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On the Book Thickness of k-Trees

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Every k-tree has book thickness at most $k+1$, and this bound is best possible for all $k \geq 3$. Vandenbussche et al. [*SIAM J. Discrete Math.*, 2009] proved that every k-tree that has a smooth degree-3 tree decomposition with width k has book thickness at most k. We prove this result is best possible for $k \geq 4$, by constructing a k-tree with book thickness $k+1$ that has a smooth degree-4 tree decomposition with width k. This solves an open problem of Vandenbussche et al.

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1 Introduction

Consider a drawing of a graph[(i)] G in which the vertices are represented by distinct points on a circle in the plane, and each edge is a chord of the circle between the corresponding points. Suppose that each edge is assigned one of k colours such that crossing edges receive distinct colours. This structure is called a k-*page book embedding* of G: one can also think of the vertices as being ordered along the spine of a book, and the edges that receive the same colour being drawn on a single page of the book without crossings. The *book thickness* of G, denoted by $\mathsf{bt}(G)$, is the minimum integer k for which there is a k-page book embedding of G. Book embeddings, first defined by Ollmann (1973), are ubiquitous structures with a variety of applications; see (Dujmović and Wood, 2004) for a survey with over 50 references. A book embedding is also called a *stack layout*, and book thickness is also called *stacknumber*, *pagenumber* and *fixed outerthickness*.

This paper focuses on the book thickness of k-trees. A vertex v in a graph G is k-*simplicial* if its neighbourhood, $N_G(v)$, is a k-clique. For $k \geq 1$, a k-*tree* is a graph G such that either $G \simeq K_{k+1}$, or G has a k-simplicial vertex v and $G - v$ is a k-tree. In the latter case, we say that G is obtained from $G - v$ by *adding* v *onto* the k-clique $N_G(v)$.

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(i) We consider simple, finite, undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. We employ standard graph-theoretic terminology; see (Diestel, 2000). For disjoint $A, B \subseteq V(G)$, let $G[A;B]$ denote the bipartite subgraph of G with vertex set $A \cup B$ and edge set $\{vw \in E(G) : v \in A, w \in B\}$.

What is the maximum book thickness of a k-tree? Observe that 1-trees are precisely the trees. Bernhart and Kainen (1979) proved that every 1-tree has a 1-page book embedding. In fact, a graph has a 1-page book embedding if and only if it is outerplanar (Bernhart and Kainen, 1979). 2-trees are the edge-maximal series-parallel graphs. Rengarajan and Veni Madhavan (1995) proved that every series parallel graph, and thus every 2-tree, has a 2-page book embedding (also see (Di Giacomo et al., 2006)). This bound is best possible, since $K_{2,3}$ is series parallel and is not outerplanar. Ganley and Heath (2001) proved that every k-tree has a $(k+1)$-page book embedding; see (Dujmović and Wood, 2007) for an alternative proof. Ganley and Heath (2001) also conjectured that every k-tree has a k-page book embedding. This conjecture was refuted by Dujmović and Wood (2007), who constructed a k-tree with book thickness $k+1$ for all $k \geq 3$. Vandenbussche et al. (2009) independently proved the same result. Therefore the maximum book thickness of a k-tree is k for $k \leq 2$ and is $k+1$ for $k \geq 3$.

Which families of k-trees have k-page book embeddings? Togasaki and Yamazaki (2002) proved that every graph with pathwidth k has a k-page book embedding (and there are graphs with pathwidth k and book thickness k). This result is equivalent to saying that every k-tree that has a smooth degree-2 tree decomposition[(ii)] of width k has a k-page book embedding. Vandenbussche et al. (2009) extended this result by showing that every k-tree that has a smooth degree-3 tree decomposition of width k has a k-page book embedding. Vandenbussche et al. (2009) then introduced the following natural definition. Let $m(k)$ be the maximum integer d such that every k-tree that has a smooth degree-d tree decomposition of width k has a k-page book embedding. Vandenbussche et al. (2009) proved that $3 \leq m(k) \leq k+1$, and state that determining $m(k)$ is an open problem. However, it is easily seen that the k-tree with book thickness $k+1$ constructed in (Dujmović and Wood, 2007) has a smooth degree-5 tree decomposition with width k. Thus $m(k) \leq 4$ for all $k \geq 3$. The main result of this note is to refine the construction in (Dujmović and Wood, 2007) to give a k-tree with book thickness $k+1$ that has a smooth degree-4 tree decomposition with width k for all $k \geq 4$. This proves that $m(k) = 3$ for all $k \geq 4$. It is open whether $m(3) = 3$ or 4. We conjecture that $m(3) = 3$.

