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On cordial labeling of hypertrees

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Let $f : V \to \mathbb{Z}_k$ be a vertex labeling of a hypergraph $H = (V, E)$. This labeling induces an edge labeling of H defined by $f(e) = \sum_{v \in e} f(v)$, where the sum is taken modulo k. We say that f is k-cordial if for all $a, b \in \mathbb{Z}_k$ the number of vertices with label a differs by at most 1 from the number of vertices with label b and the analogous condition holds also for labels of edges. If H admits a k-cordial labeling then H is called k-cordial. The existence of k-cordial labelings has been investigated for graphs for decades. Hovey (1991) conjectured that every tree T is k-cordial for every $k \geq 2$. Cichacz, Görlich and Tuza (2013) were first to investigate the analogous problem for hypertrees, that is, connected hypergraphs without cycles. The main results of their work are that every k-uniform hypertree is k-cordial for every $k \geq 2$ and that every hypertree with n or m odd is 2-cordial. Moreover, they conjectured that in fact all hypertrees are 2-cordial. In this article, we confirm the conjecture of Cichacz et al. and make a step further by proving that for $k \in \{2, 3\}$ every hypertree is k-cordial.

Keywords: k-cordial graph, hypergraph, hypergraph labeling, hypertree

1 Introduction

Graph labeling problems have been intensively studied for decades since the initiatory work of Rosa [Ros67]. However, much less is known about labelings of hypergraphs. In this article we consider the problem of cordial labeling of hypergraphs introduced by Cichacz, Görlich and Tuza [CGT13].

Let $f : V \to \mathbb{Z}_k$ be a vertex labeling of a hypergraph $H = (V, E)$. This labeling induces an edge labeling of H defined by $f(e) = \sum_{v \in e} f(v)$, where the sum is taken modulo k. We say that f is k-cordial if for all $a, b \in \mathbb{Z}_k$ the number of vertices with label a differs by at most 1 from the number of vertices with label b and the analogous condition holds also for labels of edges. If H admits a k-cordial labeling then H is called k-cordial. 2-cordial labelings in the case of graphs were introduced by Cahit [Cah87] (under the name of cordial labeling) as a weakened version of well known graceful and harmonious labelings. On the other hand, harmonious labeling and elegant labeling, other known concepts, are special cases of k-cordial labeling. More precisely, harmonious labeling can be defined as $|E|$-cordial labeling and elegant labeling can be defined as $|V|$-cordial labeling. For more information about various graph and hypergraph labeling problems, we refer to an extensive dynamic survey of Gallian [Gal14].

Since the work of Cahit, k-cordial labelings have been studied in numerous publications (see [Gal14]). Lee and Liu [LL91] and Du [Du97] proved that a complete ℓ-partite graph is 2-cordial if and only if at most three of its partite sets have odd cardinality. Cairnie and Edwards [CE00] showed that in general

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the problem of deciding whether a graph is 2-cordial is NP-complete (the authors suggest that the problem is NP-complete even in the class of connected graphs of diameter 2). Cordial labeling seems to be particularly interesting for trees. The original motivation for Cahit to investigate cordial labeling were the Graceful Tree Conjecture of Rosa [Ros67] and the Harmonious Tree Conjecture of Graham and Sloane [GS80]. Both conjectures were (and still are) far from being solved. However, Cahit was able to prove a common relaxation: all trees are 2-cordial. Hovey [Hov91] conjectured that every tree is k-cordial for any $k \geq 2$ and proved this to be true for $k = 3, 4, 5$. Note that positive solution to his conjecture would imply the Harmonious Tree Conjecture. Hegde and Murthy [HM14] showed that every tree is k-cordial for prime k provided that k is not smaller than the number of vertices. Recently, it has been showed by Driscoll, Krop and Nguyen [DKN17] that the Hovey's conjecture is true for $k = 6$. For other values of k, the conjecture is still open.

Generalizations of labeling problems to hypergraphs have been studied for example for magic labelings [Tre01a, Tre01b], antimagic labelings [Cic16, JB12, Son02] and sum number and integral sum number [ST00, ST01]. Cichacz, Görlich and Tuza [CGT13] were first to consider k-cordial labelings of hypergraphs. They provided some sufficient conditions for a hypergraph to be k-cordial. Their main results state that every k-uniform hypertree is k-cordial for every $k \geq 2$ and that every hypertree H with $|V(H)|$ or $|E(H)|$ odd is 2-cordial (for the definition of a hypertree, see next section). The second result is a partial answer to the following conjecture posed in their article:

Conjecture 1 ([CGT13])**.** *All hypertrees are* 2*-cordial.*

Note that one cannot hope for 2-cordiality of forests, since the forest consisting of 2 disjoint edges of size 2 is not 2-cordial. In this article we settle Conjecture 1 and make a step further by proving a stronger statement:

Theorem 2. *All hypertrees are k-cordial for $k \in \{2, 3\}$.*

Theorem 2 generalizes results of Cahit and Hovey on k-cordiality of tress for $k \in \{2, 3\}$. We find this theorem to be a good indication that the following generalization of the Hovey's conjecture (in fact, already tentatively suggested in [CGT13]) can be true:

Conjecture 3. *All hypertrees are k-cordial for every $k \geq 2$.*

We prove three cases of Theorem 2 separately. They both follow the same method, although the case of $k = 3$ is more complicated and requires some additional notions. The main idea is the following. For a given hypertree, we choose a certain configuration S of k edges and k vertices and we inductively label $H - S$. Having this partial labeling f, we try to extend it to entire H. Sometimes this is not possible, however, it appears that then it is enough to modify f on small parts of $H - S$ to succeed.

