# *The generalized 3-connectivity of Cartesian product graphs*[†]

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The generalized connectivity of a graph, which was introduced by Chartrand et al. in 1984, is a generalization of the concept of vertex connectivity. Let $S$ be a nonempty set of vertices of $G$, a collection $\{T_1, T_2, \ldots, T_r\}$ of trees in $G$ is said to be internally disjoint trees connecting $S$ if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of distinct integers $i, j$, where $1 \leq i, j \leq r$. For an integer $k$ with $2 \leq k \leq n$, the $k$-connectivity $\kappa_k(G)$ of $G$ is the greatest positive integer $r$ for which $G$ contains at least $r$ internally disjoint trees connecting $S$ for any set $S$ of $k$ vertices of $G$. Obviously, $\kappa_2(G) = \kappa(G)$ is the connectivity of $G$. Sabidussi's Theorem showed that $\kappa(G \,\square\, H) \geq \kappa(G) + \kappa(H)$ for any two connected graphs $G$ and $H$. In this paper, we prove that for any two connected graphs $G$ and $H$ with $\kappa_3(G) \geq \kappa_3(H)$, if $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G \,\square\, H) \geq \kappa_3(G) + \kappa_3(H)$; if $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G \,\square\, H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Our result could be seen as an extension of Sabidussi's Theorem. Moreover, all the bounds are sharp.

**Keywords:** Connectivity, Generalized connectivity, Internally disjoint path, Internally disjoint trees.

## 1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretic notations and terminologies not described here. For any graph $G$, the connectivity $\kappa(G)$ of a graph $G$ is defined as $min\{|S| \ : S \subseteq V(G)$ and $G - S$ is disconnected or trivial$\}$. Whitney [14] showed an equivalent definition of the connectivity of a graph. For each pair of vertices $x, y$ of $G$, let $\kappa(x, y)$ denote the maximum number of internally disjoint paths connecting $x$ and $y$ in $G$. Then the connectivity $\kappa(G)$ of $G$ is $min\{\kappa(x, y) : x, y$ are distinct vertices of $G\}$.

The Cartesian product of graphs is an important method to construct a bigger graph, and plays a key role in design and analysis of networks. In the past several decades, many authors have studied the (edge) connectivity of the Cartesian product graphs. Specially, Sabidussi in [11] derived the following prefect and well-known theorem on the connectivity of Cartesian product graphs.

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**Theorem 1.1 (Sabidussi's Theorem [11])** *Let $G$ and $H$ be two connected graphs. Then*

$$\kappa(G \,\square\, H) \geq \kappa(G) + \kappa(H).$$

More information about the (edge) connectivity of the Cartesian product graphs can be found in [4, 5, 6, 11, 12, 15, 16].

The generalized connectivity of a graph $G$, which was introduced by Chartrand et al. in [2], is a natural and nice generalization of the concept of vertex connectivity. A tree $T$ is called an *$S$-tree* (*$\{u_1, u_2, \ldots, u_k\}$-tree*) if $S \subseteq V(T)$, where $S = \{u_1, u_2, \ldots, u_k\} \in V(G)$. A family of trees $T_1, T_2, \ldots, T_r$ are *internally disjoint $S$-trees* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of integers $i$ and $j$, where $1 \leq i < j \leq r$. We use $\kappa(S)$ to denote the greatest number of internally disjoint $S$-trees. For an integer $k$ with $2 \leq k \leq n$, the *$k$-connectivity* $\kappa_k(G)$ of $G$ is defined as $min\{\kappa(S) \,|\, S \in V(G)\ and\ |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of $G$, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of $G$. By convention, for a connected graph $G$ with less than $k$ vertices, we set $\kappa_k(G) = 1$. For any graph $G$, clearly, $\kappa(G) \geq 1$ if and only if $\kappa_k(G) \geq 1$.

In addition to being a natural combinatorial measure, the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that $G$ represents a network. If one considers to connect a pair of vertices of $G$, then a path is used to connect them. However, if one wants to connect a set $S$ of vertices of $G$ with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI, see [13]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized $k$-connectivity can serve for measuring the capability of a network $G$ to connect any $k$ vertices in $G$.

In [8], Li and Li investigated the complexity of determining the generalized connectivity and derived that for any fixed integer $k \geq 2$, given a graph $G$ and a subset $S$ of $V(G)$, deciding whether there are $k$ internally disjoint trees connecting $S$, namely deciding whether $\kappa(S) \geq k$, is NP-complete. The generalized connectivity of complete bipartite graphs was studied by Okamoto and Zhang in [10], and Li and Li in [7].

