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The join of two varieties is the smallest variety containing both. In finite semigroup theory, the varieties of $R$-trivial and $L$-trivial monoids are two of the most prominent classes of finite monoids. Their join is known to be decidable due to a result of Almeida and Azevedo. In this paper, we give a new proof for Almeida and Azevedo’s effective characterization of the join of $R$-trivial and $L$-trivial monoids. This characterization is a single identity of $\omega$-terms using three variables.

Keywords: finite semigroup theory, join of pseudovarieties, Green’s relations, combinatorics on words

1 Introduction

Green’s relations $R$ and $L$ are a standard tool in the study of semigroups [5]. In the context of finite monoids, among other results, they have been used to give effective characterizations of language classes such as star-free languages [3,11] and piecewise testable languages [6,12]. A deterministic extension of piecewise testable languages yields the class of languages corresponding to $R$-trivial monoids, and a codeterministic extension corresponds to $L$-trivial monoids [4,9].

Almeida and Azevedo gave an effective characterization for the least variety of finite monoids containing all $R$-trivial and all $L$-trivial monoids [2], i.e., for the join of the two varieties. Their proof is based on sophisticated algebraic techniques, on Reiterman’s Theorem [10], and on a combinatorial result of König [7]. In this paper, we give a new proof of Almeida and Azevedo’s Theorem. The current proof was inspired by another proof of the authors [8], which in turn uses ideas of Klíma [6]. The main ingredient is a system of congruences which relies on simple combinatorics on words.

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2 Preliminaries

Let $A$ be a finite alphabet. The set of finite words over $A$ is denoted by $A^*$. It is the free monoid over $A$. The empty word is 1. The content of a word $u = a_1 \cdots a_n$ with $a_i \in A$ is $\alpha(u) = \{a_1, \ldots, a_n\}$, and its length is $|u| = n$. The length of the empty word is 0. A word $u$ is a prefix (respectively suffix) of $v$ if there exists $x \in A^*$ such that $ux = v$ (respectively $xu = v$); if $x \neq 1$, then $u$ is a proper prefix.

For more details concerning the algebraic concepts introduced in the remainder of this section, we refer the reader to textbooks such as [1, 4, 9]. Green’s relations $R$ and $L$ are important tools in the study of finite monoids. Let $M$ be a finite monoid. We set $uRv$ for $u, v \in M$ if $uM = vM$, and the latter condition is equivalent to the existence of $x, y \in M$ with $u = vx$ and $v = uy$. Symmetrically, $uLv$ if $Mu = Mv$. The monoid $M$ is $R$-trivial (respectively $L$-trivial) if $R$ (respectively $L$) is the identity relation on $M$. We write $u <_R v$ if $uM \subseteq vM$, and we write $u <_R v$ if $Mu \subseteq Mv$.

A variety of finite monoids is a class of monoids closed under finite direct products, submonoids, and quotients. A variety of finite monoids is often called a pseudovariety in order to distinguish from varieties in Birkhoff’s sense. Since we do not need this distinction in the current paper, whenever we use the term variety we mean a variety of finite monoids. The join $V_1 \lor V_2$ of two varieties $V_1$ and $V_2$ is the smallest variety containing $V_1 \cup V_2$. A monoid $M$ is in $V_1 \lor V_2$ if and only if there exist $M_1 \in V_1$ and $M_2 \in V_2$ such that $M$ is a quotient of a submonoid of $M_1 \times M_2$. If $M$ is a finite monoid, then there exists an integer $\omega_M \geq 1$ such that, for all $u \in M$, the element $u^{\omega_M}$ is idempotent. Moreover, the element $u^{\omega_M}$ is the unique idempotent generated by $u$. Usually, the monoid $M$ is clear from the context and thus, we simply write $\omega$ instead of $\omega_M$. This leads to the following definition. An $\omega$-term over a finite alphabet $X$ is either a word in $X^\omega$, or of the form $t|^\omega$ for some $\omega$-term $t$, or the concatenation $t_1t_2$ of two $\omega$-terms $t_1, t_2$. A homomorphism $\varphi : X^* \rightarrow M$ to a finite monoid $M$ uniquely extends to $\omega$-terms over $X$ by setting $\varphi(t)^\omega = \varphi(t)|^\omega M$. Let $u, v$ be two $\omega$-terms over $X$. A finite monoid $M$ satisfies the identity $u = v$ if $\varphi(u) = \varphi(v)$ for all homomorphisms $\varphi : X^* \rightarrow M$. The class of finite monoids satisfying the identity $u = v$ is denoted by $[u = v]$. For all $\omega$-terms $u, v$, the class $[u = v]$ forms a variety. We need the following three varieties in this paper:

\begin{align*}
R &= [(xx)^\omega x = (xy)^\omega], \\
L &= [x(zx)^\omega = (zx)^\omega], \\
W &= [(xy)^\omega x(zx)^\omega = (xy)^\omega (zx)^\omega].
\end{align*}

A monoid is in $R$ if and only if it is $R$-trivial. Symmetrically, a monoid is in $L$ if and only if it is $L$-trivial. The aim of this paper is to give a new proof of Almeida and Azevedo’s result $R \lor L = W$. The inclusion $R \lor L \subseteq W$ is trivial since $R \cup L \subseteq W$ and $W$ is a variety.