2 Construction

Theorem 1 *For all $k \geq 4$ and $n \geq 11(2k^2+1)+k$, there is an n-vertex k-tree Q, such that $\mathsf{bt}(Q) = k+1$ and Q has a smooth degree-4 tree decomposition of width k.*

Proof: Start with the complete split graph $K^\star_{k,2k^2+1}$. That is, $K^\star_{k,2k^2+1}$ is the k-tree obtained by adding a set S of $2k^2+1$ vertices onto a k-clique $K = \{u_1, u_2, \dots, u_k\}$, as illustrated in Figure 1. For each vertex $v \in S$ add a vertex onto the k-clique $(K \cup \{v\}) \setminus \{u_1\}$. Let T be the set of vertices added in this step. For each $w \in T$, if v is the neighbour of w in S, then add a set $T_2(w)$ of three simplicial vertices onto the k-clique $(K \cup \{v, w\}) \setminus \{u_1, u_2\}$, add a set $T_3(w)$ of three simplicial vertices onto the k-clique $(K \cup \{v, w\}) \setminus \{u_1, u_3\}$, and add a set $T_4(w)$ of three simplicial vertices onto the k-clique $(K \cup \{v, w\}) \setminus \{u_1, u_4\}$. This step is well defined since $k \geq 4$. For each $w \in T$, let $T(w) := T_2(w) \cup T_3(w) \cup T_4(w)$. By construction, Q is a k-tree, and as illustrated in Figure 2, Q has a smooth degree-4 tree decomposition of width k.

[(ii)] See (Diestel, 2000) for the definition of tree decomposition and treewidth. Note that k-trees are the edge maximal graphs with treewidth k. A tree decomposition of width k is *smooth* if every bag has size exactly $k+1$ and any two adjacent bags have exactly k vertices in common. Any tree decomposition of a graph G can be converted into a smooth tree decomposition of G with the same width. A tree decomposition is *degree-d* if the host tree has maximum degree at most d.

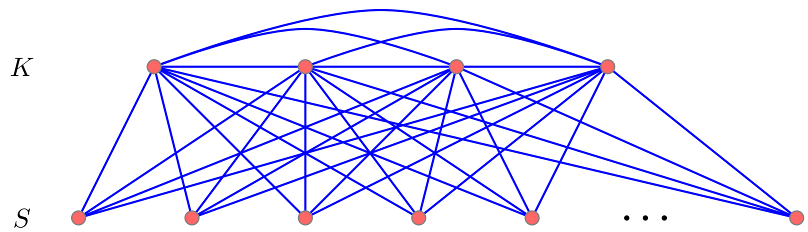

Fig. 1: The complete split graph $K^{\star}_{4,|S|}$.

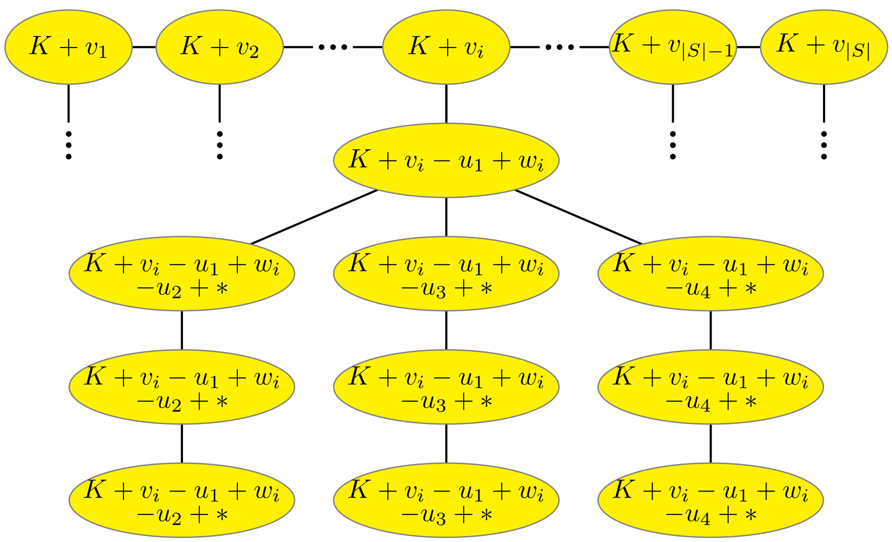

Fig. 2: A smooth degree-4 tree decomposition of Q.

It remains to prove that $\mathsf{bt}(Q) \geq k+1$. Suppose, for the sake of contradiction, that Q has a k-page book embedding. Say the edge colours are $1, 2, \dots, k$. For each ordered pair of vertices $v, w \in V(Q)$, let $\widehat{vw}$ be the list of vertices in clockwise order from v to w (not including v and w).

Say $K = (u_1, u_2, \dots, u_k)$ in anticlockwise order. Since there are $2k^2 + 1$ vertices in S, by the pigeonhole principle, without loss of generality, there are at least $2k+1$ vertices in $S \cap \widehat{u_1 u_k}$. Let $(v_1, v_2, \dots, v_{2k+1})$ be $2k+1$ vertices in $S \cap \widehat{u_1 u_k}$ in clockwise order.