2 Preliminaries

A *hypergraph* H is a pair $H = (V, E)$, where V is the set of vertices and E is a set of non-empty subsets of V called edges. We consider finite (not necessarily uniform) hypergraphs with edges of cardinality at least 2. Hypergraphs with no edges are called *trivial*. For a hypergraph H, by $n(H)$ and $m(H)$ we denote the number of vertices and edges of H, respectively. The *degree* of vertex v, denoted by $d(v)$, is the number of edges containing v. An *isolated* vertex is a vertex of degree 0.

The *incidence graph* G_H of a hypergraph H is a bipartite graph with the vertex set $V(H) \cup E(H)$ and the edge set $\{ve : v \in V(H), e \in E(H), v \in e\}$. By a *cycle* in a hypergraph H we understand a cycle in G_H. We say that a hypergraph H is *connected* if G_H is connected. A *hypertree* is a connected hypergraph with no cycles. Equivalently, a hypergraph H is a hypertree if G_H is a tree. Observe that if two edges e and e' have two common vertices v and v' then $vev'e'v$ is a cycle. Thus our definition implies that a hypertree is a linear hypergraph, that is, every two edges can have at most one common vertex.

Let $H = (V, E)$ be a hypergraph. For a set of vertices $W \subset V$ and a set of edges $F \subset E$, we denote $H - W = (V - W, E - \{e : \exists v \in e, v \in W\})$ and $H - F = (V, E - F)$. For a vertex v or an edge e, we simply write $H - v$ for $H - \{v\}$ and $H - e$ for $H - \{e\}$. For a set of edges $F \subset E$, denote by $H \ominus F$ the hypergraph obtained by removing all isolated vertices from the hypergraph $H - F$.

An edge in a hypergraph is a *leaf-edge* if it contains at most one vertex of degree greater than 1, otherwise we call this edge *internal*. A vertex of degree one contained in a leaf-edge is called a *leaf*. Note that if T is a hypertree with more than one edge, then $e \in E(T)$ is a leaf-edge if and only if $T - e$ consists of a hypertree and some isolated vertices. Observe that if a hypertree has more than one edge, then it has at least two leaf-edges. A *hyperpath* is a hypertree with at most 2 leaf-edges. A *hyperstar* is a hypertree of which every edge is a leaf-edge.

We will use the following formula for the number of edges of a hypertree.

Proposition 4. *If T is a non-trivial hypertree then*

$$m(T) = 1 + \sum_{v \in V(T)} (d(v) - 1).$$

By $\mathbb{Z}_k$ we denote the ring of integers modulo k. When we compare the elements of $\mathbb{Z}_k$, we use the order $0 < \ldots < k-1$. The set of $p \times q$ matrices over $\mathbb{Z}_k$ is denoted by $M_p^q(\mathbb{Z}_k)$.

For the rest of this section, we assume that k is a fixed positive integer greater than 1. Let $f : V \to \mathbb{Z}_k$ be a vertex labeling of a hypergraph $H = (V, E)$ with n vertices and m edges. This vertex labeling induces an edge labeling of H, also denoted by f, defined by $f(e) = \sum_{v \in e} f(v)$ (the sum is taken modulo k) for $e \in E$. We allow this abuse of notation, as it is commonly used in this topic. Denote by $n_a(f)$ and $m_a(f)$ the numbers of vertices and edges, respectively, labeled with a. We say that f is *k-cordial* if for all $a, b \in \mathbb{Z}_k$ we have $|n_a(f) - n_b(f)| \le 1$ and $|m_a(f) - m_b(f)| \le 1$. If H admits a k-cordial labeling then H is called *k-cordial*.

Now we will give the definition of a sprig, a key notion in this paper. Sprigs will be used for the induction step in the main cases of our proofs. That is, in order to label a hypertree T, we will delete a certain sprig, label the smaller hypergraph by induction hypothesis, and then label the vertices of the sprig in order to obtain a cordial labeling of T.

Definition 5. *Let $H = (V, E)$ be hypergraph. A sequence $S = (e_1, \ldots, e_k; v_1, \ldots, v_k)$, where every e_i is an edge of H and every v_i is a vertex of H, is called a* sprig *if*

1. *$v_i \in e_i$ for every i,*
2. *v_i is an isolated vertex of $H - \{e_1, \ldots, e_k\}$ for every i.*

For a hypergraph H and a sprig $S = (v_1, ..., v_k; e_1, ..., e_k)$, denote by $H - S$ and $H \ominus S$ the hypergraphs $H - \{v_1, \ldots, v_k\}$ and $H \ominus \{e_1, ..., e_k\}$, respectively. A sprig S in a hypertree T is called *pendant* if $T - S$ has at most one non-trivial component.

