickness of a $k$-tree? Observe that 1-trees are precisely the trees. [Bernhart and Kainen (1979)] proved that every 1-tree has a 1-page book embedding. In fact, a graph has a 1-page book embedding if and only if it is outerplanar. [Bernhart and Kainen (1979)] showed that 2-trees are the edge-maximal series-parallel graphs. [Rengarajan and Veni Madhavan (1995)] proved that every series parallel graph, and thus every 2-tree, has a 2-page book embedding (also see [Di Giacomo et al. 2006]). This bound is best possible, since $K_{2,3}$ is series parallel and is not outerplanar. [Ganley and Heath (2001)] proved that every $k$-tree has a $(k+1)$-page book embedding; see [Dujmović and Wood (2007)] for an alternative proof. Ganley and Heath (2001) also conjectured that every $k$-tree has a $k$-page book embedding. This conjecture was refuted by Dujmović and Wood (2007), who constructed a $k$-tree with book thickness $k+1$ for all $k \geq 3$. Vandenbussche et al. (2009) independently proved the same result. Therefore the maximum book thickness of a $k$-tree is $k$ for $k \leq 2$ and is $k+1$ for $k \geq 3$.

Which families of $k$-trees have $k$-page book embeddings? [Togasaki and Yamazaki (2002)] proved that every graph with pathwidth $k$ has a $k$-page book embedding (and there are graphs with pathwidth $k$ and book thickness $k$). This result is equivalent to saying that every $k$-tree that has a smooth degree-2 tree decomposition of width $k$ has a $k$-page book embedding. [Vandenbussche et al. (2009)] extended this result by showing that every $k$-tree that has a smooth degree-3 tree decomposition of width $k$ has a $k$-page book embedding. [Vandenbussche et al. (2009)] then introduced the following natural definition. Let $m(k)$ be the maximum integer $d$ such that every $k$-tree of width $k$ has a smooth degree-$d$ tree decomposition of width $k$. [Vandenbussche et al. (2009)] proved that $3 \leq m(k) \leq k+1$, and state that determining $m(k)$ is an open problem. However, it is easily seen that the $k$-tree with book thickness $k+1$ constructed in [Dujmović and Wood (2007)] has a smooth degree-5 tree decomposition with width $k$. Thus $m(k) \leq 4$ for all $k \geq 3$. The main result of this note is to refine the construction in [Dujmović and Wood (2007)] to give a $k$-tree with book thickness $k+1$ that has a smooth degree-4 tree decomposition with width $k$ for all $k \geq 4$. This proves that $m(k) = 3$ for all $k \geq 4$. It is open whether $m(3) = 3$ or 4. We conjecture that $m(3) = 3$.

2 Construction

**Theorem 1** For all $k \geq 4$ and $n \geq 11(2k^2+1)+k$, there is an $n$-vertex $k$-tree $Q$, such that $bt(Q) = k+1$ and $Q$ has a smooth degree-4 tree decomposition of width $k$.

**Proof:** Start with the complete split graph $K_{k,2k^2+1}^*$. That is, $K_{k,2k^2+1}^*$ is the $k$-tree obtained by adding a set $S$ of $2k^2+1$ vertices onto a $k$-clique $K = \{u_1, u_2, \ldots, u_k\}$, as illustrated in Figure 1. For each vertex $v \in S$ add a vertex onto the $k$-clique $(K \cup \{v\}) \setminus \{u_1\}$. Let $T$ be the set of vertices added in this step. For each vertex $w \in T$, if $v$ is the neighbour of $w$ in $S$, then add a set $T_2(w)$ of three simplicial vertices onto the $k$-clique $(K \cup \{v, w\}) \setminus \{u_1, u_2\}$, add a set $T_3(w)$ of three simplicial vertices onto the $k$-clique $(K \cup \{v, w\}) \setminus \{u_1, u_3\}$, and add a set $T_4(w)$ of three simplicial vertices onto the $k$-clique $(K \cup \{v, w\}) \setminus \{u_1, u_4\}$. This step is well defined since $k \geq 4$. For each vertex $w \in T$, let $T(w) := T_2(w) \cup T_3(w) \cup T_4(w)$. By construction, $Q$ is a $k$-tree, and as illustrated in Figure 2, $Q$ has a smooth degree-4 tree decomposition of width $k$.