Chartrand et al. [3] got the following result for complete graphs.

**Theorem 1.2** *[3] For every two integers $n$ and $k$ with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil k/2 \rceil$.*

**Theorem 1.3** *[9] Let $G$ be a connected graph with at least three vertices. If $G$ has two adjacent vertices with minimum degree $\delta$, then $\kappa_3(G) \leq \delta - 1$.*

**Theorem 1.4** *[9] For any connected graph $G$, $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

In this paper, we study the 3-connectivity of Cartesian product graphs and get the following result.

**Theorem 1.5** *Let $G$ and $H$ be connected graphs such that $\kappa_3(G) \geq \kappa_3(H)$. The following assertions hold:*

*$(i)$ If $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G \,\square\, H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp;*

*$(ii)$ If $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G \,\square\, H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

The paper is organized as follows. In Section 2, we recall the definition and properties of Cartesian product graphs, and give some basic results about the internally disjoint $S$-trees. As usual, in order to get a general result, we begin with a special case. In Section 3, we study the 3-connectivity of the Cartesian product of a graph $G$ and a tree $T$. This section is a preparation of Section 4. In Section 4, we study the 3-connectivity of the Cartesian product of two connected graphs $G$ and $H$. Moreover, all the bounds are sharp. Our result could be seen as an extension of Theorem 1.1.

# 2 Some basic results

We use $P_n$ to denote a path with $n$ vertices. A path $P$ is called a *u-v path*, denoted by $P_{u,v}$, if $u$ and $v$ are the endpoints of $P$.

Recall that the $Cartesian\ product$ (also called the *square product*) of two graphs $G$ and $H$, written as $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices $(u, v)$ and $(u', v')$ are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \,\square\, H \cong H \,\square\, G$. The edge $(u, v)(u', v')$ is called *one-type edge* if $(u, u') \in E(G)$ and $v = v'$; similarly, the $(u, v)(u', v')$ is called *two-type edge* if $u = u'$ and $(v, v') \in E(H)$.

Let $G$ and $H$ be two graphs with $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$, respectively. We use $G(u_j, v_i)$ to denote the subgraph of $G \,\square\, H$ induced by the set $\{(u_j, v_i) \,|\, 1 \le j \le n\}$. Similarly, we use $H(u_j, v_i)$ to denote the subgraph of $G \,\square\, H$ induced by the set $\{(u_j, v_i) \,|\, 1 \le i \le m\}$. It is easy to see $G(u_{j_1}, v_i) = G(u_{j_2}, v_i)$ for different $u_{j_1}$ and $u_{j_2}$ of $G$. Thus, we can replace $G(u_j, v_i)$ by $G(v_i)$ for simplicity. Similarly, we can replace $H(u_j, v_i)$ by $H(u_j)$. For any $u, u' \in V(G)$ and $v, v' \in V(G)$, $(u, v),\ (u, v') \in V(H(u))$, $(u', v),\ (u', v') \in V(H(u'))$, $(u, v),\ (u', v) \in V(G(v))$, and $(u, v'),\ (u', v') \in V(G(v))$. We refer to $(u, v')$ and $(u', v)$ as $the\ vertices\ corresponding\ to$ $(u, v)$ in $G(v')$ $(= G(u, v')\,)$ and $H(u')$ $(= H(u', v)\,)$, respectively. Similarly, we can define the path and tree corresponding to some path and tree, respectively.

In order to show our main results, we need the following well-known theorem.

**Theorem 2.1 (Menger's Theorem [1])** *Let $G$ be a $k$-connected graph, and let $x$ and $y$ be a pair of distinct vertices in G. Then there exist $k$ internally disjoint paths $P_1, P_2, \dots, P_k$ in G connecting $x$ and $y$.*

Let $G$ be a connected graph, and $S = \{x_1, x_2, x_3\} \subseteq V(G)$. We first have the following observation about internally disjoint $S$-trees.