Let $H = (V, E)$ be a hypergraph and let $A \subset V$. We say that a sprig $S = (e_1, \ldots, e_k; v_1, \ldots, v_k)$ is:

- *containing* A if every vertex of S belongs to A,
- *fully-incident* with A if no vertex of S belongs to A and every edge of S contains a vertex from A,
- *non-incident* with A if none of the edges of S contain a vertex from A.

Let $S = (e_1, \ldots, e_k; v_1, \ldots, v_k)$ be a sprig. The *adjacency matrix of* S is a matrix $M(S) = (a_{ij})_{k\times k}$ defined by $a_{ij} = \begin{cases} 1 & \text{if } v_j \in e_i \\ 0 & \text{otherwise} \end{cases}$. Let S be a sprig and $M \in M_k^k(\mathbb{Z}_k)$. We call S an M-*sprig* if $M(S) = M$.

Let A be a set of pairwise non-adjacent vertices in a hypergraph H. We say that a cordial labeling of H is *strong on* A if every vertex of A has a different label and the numbers of edges intersecting with A labeled with a are equal for all labels $a \in \mathbb{Z}_k$.

Observe that adding isolated vertices does not change the cordiality of a hypergraph. More precisely, we have the following proposition.

Proposition 6. *Let $H = (V, E)$ be a hypergraph and let H' be a hypergraph obtained by adding any number of isolated vertices to H. If H is k-cordial, then H' is k-cordial. Moreover if $A \subset V$ and H has a k-cordial labeling strong on A, then H' has a k-cordial labeling strong on A. In particular, every hypergraph without edges is k-cordial.*

Let f be a labeling of a hypergraph $H = (V, E)$ and let $A \subset V$ and $x \in \mathbb{Z}_k$. We say that a labeling g of H is obtained from f by *adding* x *on* A if it is defined in the following way: $g(v) = \begin{cases} f(v) + x & \text{if } v \in A \\ f(v) & \text{otherwise} \end{cases}$.

3 2-cordiality of hypertrees

The aim of this section is to prove Theorem 9, which states that all hypertrees are 2-cordial. The case of $m =_2 0$ is the main part of the proof. We prove this case by induction on m. The induction step goes as follows: we delete from a hypertree T a certain sprig S, label $T - S$ by the induction hypothesis, and then label the removed vertices. In the beginning of the section, we present sprigs that will be used in this proof. Lemma 7 shows how to obtain a cordial labeling of T from a cordial labeling of $T - S$. In some cases, we cannot simply extend the labeling, but we will succeed if we change the label of a vertex of even degree first. We want to ensure that this operation will not unbalance the edge labels. Therefore, we prove a stronger statement: for any vertex of even degree u, there exists a cordial labeling of T which is strong on $\{u\}$. Lemma 8 assures that vertex of even degree always exists.

In this section we will use M_i-sprigs for $i = 1, 2$, where:

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

For an illustration, see Figure 1.

(a) M_1-sprig (b) M_2-sprig

Fig. 1: Types of sprigs

Lemma 7. *Let H be a hypergraph, u a vertex of even degree in H and S*

1. *an M_1-sprig non-incident or fully-incident with $\{u\}$,*
2. *an M_2-sprig non-incident with $\{u\}$.*

If $H - S$ has a 2-cordial labeling strong on $\{u\}$, then H also has 2-cordial labeling strong on $\{u\}$.

Proof: Let $S = (e_1, e_2; v_1, v_2)$. Notice that $d_{H-S}(u)$ is even. Let f be a 2-cordial labeling of $H - S$ strong on $\{u\}$. For $i = 1, 2$ let $Y_i = e_i - \{v_1, v_2\}$ and $y_i = \sum_{v \in Y_i} f(v)$.

Case 1: S is an M_1-sprig non-incident or fully-incident with $\{u\}$.

If $y_1 = y_2$ then we extend f to a labeling of H by defining $f(v_1) = 0$ and $f(v_2) = 1$. Then $f(e_1) = y_1 + f(v_1) = y_1$ and $f(e_2) = y_2 + f(v_2) = y_2 + 1$, so $f(e_1) \neq f(e_2)$ and f is a 2-cordial labeling strong on $\{u\}$.

Suppose that $y_1 \neq y_2$. Let $a = f(u)$. Let f' be the labeling of $H - S$ obtained from f by adding 1 on $\{u\}$. Then $m_0(f') = m_0(f)$ and $m_1(f') = m_1(f)$, hence $|m_0(f') - m_1(f')| \leq 1$. Moreover $n_a(f') = n_a(f) - 1$, $n_{a+1}(f') = n_{a+1}(f) + 1$ and thus $n_{a+1}(f') - n_a(f') \in \{1, 2, 3\}$. We extend f' to a labeling of H by defining $f'(v_1) = a$ and $f'(v_2) = a$. Then

$$f'(e_1) = y_1 + f'(v_1) = y_1 + a,$$
$$f'(e_2) = y_2 + f'(v_2) = y_2 + a$$

if S is non-incident with $\{u\}$ and

$$f'(e_1) = y_1 + 1 + f'(v_1) = y_1 + 1 + a,$$
$$f'(e_2) = y_2 + 1 + f'(v_2) = y_2 + 1 + a$$

if S is fully-incident with $\{u\}$. In both cases we have $f'(e_1) \neq f'(e_2)$, thus still $|m_0(f') - m_1(f')| \leq 1$. Moreover now we have $n_{a+1}(f') - n_a(f') \in \{-1, 0, 1\}$. Therefore f' is a 2-cordial labeling strong on $\{u\}$.

