A note on the convexity number of complementary prisms[*]

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In the geodetic convexity, a set of vertices S of a graph G is *convex* if all vertices belonging to any shortest path between two vertices of S lie in S. The cardinality $con(G)$ of a maximum proper convex set S of G is the *convexity number* of G. The *complementary prism* $G\overline{G}$ of a graph G arises from the disjoint union of the graphs G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. In this work, we prove that the decision problem related to the convexity number is NP-complete even restricted to complementary prisms, we determine $con(G\overline{G})$ when G is disconnected or G is a cograph, and we present a lower bound when $diam(G) \neq 3$.

Keywords: geodetic convexity, convex set, convexity number, complementary prism

1 Introduction

In this paper we consider finite, simple, and undirected graphs, and we use standard terminology. For a finite and simple graph G with vertex set $V(G)$, a *graph convexity* on $V(G)$ is a collection $\mathcal{C}$ of subsets of $V(G)$ such that $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* $H(S)$ in $\mathcal{C}$ of a set S of vertices of G is the smallest set in $\mathcal{C}$ containing S.

Convex sets in graphs emerged as an analogy to convex sets in the Euclidean plane. Such concepts have attracted attention in the last decades due to its versatility for modeling some disseminating processes between discrete entities. Graph convexities can model, for instance, contexts of distributed computing (Flocchini et al., 2004; Peleg, 2002).

We can consider a computer network, modeled as a graph G (the computers and their connections are represented by $V(G)$ and $E(G)$, respectively), where a fault on some computer data propagates to other computers, according to some rule of propagation. A rule of propagation may be, for instance, if a fault occurs in two computers a and b, such a fault propagates to all computers that lie in the shortest paths between a and b. A problem raised in this context is findinding a maximum subset of computers $S \subseteq V(G)$ in which faults can occur in order to ensure that the entire network does not fail. The cardinality

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of S corresponds to the parameter known as convexity number of the graph G and the propagation rule coincides with the geodetic convexity.

We may cite other contexts in which graph convexities can be applied, *e.g.* spread of disease and contamination (Balogh and Pete, 1998; Bollobás, 2006; Dreyer and Roberts, 2009), marketing strategies (Domingos and Richardson, 2001; Kempe et al., 2003, 2005), and spread of opinion (Brunetti et al., 2012; Dreyer and Roberts, 2009).

We consider the *geodetic convexity* $\mathcal{C}$ on a graph G, which is defined by means of shortest paths in G. We say that a set of vertices S of a graph G is *convex* if all vertices belonging to any shortest path between two vertices of S lie in S. The cardinality $con(G)$ of a maximum proper convex set S of G is the *convexity number* of G.

One of the first works to introduce the convexity number was published by Chartrand et al. (2002). They determine the convexity number for complete graphs, paths, cycles, trees, and present bounds for general graphs. In the same year, Canoy and Garces (2002) show results on the convexity number for graph operations like join, composition, and Cartesian product. Later on, Kim (2004) studies the parameter for k-regular graphs. Considering complexity aspects, Gimbel (2003) shows that determining the convexity number is NP-hard for general graphs, whereas Dourado et al. (2012) refine Gimbel's result showing the NP-hardness of the problem even restricted to bipartite graphs.

Motivated by the work of Canoy and Garces (2002) on the convexity number for graph operations, we study that parameter for a graph product called complementary prism. Such graph product was introduced by Haynes et al. (2007) as a variation of the well-known *prism* of a graph (Hammack et al., 2011). Let G be a graph and $\overline{G}$ its complement. For every vertex $v \in V(G)$ we denote $\overline{v} \in V(\overline{G})$ as its *corresponding vertex*. The *complementary prism* $G\overline{G}$ of a graph G arises from the disjoint union of the graph G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. A classic example of a complementary prism is the Petersen graph $C_5\overline{C}_5$.

In this paper we determine the convexity number for complementary prisms $G\overline{G}$ when G is disconnected or G is a cograph, and we present a lower bound of that parameter for complementary prisms of graphs with restricted diameter. Furthermore, given a complementary prism $H\overline{H}$, and an integer k, we prove that it is NP-complete to decide whether $con(H\overline{H}) \geq k$.

This paper is divided in more three sections. In Section 2 we define the fundamental concepts and terminology. In Section 3 we present our contributions. We close with the conclusions in Section 4.

2 Preliminaries

Let G be a graph. Given a vertex $v \in V(G)$, its *open neighborhood* is denoted by $N_G(v)$, and its *closed neighborhood*, denoted by $N_G[v]$, is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$, and $N_G[U] = N_G(U) \cup U$.