Observe that the k edges $\{u_i v_{k-i+1} : 1 \leq i \leq k\}$ are pairwise crossing, and thus receive distinct colours, as illustrated in Figure 3(a). Without loss of generality, each $u_i v_{k-i+1}$ is coloured i. As illustrated in Figure 3(b), this implies that $u_1 v_{2k+1}$ is coloured 1, since $u_1 v_{2k+1}$ crosses all of $\{u_i v_{k-i+1} : 2 \leq i \leq k\}$ which are coloured $2, 3, \dots, k$. As illustrated in Figure 3(c), this in turn implies that $u_2 v_{2k}$ is coloured 2, and so on. By an easy induction, $u_i v_{2k+2-i}$ is coloured i for each $i \in \{1, 2, \dots, k\}$, as illustrated in Figure 3(d). It follows that for all $i \in \{1, 2, \dots, k\}$ and $j \in \{k-i+1, k-i+2, \dots, 2k+2-i\}$, the edge $u_i v_j$ is coloured i, as illustrated in Figure 3(e). Moreover, as illustrated in Figure 3(f):

If $qu_i \in E(Q)$ and $q \in \widehat{v_k v_{k+2}}$, then qu_i is coloured i. $(\star)$

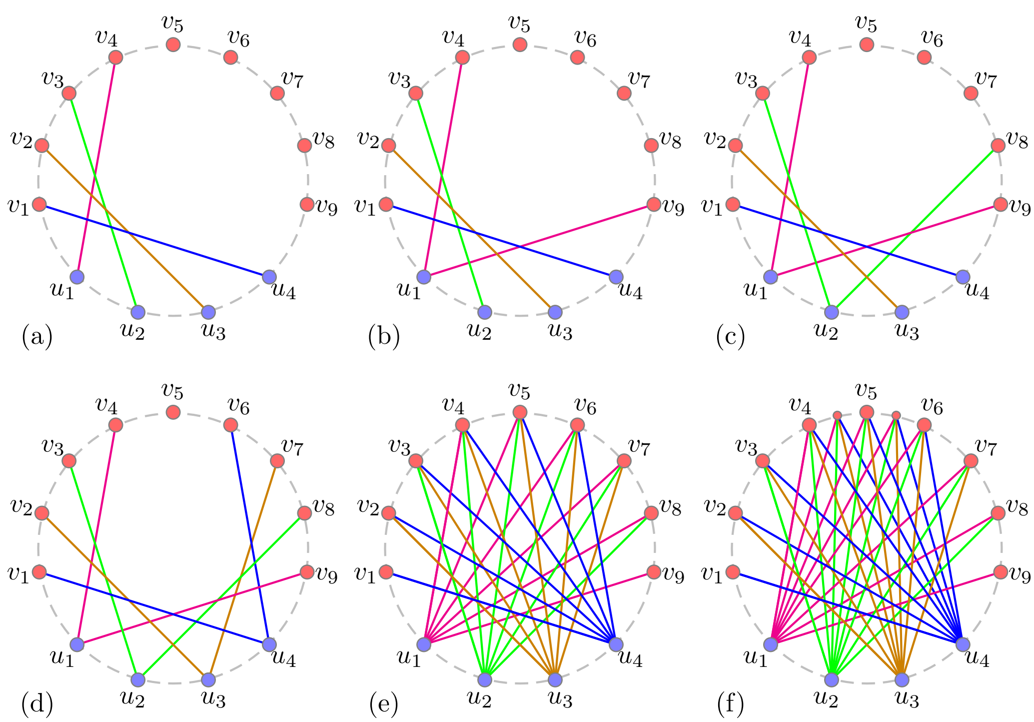

Fig. 3: Illustration of the proof of Theorem 1 with $k = 4$.

Note that the argument up to now is the same as in (Dujmović and Wood, 2007). Let w be the vertex in T adjacent to v_{k+1}. Recall that w is adjacent to each vertex in $K \setminus \{u_1\}$. Vertex w is in $\widehat{v_k v_{k+2}}$, as otherwise the edge wv_{k+1} crosses k edges of $Q[\{v_k, v_{k+2}\}; K]$ that are all coloured differently. Without loss of generality, w is in $\widehat{v_k v_{k+1}}$. Each vertex $x \in T(w)$ is in $\widehat{v_k v_{k+1}}$, as otherwise xw crosses k edges in $Q[\{v_k, v_{k+1}\}; K]$ that are all coloured differently. Therefore, all nine vertices in $T(w)$ are in $\widehat{v_k v_{k+1}}$. By the pigeonhole principle, at least one of $\widehat{v_k w}$ or $\widehat{w v_{k+1}}$ contains two vertices from $T_i(w)$ and two vertices from $T_j(w)$ for some $i, j \in \{2, 3, 4\}$ with $i \neq j$. Let x_1, x_2, x_3, x_4 be these four vertices in clockwise order in $\widehat{v_k w}$ or $\widehat{w v_{k+1}}$.