Case 2: S is an M_2-sprig non-incident with $\{u\}$.

We extend f to a 2-cordial labeling of H by defining $f(v_1)$ and $f(v_2)$. We have

$$f(e_1) = y_1 + f(v_1), \qquad f(e_2) = y_2 + f(v_1) + f(v_2).$$

Either $(y_1, y_2) = (b, b)$ or $(y_1, y_2) = (b, b + 1)$ for some $b \in \mathbb{Z}_2$. The values of $f(v_1)$ and $f(v_2)$ depending on y_1 and y_2 are given in Table 1.

y_1	y_2	$f(v_1)$	$f(v_2)$	$f(e_1)$	$f(e_2)$
b	b	0	1	b	b+1
b	b+1	1	0	b+1	b

Tab. 1

In both cases we have $f(e_1) \neq f(e_2)$, thus f is a 2-cordial labeling of H strong on $\{u\}$.

□

Lemma 8. *Every hypertree with an even number of edges has a vertex of even degree.*

Proof: Follows from Proposition 4. □

Theorem 9. *Every hypertree is* 2*-cordial.*

Proof: Let T be a hypertree with $m = m(T)$. We divide the proof into two cases.

Case 1: $m =_2 0$

By Lemma 8 every hypertree T with an even number of edges has a vertex of even degree. We prove a stronger statement: If T is a hypertree with an even number of edges and u is a vertex of even degree in T, then there exists a 2-cordial labeling of T strong on $\{u\}$.

The proof is by induction on m. For $m = 0$ the assertion obviously holds. Let T be a hypertree with $m > 0$ edges, $m =_2 0$, and let u be a vertex of even degree in T. We will find a pendant sprig S, which satisfies the assumptions of Lemma 7 and will be used in the induction step.

If $m > d(u)$, then there exists a set F containing two edges such that $T - F$ has at most one non-trivial component. Clearly, we can choose such two edges e_1, e_2 and vertices $v_i \in e_i$ for $i = 1, 2$ in a way that they can be arranged into a pendant M_1-sprig or M_2-sprig S non-incident with $\{u\}$. Notice that $d_{T\ominus S}(u) =_2 0$. Otherwise T is a hyperstar with the central vertex u. We take as S a pendant M_1-sprig fully-incident with $\{u\}$ consisting of two edges incident with u and one leaf from each of these edges. Observe that either $T \ominus S$ is the empty hypergraph or $d_{T\ominus S}(u) =_2 0$.

In each case we have found a pendant sprig S such that (by induction hypothesis and Proposition 6) T, u and S satisfy the assumptions of Lemma 7. Therefore, by Lemma 7, T has a 2-cordial labeling strong on $\{u\}$.

Case 2: $m =_2 1$

Let e be a leaf-edge in T. By Case 1 and Proposition 6, $T - e$ has a 2-cordial labeling f. Clearly, f is also a 2-cordial labeling of T, regardless of the induced value of $f(e)$.

□

4 3-cordiality of hypertrees

The case of $k = 3$ needs more careful analysis than $k = 2$. In this section, we extend our notation. In the previous section, we used a vertex of even degree to help us to extend the labeling. For the same purpose, for $k = 3$ we will use two different structures.

Let $T = (V, E)$ be a hypertree.

Definition 10. *A set* $\{u\} \subset V$ *is a* helpful 1-configuration *if* $d(u) =_3 0$.

Definition 11. *A set* $\{u_1, u_2\} \subset V$ *is a* helpful 2-set *if* u_1 *and* u_2 *are non-adjacent,* $d(u_2) =_3 2$ *and* u_1 *is a leaf.*

Notice that if $\{u_1, u_2\}$ is a helpful 2-set then $d(u_1) + d(u_2) =_3 0$.

Sometimes we need to remove a helpful 2-set A from H with a sprig containing A. In order to be able to proceed by induction, we need the hypergraph obtained by removing this sprig to be a hypertree. Before removing A we have to remove some "pendant" sprigs. Hence we introduce the following definition.

Definition 12. *Let* $A = \{u_1, u_2\}$ *be a helpful* 2*-set, where* $d(u_2) =_3 2$*. Denote by* $P_T(A) \subset E$ *the set of edges which belong to those components of* $T - u_2$ *which do not contain* u_1*. We say that* A *is a* helpful 2-configuration *if* $|P_T(A)| =_3 0$.

Notice that if A is a helpful 2-configuration then $(T - P_T(A)) - A$ has at most one non-trivial component.