**Observation 2.1** *Let $G$ be a connected graph, $S = \{x_1, x_2, x_3\} \subseteq V(G)$, and $T$ be an $S$-tree. Then there exists a subtree $T'$ of $T$ such that $T'$ is also an $S$-tree such that $1 \le d_{T'}(x_i) \le 2$, $|\{x_i \,|\, d_{T'}(x_i) = 1\}| \ge 2$ and $\{x \,|\, d_{T'}(x) = 1\} \subseteq S$. Moreover, if $|\{x_i \,|\, d_{T'}(x_i) = 1\}| = 3$, then all the vertices of $V(T') \setminus \{x_1, x_2, x_3\}$ have degree 2 except for one vertex, say $x$ with $d_{T'}(x) = 3$; if there exists one vertex of $S$, say $x_1$, of degree 2 in $T'$, then $T'$ is an $x_2$-$x_3$ path.*

**Proof:** It is easy to check that this observation holds by deleting vertices and edges of $T$. □

**Remark 2.1** $(i)$ *Since the path between any two distinct vertices is unique in $T$, the tree $T'$ obtained from $T$ in Observation 2.1 is unique. Such a tree is called a minimal $S$-tree (or minimal $\{x_1, x_2, x_3\}$-tree).*

$(ii)$ *Let $S = \{x, y, z\} \subseteq V(G)$. Throughout this paper, we can assume that each $S$-tree is a minimal $S$-tree.*

**Lemma 2.1** *Let $G$ be a graph with $\kappa_3(G) = k \geq 2$, $S = \{x, y, z\} \subseteq V(G)$. Then, the following assertions hold:*

$(i)$ *If $G[S]$ is a clique, then there exist $k$ internally disjoint $S$-trees $T_1, T_2, \ldots, T_k$, such that $E(T_i) \cap E(G[S]) = \emptyset$ for $1 \leq i \leq k-2$.*

$(ii)$ *If $G[S]$ is not a clique, then there exist $k$ internally disjoint $S$-trees $T_1, T_2, \ldots, T_k$, such that $E(T_i) \cap E(G[S]) = \emptyset$ for $1 \leq i \leq k-1$.*

**Proof:** We first prove $(i)$. Clearly, by the definition of $S$-trees, we know $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 3$. Let $\{T_1, T_2, \ldots, T_k\}$ be $k$ internally disjoint $S$-trees. If $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 2$, we are done by exchanging subscript. Thus, suppose $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| = 3$. Without loss of generality, we assume $E(T_i) \cap E(G[S]) \neq \emptyset$, where $i = k-2, k-1, k$. It is easy to check that $T_{k-2}, T_{k-1}, T_k$ must have the structures as shown in Figures 1 $a$ and $c$. But, for these two cases, we can obtain $T'_{k-2}, T'_{k-1}, T'_k$ from $T_{k-2}, T_{k-1}, T_k$, such that $E(T'_{k-2}) \cap \{xy, xz, yz\} = \emptyset$. See Figs. $1b$. and $1d$, where the tree $T'_{k-2}$ is shown by dotted lines. Thus $T_1, T_2, \ldots, T_{k-3}, T'_{k-2}, T'_{k-1}, T'_k$ are our desired $S$-trees.

The proof of $(ii)$ is similar to that of $(i)$, and thus is omitted. □

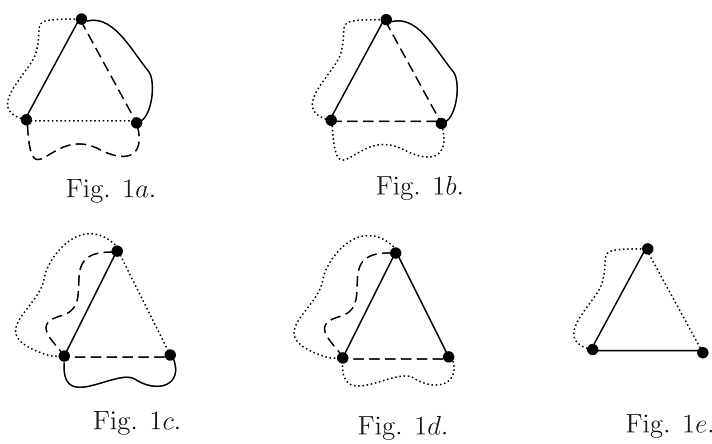


**Fig. 1:** $T'_{k-2}, T'_{k-1}, T'_k$. An edge is shown by a straight line. The edges (or paths) of a tree are shown by the same type of lines.