3 Congruences

In this section, we introduce the main combinatorial tool for our proof. It is a family of congruences $\equiv_n$ on $A^*$ for some finite alphabet $A$ such that $A^*/\equiv_n \in R \lor L$ for all integers $n \geq 0$, see Lemma 2 below. As a first step towards the definition of $\equiv_n$ we need to introduce an asymmetric, weaker congruence $\equiv_n^\#$.

Let $u, v \in A^*$. We let $u \equiv_0^\# v$ if $\alpha(u) = \alpha(v)$. For $n \geq 0$, we let $u \equiv_n^\# v$ if the following conditions hold:

1. $\alpha(u) = \alpha(v)$,
2. for all factorizations $u = u_1au_2$ and $v = v_1av_2$ with $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$ we have $u_1 \equiv_n^\# v_1$ and $u_2 \equiv_n^\# v_2$, and
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3. for all factorizations $u = u_1a_2u_2$ and $v = v_1av_2$ with $a \in A \setminus \{a(u_1) \cup a(v_1)\}$ we have $u_1 \equiv_n^R v_1$.

By a straightforward verification we see that $\equiv_n^R$ is an equivalence relation. The factorization $u_1a_2u_2$ with $a \in A \setminus a(u_1)$ is unique. Therefore, induction on $n$ shows that the index of $\equiv_n^R$ is finite. If $u \equiv_n^R v$, then $u \equiv_n^R v$. Moreover, if $u \equiv_n^R v$ and $a \in A$, then $au \equiv_n^R av$ and $ua \equiv_n^R va$. Therefore, the relation $\equiv_n^R$ is a finite index congruence on $A^*$.

**Lemma 1** For every finite alphabet $A$ and every integer $n \geq 0$ we have $A^*/\equiv_n^R \in R$.

**Proof:** It suffices to show $(xy)^{n+1}x \equiv_n^R (xy)^{n+1}$ for all words $x, y \in A^*$. We note that for $y = 1$ this yields $x^{n+2} \equiv_n^R x^{n+1}$. The proof is by induction on $n$. For $n = 0$, the claim is true since $\alpha(xy) = \alpha(xy)$. Let now $n > 0$. As before, $\alpha((xy)^{n+1}x) = \alpha((xy)^{n+1})$. Suppose $(xy)^{n+1}x = u_1a_2u_2$ and $(xy)^{n+1} = v_1av_2$ for $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$. Then $u_1 = v_1$ and $au_2$ and $av_2$ are proper prefixes of $xy$. Thus $u_2 = p(xy)p$ and $v_2 = p(xy)p$ for some $p \in A^*$. By induction $(xy)^{n-1}p \equiv_n^R (xy)^n$ and hence, $u_2 \equiv_n^R v_2$.

Suppose now $(xy)^{n+1}x = u_1a_2u_2$ and $(xy)^{n+1} = v_1av_2$ for $a \in A \setminus (\alpha(u_2) \cup \alpha(v_2))$. Then $av_2$ is a suffix of $xy$ and $au_2$ is a suffix of $yx$. We can therefore write $v_1 = (xy)^{n+1}p$ for some prefix $p$ of $xy$. Similarly, $u_1 = (xy)^{n+1}p$ for some $k \in \{n, n+1\}$ and some prefix $p$ of $xy$, i.e., we have $pq = xy$ for some $q \in A^*$. By induction, we have $(xy)^{n+1} = (xy)^n$ and thus $(xy)^{n+1}p \equiv_n^R (xy)^np$. We can therefore assume $k = n$. Without loss of generality, let $|p| \leq |p'|$, i.e., $p' = ps$ for some $s \in A^*$. It follows

$$u_1 = (pq)p \quad \text{and} \quad v_1 = (pq)^nps.$$  

Since $p' = ps$ is a prefix of $xy = pq$, the word $s$ is a prefix of $q$. In particular, there exists $t \in A^*$ such that $qp = st$. This yields

$$u_1 = p(st)^n \quad \text{and} \quad v_1 = p(st)^n.$$  

By induction, $(st)^n \equiv_n^R (st)^n$ and $u_1 \equiv_n^R v_1$. This shows $(xy)^{n+1}x \equiv_n^R (xy)^{n+1}$ which concludes the proof.