Definition 13. *A set* $A \subset V$ *is called a* helpful configuration *if it is either a helpful* 1*-configuration or a helpful* 2*-configuration.*

In order to compress the proofs in this section, we will use some matrix notation. Sequences of elements from $\mathbb{Z}_3$ will be treated sometimes purely as sequences (and then denoted with round brackets) and sometimes as elements of $M_3^1(\mathbb{Z}_3)$ (and then denoted with square brackets). The following definitions will be used for compressing the proofs in this section.

Let $\mathcal{P}$ and $\mathcal{D}$ be the set of all vectors from $M_3^1(\mathbb{Z}_3)$ containing exactly 3 and 2 distinct coordinates, respectively.

Definition 14. *Let* $M \in M_3^3(\mathbb{Z}_3)$ *and* $y \in M_3^1(\mathbb{Z}_3)$*. We say that* $x \in \mathcal{P}$ *is a* simple M-solution *for* y *if* $y + Mx \in \mathcal{P}$*.*

Definition 15. *Let* $M \in M_3^3(\mathbb{Z}_3)$ *and* $y \in M_3^1(\mathbb{Z}_3)$*. Consider* $\mathcal{X} \subset \mathcal{D}$ *and the following conditions:*

1. $|\mathcal{X}| = 3$,
2. *every* $x \in \mathcal{X}$ *satisfies* $y + Mx \in \mathcal{P}$,
3. *for every* $a \in \mathbb{Z}_3$ *there exists* $x \in \mathcal{X}$ *with two coordinates equal to* a,
4. *for every* $a \in \mathbb{Z}_3$ *there exists* $x \in \mathcal{X}$ *with no coordinate equal to* a.

We say that $\mathcal{X}$ *is: a* 1-composed M-solution *for* y *if it satisfies conditions 1, 2, 3; a* 2-composed M-solution *for* y *if it satisfies conditions 1, 2, 4; a* composed M-solution *for* y *if it satisfies conditions 1, 2, 3, 4.*

The following lemma transforms the problem of extending a partial labeling to a 3-cordial labeling to a problem of finding a certain M-solution.

Lemma 16. *Let H be a hypergraph, $S = (e_1, e_2, e_3; v_1, v_2, v_3)$ an M-sprig in H for some $M \in M_3^3(\mathbb{Z}_3)$ and A a helpful configuration in H. Assume there exists a 3-cordial labeling f of $H - S$. For $j = 1, 2, 3$ denote $Y_j = e_j - \{v_1, v_2, v_3\}$, $y_j = \sum_{v \in Y_j} f(v)$ and $y = [y_1, y_2, y_3]^T$. Assume that one of the following conditions is satisfied:*

1. *S is non-incident or fully-incident with A, f is strong on A and x is a simple M-solution for y,*
2. *S is containing A and x is a simple M-solution for y,*
3. *A is a helpful 1-configuration, S is non-incident or fully-incident with A, f is strong on A and $\mathcal{X}$ is a 1-composed M-solution for y.*
4. *A is a helpful 2-configuration, S is non-incident or fully-incident with A, f is strong on A and $\mathcal{X}$ is a 2-composed M-solution for y.*

Then there exists a 3-cordial labeling of H strong on A.

Proof: First assume that $x = [x_1, x_2, x_3]^T$ is a simple M-solution for y. Let $z = [z_1, z_2, z_3]^T = y + Mx$. We extend f to the labeling f' of H by defining $f'(v_i) = x_i$ for $i = 1, 2, 3$. Then $f'(e_i) = z_i$ for $i = 1, 2, 3$. Since both x and z are in $\mathcal{P}$, f' is a 3-cordial labeling of H. Clearly, if S is containing A, then f' is strong on A. Moreover, if f is strong on A and S is non-incident or fully-incident with A, then also f' is strong on A.

Consider the case when $A = \{u\}$ is a helpful 1-configuration and $\mathcal{X}$ is a 1-composed M-solution for y. Let $a = f(u)$. Choose $x = [x_1, x_2, x_3]^T \in \mathcal{X}$ with two coordinates equal to a and let $b \in \mathbb{Z}_3$ be the number missing in x. Let $z = [z_1, z_2, z_3]^T = y + Mx$. Let f' be the labeling obtained from f by adding $b - a$ on A. We extend f' to the labeling of H by defining $f'(v_i) = x_i$ for $i = 1, 2, 3$. Then $f'(e_i) = z_i$ if S is non-incident with A and $f'(e_i) = z_i + b - a$ if S is fully-incident with A for $i = 1, 2, 3$. By the choice of x we have $|n_p(f') - n_q(f')| \leq 1$ for all $p, q \in \mathbb{Z}_3$. Since $z \in \mathcal{P}$ then also $|m_p(f') - m_q(f')| \leq 1$ for all $p, q \in \mathbb{Z}_3$. Hence f' is a 3-cordial labeling of H. Moreover, since f is strong on A and S is non-incident or fully-incident with A, we get that f' is also strong on A.