**Remark 2.2** *Let $G$ be a graph with $\kappa_3(G) = k \geq 2$, $S = \{x, y, z\} \subseteq V(G)$. If $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \geq 2$ for any collection $\mathcal{T}$ of $k$ internally disjoint $S$-trees, then $G[S]$ is a clique. Moreover, $T_{k-1} \cup T_k$ must have the structure as shown in Figure 1e.*

## 3 The Cartesian product of a connected graph and a path

In this section, we show the following proposition.

**Proposition 3.1** *Let $G$ be a graph and $P_m$ be a path with $m$ vertices. The following assertions hold:*
$(i)$ *If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G \,\square\, P_m) \geq \kappa_3(G)$. Moreover, the bound is sharp;*
$(ii)$ *If $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G \,\square\, P_m) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp.*

We shall prove Proposition 3.1 by a series of lemmas. Since the proofs of $(i)$ and $(ii)$ are similar, we only show $(ii)$. Let $G$ be a graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$ such that $1 \leq \kappa_3(G) < \kappa(G)$, $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ such that $v_i$ and $v_j$ are adjacent if and only if $|i - j| = 1$.

Set $\kappa_3(G) = k$ for simplicity. To prove $(ii)$, we need to show that for any $S = \{x, y, z\} \subseteq V(G \,\square\, H)$, there exist $k + 1$ internally disjoint $S$-trees. We proceed our proof by the following three lemmas.

**Lemma 3.1** *If $x, y, z$ belongs to the same $V(G(v_i))$, $1 \leq i \leq m$, then there exist $k + 1$ internally disjoint $S$-trees.*

**Proof:** Without loss of generality, we assume $x, y, z \in V(G(v_1))$. Since $\kappa_3(G) = k$, there exist $k$ internally disjoint $S$-trees $T_1, T_2, \ldots, T_k$ in $G(v_1)$. We need another $S$-tree $T_{k+1}$ such that $T_{k+1}$ and $T_i$ are internally disjoint, for $i = 1, 2, \ldots, k$. Let $x', y', z'$ be the vertices corresponding to $x, y, z$ in $G(v_2)$, and $T_1'$ be the tree corresponding to $T_1$ in $G(v_2)$. Therefore, tree $T_{k+1}$ obtained from $T_1'$ by adding three edges $xx', yy', zz'$ is a desired tree. □

**Lemma 3.2** *If only two vertices of $\{x, y, z\}$ belong to some copy $G(v_i)$, then there exist $k + 1$ internally disjoint $S$-trees.*

**Proof:** We may assume $x, y \in V(G(v_1)), z \in V(G(v_2))$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let $x', y'$ be the vertices corresponding to $x, y$ in $G(v_2)$, $z'$ be the vertex corresponding to $z$ in $G(v_1)$. Consider the following two cases.

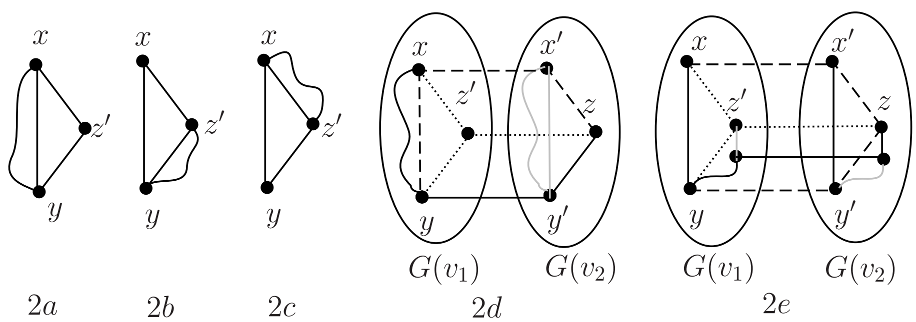


**Fig. 2:** The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in $T_i^*$.

**Case 1:** $z' \notin \{x, y\}$.

Let $S' = \{x, y, z'\}$, and $T_1, T_2, \ldots T_k$ be $k$ internally disjoint $S'$-trees in $G(v_1)$ such that $|\{T_i \mid E(T_i) \cap E(G(v_1)[S']\} \neq \emptyset|$ is as small as possible. We can assume that $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ for each $i$, where $1 \le i \le k-2$ by Lemma 2.1.

For a tree $T_i$ with $E(T_i) \cap E(G(v_1)[S']) = \emptyset$, let $T_i^*$ be the tree obtained from $T_i$ by adding $z_i z_i'$ and $z_i' z$, and deleting $z'$, where $z_i$ is any one neighbor of $z'$ in $T_i$, and $z_i'$ is the vertex corresponding to $z_i$ in $G(v_2)$.