A *clique* (resp. *independent set*) is a set of pairwise adjacent (resp. nonadjacent) vertices. A vertex of a graph G is *simplicial* in G if its neighborhood induces a clique.

The *distance* $d_G(u,v)$ of two vertices u and v in G is the minimum number of edges of a path in G between u and v. Let $A, B \subseteq V(G)$. The *distance* between A and B in G is defined by $d_G(A,B) = \min\{d_G(u,v) : x \in A, y \in B\}$. A graph G is called *connected* if any two of its vertices are linked by a path in G. Otherwise, G is called *disconnected*. A maximal connected subgraph of G is called a *connected component* or *component* of G. A component G_i of a graph G is *trivial* if $|V(G_i)| = 1$, and *non-trivial* otherwise.

Let G be a graph. For a set $X \subseteq V(G)$, we let $\overline{X}$ be the *corresponding set* of vertices in $V(\overline{G})$. We denote the set of positive integers $\{1, \ldots, k\}$ by $[k]$.

A convex set S of a graph G can be defined by a closed interval operation. The *closed interval* $I[u, v]$ of a pair $u, v \in V(G)$ consists of all vertices lying in any shortest (u, v)-path in G. For a set $S \subseteq V(G)$, the *closed interval* $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$ and if $|S| < 2$, then $I[S] = S$. We say that S is a *convex set*, if $I[S] = S$. To avoid ambiguity, sometimes a subscript can be added to the notation (*e.g.* $I_G[S]$, and $H_G(S)$) to indicate which graph G is being considered.

3 Results

Chartrand et al. (2002) provide two useful results. They proved that $con(K_n) = n - 1$, and for a noncomplete graph the result follows in Theorem 3.1.

Theorem 3.1 (Chartrand et al. (2002)) *Let G be a noncomplete connected graph of order n. Then $con(G) = n - 1$ if and only if G contains a simplicial vertex.*

We begin our contributions by determining in Theorem 3.3 the convexity number for complementary prisms of disconnected graphs. We first show Proposition 3.2 that will be useful for the subsequent results.

Proposition 3.2 *Let G be a graph, $S \subseteq V(G\overline{G})$, and $v_1 \ldots v_k$ be a path in G, for $k \geq 2$. If $\{v_1, \overline{v}_2, \ldots, \overline{v}_k\} \subseteq H(S)$, then $v_k \in H(S)$.*

Proof: The proof is by induction on k. First, let $k = 2$. Since $v_1 v_2 \in E(G)$ and $v_1, \overline{v}_2 \in H(S)$, $v_2 \in I[v_1, \overline{v}_2]$. Now, let $k > 2$. Let $v_1 \ldots v_{k-1} v_k$ be a path in G and suppose that $\{v_1, \overline{v}_2, \ldots, \overline{v}_{k-1}, \overline{v}_k\} \subseteq H(S)$. By induction hypothesis $v_{k-1} \in H(S)$, which implies that $v_k \in I[v_{k-1}, \overline{v}_k]$. Therefore, it follows that $v_k \in H(S)$, for $k \geq 2$. □

Theorem 3.3 *Let G be a disconnected graph of order n, and k be the order of a minimum component of G. Then, $con(G\overline{G}) = 2n - k$.*

Proof: Let $G_1, \ldots, G_\ell$ be the components of G, for $\ell \geq 2$. We can sort the components $G_1, \ldots, G_\ell$ of G in non-decreasing order of size (number of vertices). Then, G_1 is a component of minimum order, say $|V(G_1)| = k$. If $k = 1$, then G_1 is a trivial component. Hence, the unique vertex $v \in V(G_1)$ is a simplicial vertex, and the result $con(G\overline{G}) = n - 1$ follows from Theorem 3.1. We then consider that $|V(G_i)| \geq 2$, for every $i \in [\ell]$. Let $S = V(G\overline{G}) \setminus V(G_1)$. See Figure 1 for an illustration. We show that S is a convex set of $G\overline{G}$.

Suppose, by contradiction, that S is not convex. Then, there must be a shortest path of the form $\bar{x}xy\bar{y}$ for $x, y \in V(G_1)$. But this is a contradiction, since $d_G(\bar{x}, \bar{y}) = 2$.

Next, we show that S is maximum. By contradiction, suppose that there exists a convex set $S' \subseteq V(G\overline{G})$ such that $|S'| > |S|$.

Since $|S'| > 2n - k$, we have that $S' \cap V(G_j) \neq \emptyset$, and $S' \cap V(\overline{G}_j) \neq \emptyset$, for every $j \in [\ell]$. We divide the proof in two cases.