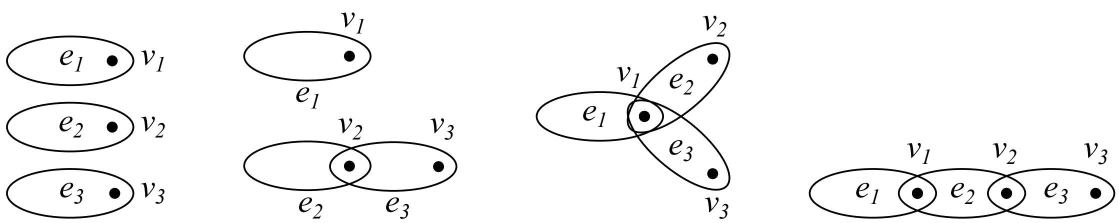
Now consider the case when $A = \{u_1, u_2\}$ is a helpful 2-configuration and $\mathcal{X}$ is a 2-composed M-solution for y. Let a be the element of $\mathbb{Z}_3$ different from $f(u_1)$ and $f(u_2)$. Choose $x = [x_1, x_2, x_3]^T \in \mathcal{X}$ with all coordinates distinct from a and let $b \in \mathbb{Z}_3$ be the number occurring on two coordinates of x. Moreover, let c be the element of $\mathbb{Z}_3$ different from a and b. Let f' be the labeling obtained from f by adding $a - c$ on A. Note that $b + a - c = c$. The rest of the proof goes as in the previous case. $\square$

The plan of the rest of this section is the same as of Section 3. The main result is Theorem 19, which states that all hypertrees are 3-cordial. Again, the most involved part is the case $m(T) =_3 0$. The proof goes similarly to the case of $m(T) =_2 0$ in Theorem 9. The difference is that here we use helpful configurations instead of even degree vertices.

In this section, we will use M_i-sprigs for $i = 1, \ldots, 4$, where:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

For an illustration, see Figure 2.



(a) M_1-sprig (b) M_2-sprig (c) M_3-sprig (d) M_4-sprig

Fig. 2: Types of sprigs

Lemma 17. *Let H be a hypergraph, A a helpful configuration in H and S be*

1. *an M_1-sprig non-incident or fully-incident with A,*
2. *an M_2-sprig non-incident with or containing A,*
3. *an M-sprig non-incident with A for $M \in \{M_3, M_4\}$.*

Assume $H - S$ has a 3-cordial labeling f. Moreover, if S is non-incident or fully-incident with A, then assume f is strong on A.

Then there exists a 3-cordial labeling of H strong on A.

Proof: Let $S = (e_1, e_2, e_3; v_1, v_2, v_3)$. For $j = 1, 2, 3$ let $Y_j = e_j - \{v_1, v_2, v_3\}$ and $y_j = \sum_{v \in Y_j} f(v)$.

By Lemma 16, to extend the 3-cordial labeling of $H - S$ into a 3-cordial labeling of H it suffices to find a suitable simple or composed M_i-solution for $y = [y_1, y_2, y_3]^T$.

Let $\mathbf{1}$ denote $[1, 1, 1]^T$. Observe that $z \in \mathcal{P} \Leftrightarrow z + a \cdot \mathbf{1} \in \mathcal{P}$ and $z \in \mathcal{D} \Leftrightarrow z + a \cdot \mathbf{1} \in \mathcal{D}$ for every $a \in \mathbb{Z}_3$. Notice also that $y - b \cdot M_i \cdot \mathbf{1} + M_i \cdot (x + b \cdot \mathbf{1}) = y + M_i \cdot x$ for every $b \in \mathbb{Z}_3$. Therefore we claim that:

There exists a simple (composed) M_i-solution for y if and only if there exists a simple (composed) M_i-solution for $y' = y - a \cdot \mathbf{1} - b \cdot M_i \cdot \mathbf{1}$ for any $a, b \in \mathbb{Z}_3$. $(*)$

Consider $i \in \{2, 3, 4\}$. Then $M_i \cdot \mathbf{1} = [1, c, 2]^T$ for some $c \in \{1, 2\}$ depending on i. By $(*)$ it is sufficient to find a simple or composed M_i-solution only for all y' of the form $[0, y_2, 0]$. For $i = 1$ we have $M_1 \cdot \mathbf{1} = \mathbf{1}$. Hence by $(*)$ it is sufficient to find a simple or composed M_1-solution only for all y' of the form $[0, y_2, y_3]^T$. We denote $x = [x_1, x_2, x_3]^T$ and $z = y + M_i x = [z_1, z_2, z_3]^T$.