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(iii) See [Diestel (2000)] for the definition of tree decomposition and treewidth. Note that $k$-trees are the edge maximal graphs with treewidth $k$. A tree decomposition of width $k$ is smooth if every bag has size exactly $k+1$ and any two adjacent bags have exactly $k$ vertices in common. Any tree decomposition of a graph $G$ can be converted into a smooth tree decomposition of $G$ with the same width. A tree decomposition is degree-$d$ if the host tree has maximum degree at most $d$. 

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It remains to prove that $\text{bt}(Q) \geq k + 1$. Suppose, for the sake of contradiction, that $Q$ has a $k$-page book embedding. Say the edge colours are $1, 2, \ldots, k$. For each ordered pair of vertices $v, w \in V(Q)$, let $\hat{v} \hat{w}$ be the list of vertices in clockwise order from $v$ to $w$ (not including $v$ and $w$).

Say $K = (u_1, u_2, \ldots, u_k)$ in anticlockwise order. Since there are $2k^2 + 1$ vertices in $S$, by the pigeonhole principle, without loss of generality, there are at least $2k + 1$ vertices in $S \cap \hat{u}_1 \hat{u}_k$. Let $(v_1, v_2, \ldots, v_{2k+1})$ be $2k + 1$ vertices in $S \cap \hat{u}_1 \hat{u}_k$ in clockwise order.

Observe that the $k$ edges $\{u_iv_{k-i+1} : 1 \leq i \leq k\}$ are pairwise crossing, and thus receive distinct colours, as illustrated in Figure 3(a). Without loss of generality, each $u_iv_{k-i+1}$ is coloured $i$. As illustrated in Figure 3(b), this implies that $u_1v_{2k+1}$ is coloured 1, since $u_3v_{2k+1}$ crosses all of $\{u_iv_{k-i+1} : 2 \leq i \leq k\}$ which are coloured $2, 3, \ldots, k$. As illustrated in Figure 3(c), this in turn implies that $u_2v_{2k}$ is coloured 2, and so on. By an easy induction, $u_iv_{2k+2-i}$ is coloured $i$ for each $i \in \{1, 2, \ldots, k\}$, as illustrated in Figure 3(d). It follows that for all $i \in \{1, 2, \ldots, k\}$ and $j \in \{k-i+1, k-i+2, \ldots, 2k+2-i\}$, the edge $u_iv_j$ is coloured $i$, as illustrated in Figure 3(e). Moreover, as illustrated in Figure 3(f):

![Figure 1: The complete split graph $K^*_4$](image)

![Figure 2: A smooth degree-4 tree decomposition of $Q$](image)
If \( qu_i \in E(Q) \) and \( q \in v_kv_{k+2} \), then \( qu_i \) is coloured \( i \).

**(a)**

![Diagram](a)

**(b)**

![Diagram](b)

**(c)**

![Diagram](c)

**(d)**

![Diagram](d)

**(e)**

![Diagram](e)

**(f)**

![Diagram](f)

Fig. 3: Illustration of the proof of Theorem 1 with \( k = 4 \).

Note that the argument up to now is the same as in [Dujmović and Wood, 2007]. Let \( w \) be the vertex in \( T \) adjacent to \( v_{k+1} \). Recall that \( w \) is adjacent to each vertex in \( K \setminus \{u_1\} \). Vertex \( w \) is in \( v_kv_{k+2} \), as otherwise the edge \( uv_{k+1} \) crosses \( k \) edges of \( Q[\{v_k, v_{k+2}\}; K] \) that are all coloured differently. Without loss of generality, \( w \) is in \( v_kv_{k+1} \). Each vertex \( x \in T(w) \) is in \( v_kv_{k+1} \), as otherwise \( xw \) crosses \( k \) edges in \( Q[\{v_k, v_{k+1}\}; K] \) that are all coloured differently. Therefore, all nine vertices in \( T(w) \) are in \( v_kv_{k+1} \). By the pigeonhole principle, at least one of \( vkw \) or \( uv_{k+1} \) contains two vertices from \( T_i(w) \) and two vertices from \( T_j(w) \) for some \( i, j \in \{2, 3, 4\} \) with \( i \neq j \). Let \( x_1, x_2, x_3, x_4 \) be these four vertices in clockwise order in \( v_kv_{k+1} \).