Case 1 *$S' \cap V(\overline{G})$ is not a clique.*

Let $\overline{x}, \overline{y} \in S' \cap V(\overline{G})$ such that $\bar{x}\bar{y} \notin E(\overline{G})$. By the definition of complementary prism, every vertex in $V(\overline{G}_i)$ is adjacent to every vertex in $V(\overline{G}_j)$, for every $i, j \in [\ell]$, $i \neq j$. This implies that $\overline{x}, \overline{y} \in V(\overline{G}_i)$, for some $i \in [\ell]$, and $V(\overline{G}) \setminus V(\overline{G}_i) \subseteq I_{G\overline{G}}[\overline{x}, \overline{y}]$.

We know that $|V(G_k)| \geq 2$, for every $k \in [\ell]$. Let $j \in [\ell] \setminus \{i\}$. Since G_j is a connected component, and $|V(G_j)| \geq 2$, there exist $u, v \in V(G_j)$ such that $uv \in E(G)$. Thus, $\bar{u}\bar{v} \notin E(\overline{G})$. Since $\bar{u}, \bar{v} \in I_{G\overline{G}}[\overline{x}, \overline{y}]$, we obtain that $V(\overline{G}_i) \subseteq I_{G\overline{G}}[\overline{u}, \overline{v}]$. Hence, $V(\overline{G}) \subseteq H_{G\overline{G}}(\{\overline{x}, \overline{y}\})$. Since S' is convex, then $V(\overline{G}) = S' \cap V(\overline{G})$. Since $S' \cap V(G_i) \neq \emptyset$, for every $i \in [\ell]$, and G_i is connected, Proposition 3.2 implies that $V(G) \subseteq H_{G\overline{G}}(S')$, a contradiction.

Case 2 *$S' \cap V(\overline{G})$ is a clique.*

Consider that $S' \cap V(\overline{G})$ is maximum clique. Let $\overline{C}_i = S' \cap V(\overline{G}_i)$, for every $i \in [\ell]$. We know that C_i is an independent set, and $\overline{C}_i$ is a clique. We claim that $S' \cap (N_G(C_i) \cup \overline{N_G(C_i)}) = \emptyset$, for every $i \in [\ell]$. In fact, we prove a stronger statement, $S' \cap (V(G_i\overline{G}_i) \setminus (C_i \cup \overline{C}_i)) = \emptyset$, for every $i \in [\ell]$.

Claim 1 *For every $i \in [\ell]$, $S' \cap (V(G_i\overline{G}_i) \setminus (C_i \cup \overline{C}_i)) = \emptyset$.*

Proof of Claim 1 First, recall that $\overline{u}, \overline{v} \in I_{G\overline{G}}[u, v]$, for every $u \in V(G_i)$ and $v \in V(G_j)$, for every $i, j \in [\ell]$, $i \neq j$. Let $i \in [\ell]$. By contradiction, suppose that there exists $v \in S' \cap (V(G_i\overline{G}_i) \setminus (C_i \cup \overline{C}_i))$.

Since $\overline{C}_i$ is maximum, if $v \in S' \cap (V(\overline{G}_i) \setminus \overline{C}_i)$ then $S' \cap V(\overline{G})$ contains two nonadjacent vertices, a contradiction. Then, suppose that $v \in S' \cap (V(G_i) \setminus C_i)$. Since $S' \cap V(G_j) \neq \emptyset$, for every $j \in [\ell]$, we have that $\overline{v} \in I_{G\overline{G}}[u, v]$, for some $u \in S' \cap V(G_j)$, $i \neq j$. Since $\overline{C}_i$ is maximum, there exists $\overline{w} \in \overline{C}_i$ such that $\bar{v}\bar{w} \notin E(\overline{G})$. Consequently $S' \cap V(\overline{G})$ contains two nonadjacent vertices, a contradiction. $\square$

Claim 2 *For every $i \in [\ell]$, $|C_i \cap S'| \leq |V(G_i) \setminus C_i|$.*

Proof of Claim 2 Let $i \in [\ell]$. By Claim 1 $S' \cap (V(G_i) \setminus C_i) = \emptyset$. Since $S' \cap V(G_i) \neq \emptyset$, we have that $|C_i \cap S'| \geq 1$. By contradiction, suppose that $|C_i \cap S'| > |V(G_i) \setminus C_i|$. Let $|C_i \cap S'| = p$.

If $p = 1$, then $|V(G_i) \setminus C_i| = 0$. Since C_i is an independent set, it follows that G_i is disconnected, a contradiction. Then, consider $p \geq 2$. Let $C_i \cap S' = \{u_1, \ldots, u_p\}$, and $V(G_i) \setminus C_i = \{v_1, \ldots, v_{p-1}\}$.