Case 1: We present M_1-solutions for $y = [0, y_2, y_3]^T$. By symmetry we can assume that $y_2 \le y_3$. Here are the corresponding formulas for z.

$$z_1 = 0 + x_1, \qquad z_2 = y_2 + x_2, \qquad z_3 = y_3 + x_3$$

The solutions are given in Table 2. For each pair of values of y_2 and y_3 (which corresponds to a single or a triple row), we give proper values of x. Simple solutions are presented as single rows, composed solutions are presented as triple rows. To make it easier to check, we also put the obtained values of z.

y_2	y_3	x_1	x_2	x_3	z_1	z_2	z_3
0	0	0	1	2	0	1	2
0	1	0	2	0	0	2	1
		0	1	1	0	1	2
		1	2	2	1	2	0
0	2	0	1	0	0	1	2
		1	2	1	1	2	0
		0	2	2	0	2	1
1	1	0	0	1	0	1	2
		1	1	2	1	2	0
		2	0	2	2	1	0
1	2	0	1	2	0	2	1
2	2	0	0	2	0	2	1
		1	0	1	1	2	0
		2	1	2	2	0	1

Tab. 2

Case 2: We present M_2-solutions for $y = [0, y_2, 0]^T$ in Table 3. Here are the corresponding formulas for z.

$$z_1 = 0 + x_1, \qquad z_2 = y_2 + x_2, \qquad z_3 = 0 + x_2 + x_3$$

y_2	x_1	x_2	x_3	z_1	z_2	z_3
0	1	0	2	1	0	2
1	1	2	0	1	0	2
2	2	1	0	2	0	1

Tab. 3

Case 3 (M_3): We present M_3-solutions for $y = [0, y_2, 0]^T$ in Table 4. Here are the corresponding formulas for z.

$$z_1 = 0 + x_1, \qquad z_2 = y_2 + x_1 + x_2, \qquad z_3 = 0 + x_1 + x_3$$

y_2	x_1	x_2	x_3	z_1	z_2	z_3
0	0	1	2	0	1	2
1	1	0	2	1	2	0
2	2	0	1	2	1	0

Tab. 4

Case 3 (M_4): We present M_4-solutions for $y = [0, y_2, 0]^T$ in Table 5. Here are the corresponding formulas for z.

$$z_1 = 0 + x_1, \qquad z_2 = y_2 + x_1 + x_2, \qquad z_3 = 0 + x_2 + x_3$$

y_2	x_1	x_2	x_3	z_1	z_2	z_3
0	1	2	0	1	0	2
1	2	0	1	2	0	1
2	1	0	2	1	0	2

Tab. 5

□

Lemma 18. *Every hypertree T with $m(T) =_3 0$ has a helpful configuration.*

Proof:

Denote $m = m(T)$. The proof is by induction on m. For $m = 0$ the assertion obviously holds. Let T be a hypertree with $m > 0$ edges, $m =_3 0$, and suppose that there is no vertex of degree divisible by 3 in T.

By Proposition 4, there exists a vertex v with $d(v) =_3 2$, denote $d = d(v)$. Let $\{e_1, \ldots, e_d\}$ be the set of edges incident with v. For $i = 1, \ldots, d$ denote by T_i the hypertree induced by e_i and the components of $T - v$ intersecting with e_i in T. Let $m_i = m(T_i)$ for $i = 1, \ldots, d$. We have $m_1 + \ldots + m_d =_3 0$. Since $d =_3 2$, not all $m_i =_3 1$. We consider two cases.

Case 1: $m_i =_3 0$ for some $i \in \{1, \ldots, d\}$

Let $T' = T_i$. As $m_i < m$, by the induction hypothesis there exists a helpful configuration A' in T'. There are no vertices of degree divisible by 3 in T', hence $|A'| = 2$. Let $u \in A'$ be the vertex which is not a leaf. If A' contains a leaf from e_i, say w, then let $A = A' - \{w\} \cup \{x\}$, where x is any leaf of T not contained in T_i. Otherwise let $A = A'$. Then A is a helpful 2-set in T. If $e_i \in P_{T'}(A')$ or if $u \in e_i$, then $P_T(A) = P_{T'}(A') \cup \big(E(T) - E(T_i)\big)$ and $|P_T(A)| = |P_{T'}(A')| + m - m_i =_3 0$. Otherwise $P_T(A) = P_{T'}(A')$. Therefore A is a helpful configuration in T.

Case 2: $m_i =_3 2$ for some $i \in \{1, \ldots, d\}$

Let $S_1, \ldots, S_q$ be the components of $T_i \ominus \{e_i\}$. Denote $p_j = m(S_j)$ for $j = 1, \ldots, q$. Note that $p_1 + \ldots + p_q = m_i - 1 =_3 1$. Hence not all $p_j =_3 0$.

Consider the case when for some j we have $p_j =_3 1$. S_j contains a leaf of T, say x. The set $A = \{v, x\}$ is a helpful 2-set in T. Moreover, $|P_T(A)| = m - d - p_j =_3 0$. Therefore A is a helpful configuration in T.

Otherwise for some j, it holds that $p_j =_3 2$. Consider the hypertree T' induced by S_j and e_i. Note that $m > m(T') =_3 0$. By induction hypothesis there exists a helpful configuration A' in T'. There are no vertices of degree divisible by 3 in T', hence $|A'| = 2$. If A' contains a leaf from e_i, say w, then let $A = A' - \{w\} \cup \{x\}$, where x is any leaf of T not contained in T_i.