**Case 1.** \( x_1, x_2, x_3, x_4 \) are in \( vkw \): By \( (*) \), the edges in \( Q[\{w\}; K] \) are coloured \( 2, 3, \ldots, k \). Thus \( x_2v_{k+1} \), which crosses all the edges in \( Q[\{w\}; K] \), is coloured 1. At least one of the vertices in \( \{x_2, x_3, x_4\} \) is adjacent to \( K \setminus \{u_1, u_i\} \) and at least one to \( K \setminus \{u_1, u_j\} \). Thus, by \( (*) \), the edges in \( Q[\{x_2, x_3, x_4\}; K] \) are coloured \( 2, 3, \ldots, k \). Thus \( x_1w \), which crosses all the edges of \( Q[\{x_2, x_3, x_4\}; K] \) is coloured 1. Thus \( x_2v_{k+1} \) and \( x_1w \) cross and are both coloured 1, which is the desired contradiction.

**Case 2.** \( x_1, x_2, x_3, x_4 \) are in \( uv_{k+1} \): As in Case 1, the edges in \( Q[\{x_2, x_3, x_4\}; K] \) are coloured \( 2, 3, \ldots, k \). Thus \( x_1v_{k+1} \), which crosses all the edges in \( Q[\{x_2, x_3, x_4\}; K] \), is coloured 2. Since the edges in \( Q[\{x_1, x_2, x_3\}; K] \) are coloured \( 2, 3, \ldots, k \), the edge \( x_3w \), which crosses all the edges of
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$Q[\{x_1, x_2, x_3\}; K]$, is coloured 1. Thus $x_1v_{k+1}$ and $x_4w$ cross and are both coloured 1, which is the desired contradiction.

Finally, observe that $|V(Q)| = |K| + |S| + |T| + \sum_{w \in Q} |T(w)| = |K| + 11|S| = k + 11(2k^2 + 1)$. Adding more $k$-simplicial vertices to $Q$ does not reduce its book thickness. Moreover, it is simple to verify that the graph obtained from $Q$ by adding simplicial vertices onto $K$ has a smooth degree-4 tree decomposition of width $k$. Thus for all $n \geq 11(2k^2 + 1) + k$, there is a $k$-tree $G$ with $n$ vertices and $\text{bt}(G) = k + 1$ that has the desired tree decomposition.

3 Final Thoughts

For $k \geq 3$, the minimum book thickness of a $k$-tree is $\lceil \frac{k+1}{2} \rceil$ (since every $k$-tree contains $K_{k+1}$, and $\text{bt}(K_{k+1}) = \lceil \frac{k+1}{2} \rceil$; see (Bernhart and Kainen, 1979)). However, we now show that the range of book thicknesses of sufficiently large $k$-trees is very limited.

**Proposition 1** Every $k$-tree $G$ with at least $\frac{1}{2} k(k + 1)$ vertices has book thickness $k - 1$, $k$ or $k + 1$.

**Proof:** Ganley and Heath (2001) proved that $\text{bt}(G) \leq k + 1$. It remains to prove that $\text{bt}(G) \geq k - 1$ assuming $|V(G)| \geq \frac{1}{2} k(k + 1)$. Numerous authors (Bernhart and Kainen, 1979; Cottafava and D’Antona, 1984; Keys, 1975) observed that $|E(G)| < (\text{bt}(G) + 1)|V(G)|$ for every graph $G$. Thus

$$(k - 1)|V(G)| \leq k|V(G)| - \frac{1}{2} k(k + 1) = |E(G)| < (\text{bt}(G) + 1)|V(G)|.$$ 

Hence $k - 1 < \text{bt}(G) + 1$. Since $k$ and $\text{bt}(G)$ are integers, $\text{bt}(G) \geq k - 1$. □

We conclude the paper by discussing some natural open problems regarding the computational complexity of calculating the book thickness for various classes of graphs.

Proposition 1 begs the question: Is there a characterisation of the $k$-trees with book thickness $k - 1$, $k$ or $k + 1$? And somewhat more generally, is there a polynomial-time algorithm to determine the book thickness of a given $k$-tree? Note that the $k$-th power of paths are an infinite class of $k$-trees with book thickness $k - 1$; see (Swaminathan et al., 1995).

$k$-trees are the edge-maximal chordal graphs with no $(k + 2)$-clique, and also are the edge-maximal graphs with treewidth $k$. Is there a polynomial-time algorithm to determine the book thickness of a given chordal graph? Is there a polynomial-time algorithm to determine the book thickness of a given graph with bounded treewidth?
References