Since C_i is an independent set, and G_i is connected, it follows that every vertex in $C_i \cap S'$ is adjacent to at least one vertex in $V(G_i) \setminus C_i$. Since $|C_i \cap S'| > |V(G_i) \setminus C_i|$, we have that there exist $j, j' \in [p]$, $j \neq j'$, such that $u_j v_q, u_{j'} v_q \in E(G)$, for some $q \in [p-1]$. Then, $v_q \in I_{G\overline{G}}[u_j, u_{j'}]$. Since S' is convex, $v_q \in S'$. But that is a contradiction, since $S' \cap (V(G_i) \setminus C_i) = \emptyset$. $\square$

To conclude the proof, we show that $|S'| \leq n$. In view of the above statements, for every $i \in [\ell]$, it follows that

$$\begin{aligned}
|S' \cap V(G_i\overline{G}_i)| &= |S' \cap V(G_i)| + |S' \cap V(\overline{G}_i)| \\
&= |C_i \cap S'| + |\overline{C_i}| \\
&= |C_i \cap S'| + |V(\overline{G}_i)| - |V(\overline{G}_i) \setminus \overline{C}_i| \\
&\leq |V(G_i) \setminus C_i| + |V(\overline{G}_i)| - |V(\overline{G}_i) \setminus \overline{C}_i| && \text{(by Claim 2)} \\
&= |V(G_i)|.
\end{aligned}$$

Since $|S' \cap V(G_i\overline{G}_i)| \leq |V(G_i)|$, for every $i \in [\ell]$, we obtain that

$$|S'| \leq \sum_{i=1}^{\ell} |V(G_i)| = n,$$

a contradiction. Therefore S is a maximum convex set of $G\overline{G}$, and $con(G\overline{G}) \leq 2n - k$, which completes the proof. □

Figure 1 shows an illustration of a proper convex set of $G\overline{G}$, represented by the black vertices. Consider G_1 the component of minimum order of G.

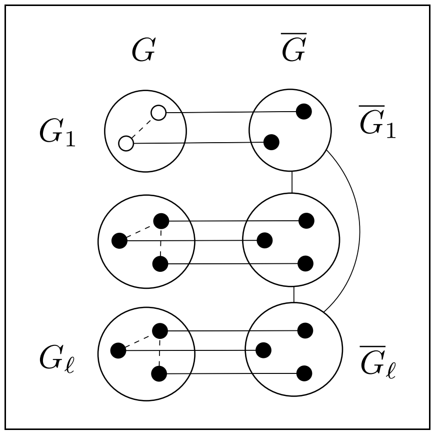

Fig. 1: Example of a proper convex set of $G\overline{G}$.

As a consequence of Theorem 3.3, we have:

Corollary 3.4 *Let G be a connected cograph of order n, and k be the order of a minimum component of $\overline{G}$. Then, $con(G\overline{G}) = 2n - k$.*

Proof: Since a cograph G is connected if and only if $\overline{G}$ is disconnected (Corneil et al., 1981), and $G\overline{G}$ is isomorphic to $\overline{G}G$, the result follows from Theorem 3.3. □

3.1 NP-Completeness

Next, we show a hardness result of the geodetic number for complementary prims. We first present Proposition 3.5, and we define the two decision problems to be considered.

Proposition 3.5 *Let G be a graph, S be a convex set of $G\overline{G}$, $S' = S \cap V(\overline{G})$. If $\overline{A} = N_{\overline{G}}(S')$ then $S \cap A = \emptyset$.*

Proof: Suppose that $u \in S \cap A$. Since $\bar{u}\bar{v} \in E(\overline{G})$, for some $\overline{v} \in S'$, we have that $\overline{u} \in I_{G\overline{G}}[u, \overline{v}]$. We know that S is convex, which implies that $\overline{u} \in S$. But this is a contradiction, since $\overline{u} \notin S \cap V(\overline{G})$. □

Problem 3.6 CLIQUE *(Garey and Johnson, 1979).*
Instance: *A graph G and an integer k.*
Question: *Does G have a clique of order at least k?*

Problem 3.7 CONVEXITY NUMBER *(Dourado et al., 2012).*
Instance: *A graph G and an integer k.*
Question: *Does G have a proper convex set of order at least k?*

Theorem 3.8 CONVEXITY NUMBER *is* NP*-complete even restricted to complementary prisms.*

Proof: Since computing the convex hull of a set of vertices can be done in polynomial time, CONVEXITY NUMBER is in NP.