Otherwise let $A = A'$. Then A is a helpful 2-set in T. If $e_i \in P_{T'}(A')$ or if $u \in e_i$, then $P_T(A) = P_{T'}(A') \cup \big(E(T) - E(T_i)\big) \cup \big(E(T_i) - E(T')\big)$ and $|P_T(A)| = |P_{T'}(A')| + m - m_i + m_i - p_j - 1 =_3 0$. Otherwise $P_T(A) = P_{T'}(A')$. Therefore A is a helpful configuration in T.

□

Theorem 19. *Every hypertree is* 3*-cordial.*

Proof: Denote $m = m(T)$. We divide the proof into three cases.

Case 1: $m =_3 0$

By Lemma 18 every hypertree T with $m =_3 0$ has a helpful configuration. We prove a stronger statement: if T is a hypertree with with $m =_3 0$ and A is a helpful configuration in T, then there exists a 3-cordial labeling of T strong on A.

The proof is by induction on m. For $m = 0$ the assertion obviously holds. Let T be a hypertree with $m > 0$ edges, $m =_3 0$, and let A be a helpful configuration in T. We will find a pendant sprig S, which satisfies the assumptions of Lemma 17 and will be used in the induction step.

First, we consider the case when A is a helpful 1-configuration. Let $A = \{u\}$. If $m > d(u)$, then there exist a set F containing three edges non-incident with u such that $T \ominus F$ has at most one non-trivial component. Hence we can choose such three edges e_1, e_2, e_3 and vertices $v_i \in e_i$ for $i = 1, 2, 3$ such that they can be arranged into a pendant M-sprig $S = (e_1, e_2, e_3; v_1, v_2, v_3)$ non-incident with A, where $M \in \{M_1, \ldots, M_4\}$. Notice that A is a helpful configuration in $T \ominus S$. Otherwise, $m = d(u)$ and T is a hyperstar with the central vertex u. We take as S a pendant M_1-sprig fully-incident with A consisting of three edges incident with u and one leaf from each of these edges. Observe that either $T \ominus S$ is the empty hypergraph or A is a helpful configuration in $T \ominus S$.

Now we consider the case when A is a helpful 2-configuration. Let $A = \{u, v\}$ and $d(u) =_3 2$. If $|P_T(A)| > 0$, then there exists a set F containing three edges non-incident with any vertex of A such that $T - F$ has at most one non-trivial component, thus there exists a pendant M-sprig S non-incident with A, where $M \in \{M_1, \ldots, M_4\}$. If $|P_T(A)| = 0$ and $d(u) > 2$ then we take as S a pendant M_1-sprig S fully-incident with A (consisting of three leaf-edges incident with u and suitably selected vertices). Observe that in both situations A is a helpful configuration in $T \ominus S$. If $|P_T(A)| = 0$ and $d(u) = 2$ then we take as S a pendant M_2-sprig containing A (consisting of the two edges incident with u, the edge incident with v and suitably selected vertices).

In each case we have found a pendant sprig S such that (by induction hypothesis and Proposition 6) T, A and S satisfy the assumptions of Lemma 17. Therefore, by Lemma 17, T has a 3-cordial labeling strong on A.

Case 2: $m =_3 1$

Let e be a leaf-edge in T. By Case 1 and Proposition 6, $T - e$ has a 3-cordial labeling f. Clearly, f is also a 3-cordial labeling of T, regardless of the induced value of $f(e)$.

Case 3: $m =_3 2$

Let e_1, e_2 be two leaf-edges in T, and let v_i be a leaf from e_i for $i = 1, 2$. Let $Y_i = e_i - \{v_i\}$ for $i = 1, 2$. Let $T' = T - \{v_1, v_2\}$. By Case 1 and Proposition 6, T' has a 3-cordial labeling f.

For $i = 1, 2$ let $y_i = \sum_{u \in Y_i} f(u)$. We have $n_a(f) \leq n_{a+1}(f) \leq n_{a+2}(f)$ for some $a \in \mathbb{Z}_3$. We extend f to a labeling of T by defining $f(v_1)$ and $f(v_2)$. The values of $f(v_1)$ and $f(v_2)$ depending on y_1 and y_2 are given in Table 6.

y_1	y_2	$f(v_1)$	$f(v_2)$	$f(e_1)$	$f(e_2)$
b	b	a	a+1	a+b	a+b+1
b	b+1	a	a+1	a+b	a+b+1
b	b+2	a+1	a	a+b+1	a+b+2

Tab. 6

□

5 Conclusions

We believe that our method can work to prove k-cordiality of hypertrees for larger values of k. However, the complication of the arguments is growing and some structures may need a special treatment. Therefore, one probably need some new ideas to work with larger values of k.

A hypergraph H is called *d-degenerate* if every subhypergraph H' of H (meaning $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$) has a vertex of degree at most d. In case of graphs, 1-degeneracy coincides with being a forest. However, in general hypergraphs, the class of 1-degenerated hypergraphs is much wider than the class of hyperforests (where a hyperforest is understood as a disjoint union of hypertrees). In this paper, we proved that all hypertrees are 2-cordial. It seems to be a natural next step to determine if the following conjecture is true:

Conjecture 20. *All* 1*-degenerated connected hypergraphs are* 2*-cordial.*

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