There is a left-right symmetric congruence $\equiv_n^R$ on $A^*$. It can be defined by setting $u \equiv_n^R v$ if and only if $u^p \equiv_n^R v^p$. Here, $u^p = a_n \cdots a_1$ is the reversal of the word $u = a_1 \cdots a_n$ with $a_i \in A$. It satisfies $A^*/\equiv_n^R \in L$ for every $n \geq 0$. We define $u \equiv_n v$ if and only if both $u \equiv_n^R v$ and $u \equiv_n^R v$. The following lemma puts together some properties of the finite index congruence $\equiv_n$.

**Lemma 2** For every finite alphabet $A$ and every integer $n \geq 0$ the following properties hold:

1. $A^*/\equiv_n \in R \lor L$
2. If $u_1au_2 \equiv_n v_1av_2$ for $a \in A \setminus (\alpha(u_1) \cup \alpha(v_1))$, then $u_1 \equiv_n v_1$ and $u_2 \equiv_n v_2$.
3. If $u_1au_2 \equiv_n v_1av_2$ for $a \in A \setminus (\alpha(u_2) \cup \alpha(v_2))$, then $u_1 \equiv_n v_1$ and $u_2 \equiv_n v_2$.

**Proof:** We have $A^*/\equiv_n \in R \lor L$ since it is a submonoid of $(A^*/\equiv_n^R) \times (A^*/\equiv_n^L)$, and $A^*/\equiv_n \in R$ and $A^*/\equiv_n \in L$ by Lemma 1 and its left-right dual. The properties 2 and 3 trivially follow from the definition of $\equiv_n$.  \[\square\]
4 An Equation for the Join

The goal of this section is to prove $W \subseteq R \lor L$. By Lemma 2, it suffices to show that for every $A$-generated monoid $M \in W$ there exists an integer $n \geq 0$ such that $M$ is a quotient of $A^* / \equiv_n$. The outline of the proof is as follows. First, in Lemma 3 we give a substitution rule valid in $W$. Then, in Lemma 4 we show that $\equiv_n$-equivalence allows a factorization satisfying the premise for applying this substitution rule; this relies on a property of $W$ shown in Lemma 5. Finally, in Theorem 6 all the ingredients are put together.

Lemma 3 Let $M \in W$ and let $u, v, x \in M$. If $u \not\equiv R ux$ and $v \not\equiv L xv$, then $uxv = uv$.

Proof: Since $u \not\equiv R ux$ and $v \not\equiv L xv$, there exist $y, z \in M$ with $u = uxy$ and $v = zvx$. In particular, we have $u = u(xy)^{\omega}$ and $v = (zx)^{\omega}$. By $M \in W$ we conclude $uxv = u(xy)^{\omega}(zx)^{\omega}v = u(xy)^{\omega}(zx)^{\omega}v = uv$. \qed

We will apply the previous lemma as follows. Let $M \in W$ and $u, v, s, t \in M$ such that $u \not\equiv R us$ and $v \not\equiv L sv$. Then $usv = utv$ since $usv = uv$ and $utv = uv$ by Lemma 3. The $R$-equivalences and $L$-equivalences for being able to apply this substitution rule are established in Lemma 5. Before we give a simple property of $W$. It is the link between Green’s relations and the congruence $\equiv_n$.

Lemma 4 Let $M \in W$ and let $u, v, a \in M$. If $u \not\equiv R v \not\equiv R va$, then $u \not\equiv R ua$. If $u \not\equiv L v \not\equiv L av$, then $u \not\equiv L au$.

Proof: Since $u \not\equiv R v$ and $u \not\equiv R va$, there exist $x, y \in M$ with $u = uxy$ and $v = vay$. Now, $u = uxy = u(xay)^{2\omega+1} = u(xay)^{\omega}(axy)^{\omega}ay = u(xay)^{\omega}(axy)^{\omega}ay = u(axy)^{\omega}ay \in uAM$ where the fourth equality uses $M \in W$. This shows $uM \subseteq uAM$ and thus $u \not\equiv R ua$. The second implication is left-right symmetric. \qed

The intuitive interpretation of the algebraic statement in Lemma 4 is the following: For $M \in W$ it only depends on the element $a$ and the $R$-class of $u$ whether $u \not\equiv R ua$ or not (but not on the element $u$ itself). The statement for $L$-classes is analogous.