In order to prove NP-completeness, we describe a polynomial reduction from the NP-complete problem CLIQUE (Garey and Johnson, 1979).

Let (G, k) be an instance of CLIQUE. We may assume that G is connected, and $k \geq 3$. Let $|V(G)| = n$. We construct a graph H arising from G as follows. Add to H three cliques U, X, and Z, respectively on n, $4n$ and 2 vertices, say $U = \{u_1, \dots, u_n\}$, $X = \{x_1, \dots, x_{4n}\}$, and $Z = \{z_1, z_2\}$. Add to H an independent set $Y = \{y_1, y_2\}$. Join every vertex in $U \cup Z$ to every vertex in $V(G)$. Also join every vertex in X to $U \cup Y$. This completes the construction of H. Use the graph H to create the complementary prism $H\overline{H}$. See an example in Figure 2. We prove that G has a clique of order at least k if and only if $H\overline{H}$ has a proper convex set of order at least $k + 5n + 3$.

First, we assume that G has a clique C of order at least k. Let $S \subseteq V(H\overline{H})$ such that $S = C \cup U \cup X \cup Y \cup \{\overline{u}_1\}$. Notice that $|S| = k + 5n + 3$. We show that S is a convex set of $H\overline{H}$.

Let $w \in S \cap V(H)$. Since $d_{H\overline{H}}(w, \overline{u}_1) \leq 2$, we have that $I_{H\overline{H}}[w, \overline{u}_1] = \{w, \overline{u}_1\}$, if $w = u_1$, or $I_{H\overline{H}}[w, \overline{u}_1] = \{w, u_1, \overline{u}_1\}$, otherwise. Now, let $w, w' \in S \cap V(H)$. Since $d_H(w, w') \leq 2$, and C is a clique in G, we have that $I_{H\overline{H}}[w, w'] \subseteq S \cap V(H)$. Therefore $I_{H\overline{H}}[S] = S$, and S is a convex set of $H\overline{H}$.

For the converse, we show two useful claims first.

Claim 1 *Let S be a proper convex set of $H\overline{H}$. Then, for every $w, w' \in S$, $d_{H\overline{H}}(w, w') \leq 2$.*

Proof of Claim 1 By contradiction, suppose that there exist $w, w' \in S$ such that $d_{H\overline{H}}(w, w') > 2$. By the definition of complementary prism, we know that either $w, w' \in V(H)$ or $w, w' \in V(\overline{H})$.

Let $w, w' \in S \cap V(H)$. By the construction of H, $d_H(X \cup Y, Z) = 3$; then we can consider that $w \in X \cup Y$ and $w' \in Z$. That way, we have that $U \cup V(G) \cup \{\overline{w}, \overline{w'}\} \subseteq I_{H\overline{H}}[w, w']$. Since $\bar{v}\bar{w} \in E(\overline{H})$ for every $v \in V(G)$, we have that $\overline{v} \in I_{H\overline{H}}[v, \overline{w}]$. Consequently $V(\overline{G}) \subseteq I_{H\overline{H}}[V(G) \cup \{\overline{w}\}]$. By symmetry, $\overline{U} \subseteq I_{H\overline{H}}[U \cup \{\overline{w'}\}]$. Since G is connected, there exist two nonadjacent vertices $\overline{v}_1, \overline{v}_2 \in V(\overline{G})$. This implies that $\overline{X} \cup \overline{Y} \subseteq I_{H\overline{H}}[\overline{v}_1, \overline{v}_2]$. Similarly, since $\overline{U}$ is an independent set, $\overline{Z} \subseteq I_{H\overline{H}}[\overline{U}]$. Since $V(\overline{H}) \subseteq H_{H\overline{H}}(S)$, Proposition 3.2 implies that $V(H) \subseteq H_{H\overline{H}}(S)$, a contradiction.

Now, let $\overline{w}, \overline{w}' \in S \cap V(\overline{H})$. By the construction of H, we can select $\overline{w} \in U$, and $\overline{w}' \in V(\overline{G})$. We have that $\overline{X} \cup \overline{Y} \cup \overline{Z} \cup \{w, w'\} \subseteq I_{H\overline{H}}[\overline{w}, \overline{w}']$. Consequently, $\overline{U} \subseteq I_{H\overline{H}}[\bar{Z}]$, and $V(\overline{G}) \subseteq I_{H\overline{H}}[\bar{X}]$. Since $V(\overline{H}) \cup \{w, w'\} \subseteq H_{H\overline{H}}(S)$, Proposition 3.2 implies that $V(H) \subseteq H_{H\overline{H}}(S)$, a contradiction. □

Claim 2 *Let S be a proper convex set of $H\overline{H}$. Then $S \cap V(\overline{H})$ is a clique.*

Proof of Claim 2 By contradiction, suppose that there exist $\overline{w}, \overline{w}' \in S \cap V(\overline{H})$ such that $\bar{w}\bar{w}' \notin E(\overline{H})$. Then, we have the following cases.