Case 1. x_1, x_2, x_3 and x_4 are in $\widehat{v_k w}$: By $(\star)$, the edges in $Q[\{w\}; K]$ are coloured $2, 3, \dots, k$. Thus $x_2 v_{k+1}$, which crosses all the edges in $Q[\{w\}; K]$, is coloured 1. At least one of the vertices in $\{x_2, x_3, x_4\}$ is adjacent to $\{K \setminus \{u_1, u_i\}\}$ and at least one to $\{K \setminus \{u_1, u_j\}\}$. Thus, by $(\star)$, the edges in $Q[\{x_2, x_3, x_4\}; K]$ are coloured $2, 3, \dots, k$. Thus $x_1 w$, which crosses all the edges of $Q[\{x_2, x_3, x_4\}; K]$ is coloured 1. Thus $x_2 v_{k+1}$ and $x_1 w$ cross and are both coloured 1, which is the desired contradiction.

Case 2. x_1, x_2, x_3 and x_4 are in $\widehat{w v_{k+1}}$: As in Case 1, the edges in $Q[\{x_2, x_3, x_4\}; K]$ are coloured $2, 3, \dots, k$. Thus $x_1 v_{k+1}$, which crosses all the edges in $Q[\{x_2, x_3, x_4\}; K]$, is coloured 1. Since the edges in $Q[\{x_1, x_2, x_3\}; K]$ are coloured $2, 3, \dots, k$, the edge $x_4 w$, which crosses all the edges of

$Q[\{x_1, x_2, x_3\}; K]$, is coloured 1. Thus $x_1 v_{k+1}$ and $x_4 w$ cross and are both coloured 1, which is the desired contradiction.

Finally, observe that $|V(Q)| = |K| + |S| + |T| + \sum_{w \in Q} |T(w)| = |K| + 11|S| = k + 11(2k^2 + 1)$. Adding more k-simplicial vertices to Q does not reduce its book thickness. Moreover, it is simple to verify that the graph obtained from Q by adding simplicial vertices onto K has a smooth degree-4 tree decomposition of width k. Thus for all $n \geq 11(2k^2 + 1) + k$, there is a k-tree G with n vertices and $\mathsf{bt}(G) = k + 1$ that has the desired tree decomposition. □

3 Final Thoughts

For $k \geq 3$, the minimum book thickness of a k-tree is $\lceil \frac{k+1}{2} \rceil$ (since every k-tree contains K_{k+1}, and $\mathsf{bt}(K_{k+1}) = \lceil \frac{k+1}{2} \rceil$; see (Bernhart and Kainen, 1979)). However, we now show that the range of book thicknesses of sufficiently large k-trees is very limited.

Proposition 1 *Every k-tree G with at least $\frac{1}{2}k(k+1)$ vertices has book thickness $k-1$, k or $k+1$.*

Proof: Ganley and Heath (2001) proved that $\mathsf{bt}(G) \leq k + 1$. It remains to prove that $\mathsf{bt}(G) \geq k - 1$ assuming $|V(G)| \geq \frac{k(k+1)}{2}$. Numerous authors (Bernhart and Kainen, 1979; Cottafava and D'Antona, 1984; Keys, 1975) observed that $|E(G)| < (\mathsf{bt}(G) + 1)|V(G)|$ for every graph G. Thus

$$(k-1)|V(G)| \leq k|V(G)| - \tfrac{1}{2}k(k+1) = |E(G)| < (\mathsf{bt}(G) + 1)|V(G)| \ .$$

Hence $k - 1 < \mathsf{bt}(G) + 1$. Since k and $\mathsf{bt}(G)$ are integers, $\mathsf{bt}(G) \geq k - 1$. □

We conclude the paper by discussing some natural open problems regarding the computational complexity of calculating the book thickness for various classes of graphs.

Proposition 1 begs the question: Is there a characterisation of the k-trees with book thickness $k-1$, k or $k+1$? And somewhat more generally, is there a polynomial-time algorithm to determine the book thickness of a given k-tree? Note that the k-th power of paths are an infinite class of k-trees with book thickness $k-1$; see (Swaminathan et al., 1995).

k-trees are the edge-maximal chordal graphs with no $(k+2)$-clique, and also are the edge-maximal graphs with treewidth k. Is there a polynomial-time algorithm to determine the book thickness of a given chordal graph? Is there a polynomial-time algorithm to determine the book thickness of a given graph with bounded treewidth?

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