Lemma 5 Let $M \in W$ and let $\varphi : A^* \to M$ be a homomorphism. If $u \equiv_n v$ for $n \geq 2|M|$, then there exist factorizations $u = a_1s_1 \cdots a_{\ell-1}s_{\ell-1}a_\ell$ and $v = a_1t_1 \cdots a_{\ell-1}t_{\ell-1}a_\ell$ with $a_i \in A$ and $s_i, t_i \in A^*$ and with $\ell \leq 2|M|$ such that for all $i \in \{1, \ldots, \ell-1\}$ we have:

\[
\varphi(a_is_1 \cdots a_{i-1}s_{i-1}a_i) \equiv \varphi(a_1s_1 \cdots a_is_i) \equiv \varphi(a_1s_1 \cdots a_{i-1}s_{i-1}a_\ell),
\]

\[
\varphi(a_1t_1 \cdots a_{\ell-1}t_{\ell-1}a_\ell) \equiv \varphi(a_1t_1 \cdots a_{i-1}t_{i-1}a_i) \equiv \varphi(s_is_{i+1} \cdots s_{\ell-1}a_\ell).
\]

Proof: To simplify notation, for some relation $\not\equiv$ on $M$ we write $u \not\equiv v$ for words $u, v \in A^*$ if $\varphi(u) \not\equiv \varphi(v)$. Consider the $R$-factorization of $u$, i.e., let $u = b_1u_1 \cdots b_ku_k$ with $b_i \in A$ such that

\[
b_1u_1 \cdots b_i \equiv_R b_1u_1 \cdots b_iu_i \quad \text{for all } i \in \{1, \ldots, k\},
\]

\[
b_1u_1 \cdots b_iu_i \equiv_R b_1u_1 \cdots b_iu_{i+1} \quad \text{for all } i \in \{1, \ldots, k-1\}.
\]

Similarly, let $v = v_1c_1 \cdots v_{k'}c_{k'}$ be the $L$-factorization of $v$, i.e., we have $c_i \in A$ and

\[
c_i \cdots v_{k'}c_{k'} \equiv_L v_ic_i \cdots v_{k'}c_{k'} \quad \text{for all } i \in \{1, \ldots, k'\},
\]

\[
v_ic_i \cdots v_{k'}c_{k'} \equiv_L c_{i-1}v_ic_i \cdots v_{k'}c_{k'} \quad \text{for all } i \in \{2, \ldots, k'\}.
\]
We have $k, k' \leq |M|$ because neither the number of $R$-classes nor the number of $L$-classes can exceed $|M|$. By Lemma 3, we have $b_i \notin \alpha(u_{i-1})$ for all $i \in \{2, \ldots, k\}$ and $c_i \notin \alpha(v_{i+1})$ for all $i \in \{1, \ldots, k' - 1\}$. We use these properties to convert the $R$-factorization of $u$ to $v$ and to convert the $L$-factorization of $v$ to $u$: Let $v = b_1 v'_1 \cdots b_k v'_k$ such that $b_i \notin \alpha(v'_{i+1})$, and let $u = u_1 c_1 \cdots u_k' c_k'$ with $c_i \notin \alpha(u'_{i+1})$. These factorizations exist because $u \equiv_n v$; in particular, by Lemma 2,

$$u_{i} b_{i+1} u_{i+1} \cdots b_k v_k = n^{-i} v'_i b_{i+1} v'_{i+1} \cdots b_k v_k$$

$$v_1 c_1 \cdots v_{j-1} c_{j-1} v_j = n^{k'-i-j} u_1 c_1 \cdots u'_{j-1} c_{j-1} u_j$$

for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, k'\}$. Moreover, we see that $\alpha(u_i) = \alpha(v'_i)$ and $\alpha(v_j) = \alpha(u'_j)$.

We now show that the relative positions of the $b_i$'s and $c_j$'s in the above factorizations are the same in $u$ and $v$. Let $p$ be the position of $b_1$ in the $R$-factorization of $u$ and let $q$ be the position of $c_1$ in the above factorization of $u$. Similarly, let $p'$ be the position of $b_1$ in $v$ and let $q'$ be the position of $c_1$ in $v$. First, suppose $p < q$. Let

$$u = b_1 u_1 \cdots b_{i-1} u_{i-1} b_i u'_{i+1} c_j u'_{j+1} \cdots u_k' c_k'$$