Case 1.1 $\overline{w}, \overline{w}' \in \overline{Z}$.

This implies that $\overline{X} \cup \overline{Y} \cup \overline{U} \subseteq I_{H\overline{H}}[\overline{w}, \overline{w}']$. Consequently, $V(\overline{G}) \subseteq I_{H\overline{H}}[\bar{X}]$. Since $d_{H\overline{H}}(\overline{U}, V(\overline{G})) = 3$, by Claim 1, S is not a proper convex set, a contradiction.

Case 1.2 $\overline{w}, \overline{w}' \in \overline{X} \cup \overline{Y} \cup \overline{U}$.

In this case, $\overline{Z} \in I_{H\overline{H}}[\overline{w}, \overline{w}']$, then the proof follows by Case 1.1.

Case 1.3 $\overline{w}, \overline{w}' \in V(\overline{G}) \cup \overline{Z}$.

We have that $\overline{X} \in I_{H\overline{H}}[\overline{w}, \overline{w}']$, then the proof follows by Case 1.2. □

Let S be a proper convex set of $H\overline{H}$ of order at least $k + 5n + 3$. By Claim 2, we know that $S \cap V(\overline{H})$ is a clique. Let $\overline{C} = S \cap V(\overline{H})$. To proceed with the proof, we show that $\overline{C} \cap (V(\overline{G}) \cup \overline{X} \cup \overline{Y} \cup \overline{Z}) = \emptyset$. For that, we consider two cases, $\overline{C}$ does not contain vertices from $V(\overline{G}) \cup \overline{Z}$, and $\overline{C}$ does not contain vertices from $\overline{X} \cup \overline{Y}$.

Claim 2.1 $\overline{C} \cap (V(\overline{G}) \cup \overline{Z}) = \emptyset$.

Proof of Claim 2.1 Suppose, by contradiction, that $\overline{C}$ contains a vertex in $V(\overline{G})$ or in $\overline{Z}$. If $\overline{C} \cap \overline{X} = \emptyset$, then $\overline{X} \subseteq N_{\overline{H}}(\overline{C})$. Otherwise, we have that $\overline{X} \setminus \{\overline{x}_i\} \subseteq N_{\overline{H}}(\overline{C})$, for some $i \in [4n]$. In both cases, we have that $|N_{\overline{H}}(\overline{C})| \geq 4n - 1$. For $\overline{A} = N_{\overline{H}}(\overline{C})$, Proposition 3.5 implies that $S \cap A = \emptyset$ then $|S \cap V(H)| = |V(H)| - |A| \leq 6n + 4 - (4n - 1) = 2n + 5$. □

So far, we conclude that the number of vertices from S in H is at most $2n + 5$. It remains to show the maximum number of vertices from S in $\overline{H}$. By the construction of H, a clique in $\overline{H}$ of maximum order is a proper subset of $V(\overline{G}) \cup \overline{Y}$, hence $|S \cap V(\overline{H})| < n+2$. Consequently, $|S| = |S \cap V(H)| + |S \cap V(\overline{H})| < 2n + 5 + n + 2 = 3n + 7$, a contradiction.

Claim 2.2 $\overline{C} \cap (\overline{X} \cup \overline{Y}) = \emptyset$.

Proof of Claim 2.2 We know by Claim 2.1 that $\overline{C} \cap (V(\overline{G}) \cup \overline{Z}) = \emptyset$. By contradiction, suppose that $\overline{C}$ contains a vertex from $\overline{X}$ or $\overline{Y}$. Let $\overline{A} = N_{\overline{H}}(\overline{C})$. In this case, we have that $V(\overline{G}) \cup \overline{Z} \subseteq \overline{A}$. Hence, $|\overline{A}| \geq n+2$. It follows from Proposition 3.5 that $|S \cap V(H)| = |V(H)| - |A| \leq 6n+4-(n+2) = 5n+2$.
Since $\overline{C} \cap (V(\overline{G}) \cup \overline{Z}) = \emptyset$ and $\overline{U} \cup \overline{X}$ is an independent set, the maximum order of a clique in $V(\overline{H})$ is $|\overline{Y}| = 2$; then $|S \cup V(\overline{H})| \leq 2$. This implies that $|S| = |S \cap V(H)| + |S \cap V(\overline{H})| \leq 5n+2+2 = 5n+4$. Since $k \geq 3$, we have that $|S| \geq k + 5n + 3 = 5n + 6$, a contradiction. □

By Claims 2.1 and 2.2, we have that a proper convex set S of $H\overline{H}$ of order at least $k + 5n + 3$ is such that $S \cap \overline{U} \neq \emptyset$ or $S \cap V(\overline{H}) = \emptyset$. We show that, in both cases, the order of S implies in $|S \cap V(G)| \geq k$.

Case 3.1 $S \cap \overline{U} \neq \emptyset$.

Since $\overline{U}$ is an independent set, $|S \cap \overline{U}| \leq 1$. So, let $i \in [n]$, and consider that $\overline{u}_i \in S$. We have that $\overline{A} = N_{\overline{H}}(\overline{u}_i) = \overline{Z}$. By Proposition 3.5, $S \cap Z = \emptyset$. By the construction of H, we have that $|U \cup X \cup Y| = 5n + 2$. Since $|S \cap V(H)| = |S| - 1 = k + 5n + 2$, we have that $|S \cap V(G)| \geq k$.

Case 3.2 $S \cap V(\overline{H}) = \emptyset$.

By Claim 1, we have that

$$\begin{array}{lll} \text{either} & S \cap V(H) \subseteq (U \cup V(G) \cup X \cup Y) & \text{(I)} \\ \text{or} & S \cap V(H) \subseteq (U \cup V(G) \cup Z) & \text{(II).} \end{array}$$

Condition II implies that the order of S is at most $2n+2$, a contradiction. Then, consider that $S \subseteq (U \cup V(G) \cup X \cup Y)$. Still by the construction of H, $|U \cup X \cup Y| = 5n+2$. Since $|S \cap V(H)| = |S| = k+5n+3$, we obtain that $|S \cap V(G)| \geq k + 1$.

By Cases 3.1 and 3.2 we have that $|S \cap V(G)| \geq k$. It remains to show that $S \cap V(G)$ is a clique. Suppose, by contradiction, that $S \cap V(G)$ is not a clique. Then, there exist $v_1, v_2 \in V(G)$ such that $v_1v_2 \notin E(G)$. This implies that $Z \subseteq I_{H\overline{H}}[v_1, v_2]$. But, in both cases, $S \cap \overline{U} \neq \emptyset$ and $S \cap V(\overline{H}) = \emptyset$; thus we have that $S \cap Z = \emptyset$, a contradiction. Therefore $S \cap V(G)$ is a clique of order at least k, which completes the proof. □

Figure 2 contains an example of graph $H\overline{H}$ constructed for Theorem 3.8. Every edge joining two rectangles A and B represents the set of all edges joining every pair of vertices $a \in A$ and $b \in B$. The black vertices correspond to the convex set S except the vertices from $V(G)$. For convenience, the edges joining corresponding vertices from G to $\overline{G}$ are not depicted in the figure.

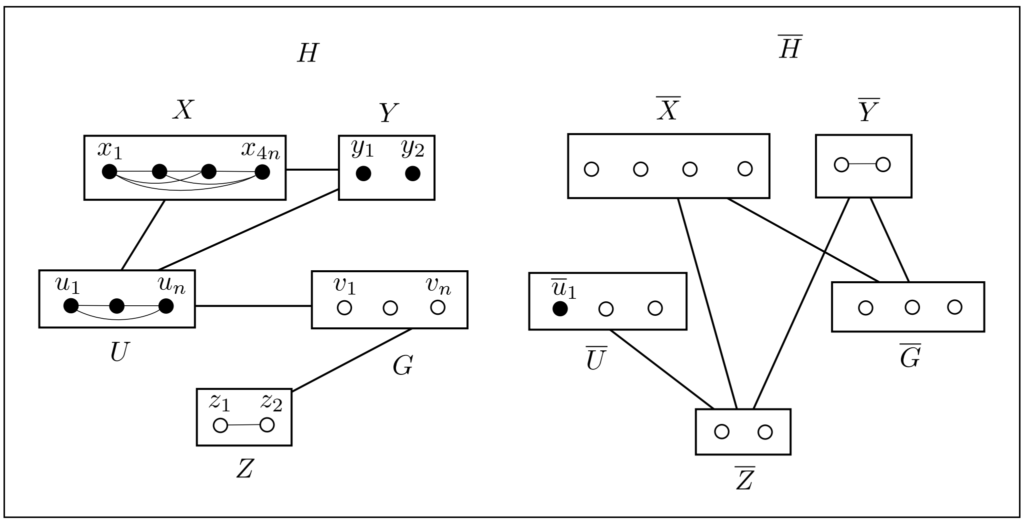

Fig. 2: Graph $H\overline{H}$ constructed for the reduction of Theorem 3.8.