By an $i$-fold application of property 1 in Lemma 2 with $a \in \{b_1, \ldots, b_i\}$ (which is possible for $u$) we obtain $v = b_1 v'_1 \cdots b_{i-1} v'_{i-1} b_i z$ with $z = n^{-i} u' c_j u'_{i+1} \cdots u_k' c_k'$. By a $(k' + 1 - j)$-fold application of property 1 in Lemma 2 with $a \in \{c_{j'}, \ldots, c_{j}\}$ (which is possible for the word $u' c_j u'_{j+1} \cdots u_k' c_k'$) we obtain $z = v c_j v'_{j+1} \cdots v_k c_k'$. Thus

$$v = b_1 v'_1 \cdots b_{i-1} v'_{i-1} b_i v' c_j v'_{j+1} \cdots v_k c_k'$$

showing that $p' < q'$. Symmetrically, one shows that $p' < q'$ implies $p < q$. We conclude $p < q$ if and only if $p' < q'$. Similarly, we have $p = q$ if $q = q'$ if it follows that the relative order of the $b_i$'s and $c_j$'s in $u$ and $v$ is the same. By factoring $u$ and $v$ at all $b_i$'s and $c_j$'s, we obtain $u = a_1 s_1 \cdots a_{i-1} s_{i-1} a_i$ and $v = a_1 t_1 \cdots a_{k'} t_{k'-1} a_k$ with $a_i \in A$ and $\ell \leq k + k' \leq 2 |M|$. We have $a_1 s_1 \cdots a_{i-1} s_{i-1} a_i R a_1 s_1 \cdots a_{i-1} s_{i-1} a_i s_{i}$ since the factorization $u = a_1 s_1 \cdots a_{i-1} s_{i-1} a_i$ is a refinement of the $R$-factorization. Note that we cannot assume $\alpha(s_i) = \alpha(t_i)$. But each $t_i$ is a factor of some $v'_j$, and at the same time $s_i$ is a factor of $u_j$. More precisely, there exists $m \leq i$ such that

$$b_1 v'_1 \cdots b_{j-1} v'_{j-1} b_j = a_1 t_1 \cdots a_{m-1} t_{m-1} a_m \quad \text{and} \quad t_{m+1} a_m \cdots t_{i-1} a_i$$

Furthermore, $s_m a_{m+1} \cdots s_{j-1} a_i s_i$ is a prefix of $v_j$. Now, $\alpha(t_i) \subseteq \alpha(v'_j) = \alpha(u_j)$ and, by Lemma 4 for all words $z$ with $\alpha(z) \subseteq \alpha(u_j)$ we have $a_1 s_1 \cdots a_{i-1} s_{i-1} a_i R a_1 s_1 \cdots a_{i-1} s_{i-1} a_i z$. Symmetrically we see $a_{i+1} t_{i+1} \cdots a_{k'} t_{k'-1} a_k L s_{i+1} t_{i+1} \cdots a_{i-1} t_{i-1} a_i$. 

\[\square\]

**Theorem 6 (Almeida/Azevedo, 1989)**

\[R \lor L = [(xy)^\omega x(xy)^\omega = (xy)^\omega (xy)^\omega]\]

**Proof:** The inclusion $R \lor L \subseteq W$ is trivial since $R \cup L \subseteq W$ and $W$ is a variety of finite monoids. Let $M \in W$ be generated by $A$, and let $\varphi : A^* \rightarrow M$ be the homomorphism induced by $A \subseteq M$. Let $n = 2 |M|$ and
suppose \( u \equiv_n v \). Let 
\[ u = a_1 s_1 \cdots a_{\ell - 1} s_{\ell - 1} a_\ell \] and 
\[ v = a_1 t_1 \cdots a_{\ell - 1} t_{\ell - 1} a_\ell \] be the factorizations from Lemma 5. Applying Lemma 3 repeatedly, we get 
\[ \varphi(v) = \varphi(a_1 a_2 t_2 \cdots a_{\ell - 2} t_{\ell - 2} a_{\ell - 1} a_\ell) = \varphi(a_1 s_1 a_2 s_2 \cdots a_{\ell - 2} s_{\ell - 2} a_{\ell - 1} a_\ell) = \varphi(a_1 s_1 a_2 s_2 \cdots a_{\ell - 2} s_{\ell - 2} s_{\ell - 1} a_\ell) = \varphi(a_1 s_1 a_2 s_2 \cdots a_{\ell - 2} s_{\ell - 2} s_{\ell - 1} s_{\ell - 1} s_{\ell - 1} a_\ell) = \varphi(u). \]

Note that the substitution rules \( t_i \to s_i \) are \( \varphi \)-invariant only when applied from left to right. This shows that \( M \) is a quotient of \( A^\ast / \equiv_n \), and the latter is in \( R \lor L \) by Lemma 2. Thus \( M \in R \lor L \). \( \square \)

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