Notice that the graph H constructed for Theorem 3.8 has diameter 3. In view of the complexity result of that theorem, we finish by showing a lower bound of this parameter for graphs with restricted diameter.

Theorem 3.9 *Let G be a graph of order n. If $diam(G) \neq 3$, then $con(G\overline{G}) \geq n$.*

Proof: Consider first that $diam(G) \leq 2$. Let $u, v \in V(G)$. Since $diam(G) \leq 2$, and every (u, v)-path passing through $V(\overline{G})$ has length at least 3, we obtain that $I_{G\overline{G}}[u, v] \cap V(\overline{G}) = \emptyset$. Hence, $V(G)$ is a convex set of $G\overline{G}$. Since $V(G) \subset V(G\overline{G})$, it follows that $con(G\overline{G}) \geq |V(G)| = n$.

Now, let $diam(G) \geq 4$. According to Goddard and Oellermann (2011), $diam(G) > 3$ implies that $diam(\overline{G}) \leq 2$. Since $G\overline{G}$ is isomorphic to $\overline{G}G$, the result follows from the above case. □

4 Conclusions

We have considered the convexity number in the geodetic convexity for complementary prisms $G\overline{G}$. When G is disconnected or G is a cograph we provided an equality. When $diam(G) \neq 3$ we have presented a lower bound. From the complexity point of view, we have proved that, given a complementary prism $H\overline{H}$, and an integer k, it is NP-complete to decide whether $con(H\overline{H}) \geq k$.

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References

J. Balogh and G. Pete. Random disease on the square grid. *Random Structures and Algorithms*, 13(3-4): 409–422, 1998.

B. Bollobás. *The Art of Mathematics: Coffee Time in Memphis*. Cambridge University Press, 2006.

S. Brunetti, G. Cordasco, L. Gargano, E. Lodi, and W. Quattrociocchi. Minimum weight dynamo and fast opinion spreading. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 249–261. Springer, 2012.

S. R. Canoy, Jr. and I. Garces. Convex sets under some graph operations. *Graphs and Combinatorics*, 18 (4):787–793, Dec 2002.

G. Chartrand, C. E. Wall, and P. Zhang. The convexity number of a graph. *Graphs and Combinatorics*, 18(2):209–217, 2002.

D. Corneil, H. Lerchs, and L. Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3(3):163 – 174, 1981. ISSN 0166-218X.

P. Domingos and M. Richardson. Mining the network value of customers. In *Proceedings of the Seventh ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '01, pages 57–66, New York, NY, USA, 2001. ACM. ISBN 1-58113-391-X.

M. C. Dourado, F. Protti, D. Rautenbach, and J. L. Szwarcfiter. On the convexity number of graphs. *Graphs and Combinatorics*, 28(3):333–345, 2012.

P. A. Dreyer and F. S. Roberts. Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion. *Discrete Applied Mathematics*, 157(7):1615–1627, 2009.

P. Flocchini, E. Lodi, F. Luccio, L. Pagli, and N. Santoro. Dynamic monopolies in tori. *Discrete applied mathematics*, 137(2):197–212, 2004.

M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979. ISBN 0716710447.

J. Gimbel. Some remarks on the convexity number of a graph. *Graphs and Combinatorics*, 19(3):357–361, 2003.

W. Goddard and O. R. Oellermann. Distance in graphs. In *Structural Analysis of Complex Networks*, pages 49–72. Springer, 2011.

R. Hammack, W. Imrich, and S. Klavžar. *Handbook of Product Graphs*. CRC press, 2011.

T. W. Haynes, M. A. Henning, P. J. Slater, and L. C. van der Merwe. The complementary product of two graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 51:21–30, 2007.

D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD '03, pages 137–146, New York, NY, USA, 2003. ACM. ISBN 1-58113-737-0.

D. Kempe, J. Kleinberg, and É. Tardos. Influential nodes in a diffusion model for social networks. In *International Colloquium on Automata, Languages, and Programming*, pages 1127–1138. Springer, 2005.

B. K. Kim. A lower bound for the convexity number of some graphs. *Journal of Applied Mathematics and Computing*, 14(1-2):185–191, 2004.

D. Peleg. Local majorities, coalitions and monopolies in graphs: a review. *Theoretical Computer Science*, 282(2):231–257, 2002.