If $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$, say $yz' \in E(T_k) \cap E(G(v_1)[S'])$. Let $T_k^* = T_k + zz'$ and $T_{k+1}^* = T_k' + xx' + yy'$, where $T_k'$ is the tree corresponding to $T_k$ in $G(v_2)$.

If $E(T_{k-1}) \cap E(G(v_1)[S']) \neq \emptyset$ and $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$. Then $T_{k-1} \cup T_k$ must have one of the structures as shown in Figures 2 a, b and c by Remark 2.2. If $T_{k-1}$ and $T_k$ have the structures as shown in Figure 2a, then we can obtain trees $T_{k-1}^*, T_k^*$ and $T_{k+1}^*$ as shown in Figure 2d. If $T_{k-1}$ and $T_k$ have the structures as shown in Figure 2b, then we can obtain trees $T_{k-1}^*, T_k^*$ and $T_{k+1}^*$ as shown in Figure 2e. If $T_{k-1}$ and $T_k$ have the structures as shown in Figure 2c, then we can obtain trees $T_{k-1}^*, T_k^*$ and $T_{k+1}^*$ similar to those in Figure 2d.

**Case 2:** $z' \in \{x, y\}$.

Without loss of generality, assume $z' = y$. Since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, there exist at least $k+1$ internally disjoint *x*-*y* paths $P^1, P^2, \ldots, P^{k+1}$ in $G(v_1)$. Assume that $y_i$ is the only neighbor of $y$ in $P^i$, and that $y_i'$ is the vertex corresponding to $y_i$ in $G(v_2)$. If $x$ and $y$ are nonadjacent in $P^i$, let $T_i$ be the tree obtained from $P^i$ by adding $y_i y_i'$ and $y_i' z$. If $x$ and $y$ are adjacent in $P^i$, let $T_i$ be the tree obtained from $P^i$ by adding $yz$. Since $G$ is a simple graph, there exists at most one path $P^i$ such that $x$ and $y$ are adjacent on $P^i$. Thus $T_i, 1 \le i \le k+1$, are $k+1$ internally disjoint $S$-trees.

□

**Lemma 3.3** *If $x, y, z$ are contained in distinct $G(v_i)$s, then there exist $k+1$ internally disjoint $S$-trees.*

**Proof:** We may assume that $x \in V(G(v_1)), y \in V(G(v_2)), z \in V(G(v_3))$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let $y', z'$ be the vertices corresponding to $y, z$ in $G(v_1)$, $x', z''$ be the vertices corresponding to $x, z$ in $G(v_2)$ and $x'', y''$ be the vertices corresponding to $x, y$ in $G(v_3)$. We consider the following three cases.

**Case 1:** $x, y', z'$ are distinct vertices in $G(v_1)$

Let $S' = \{x, y', z'\}$, and $T_1, T_2, \ldots T_k$ be $k$ internally disjoint $S'$-trees in $G(v_1)$ such that $|\{T_i \mid E(T_i) \cap E(G(v_1)[S']) \neq \emptyset\}|$ is as small as possible. We can assume that $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ for each $i$, where $1 \le i \le k-2$ by Lemma 2.1. For each $T_i$ such that $E(T_i) \cap E(G(v_1)[S']) = \emptyset$, we can obtain an $S$-tree $T_i^*$ from $T_i$ similar to that in Subcase 1.1 of Lemma 3.2.

If $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$ or $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$. Without loss of generality, we assume $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$. Let $T_k^*$ be the tree obtained from $T_k$ by adding edges

$y'y, z'z''$ and $z''z$, $T^*_{k+1}$ be the tree obtained from $T''_k$ by adding $x''x', x'x$ and $y''y$, where $T''_k$ is the tree corresponding to $T_k$ in $G(v_3)$. Thus, $T^*_i$s, $1 \leq i \leq k+1$, are $k$ internally disjoint $S$-tree.

Otherwise, that is, $E(T_{k-1}) \cap E(G(v_1)[S']) \neq \emptyset$ and $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$. Then $T_{k-1}$ and $T_k$ must have the structures as shown in Figure 3a, b and c. If $T_{k-1}$ and $T_k$ have the structures as shown in Figure 3a, then we can obtain trees $T^*_{k-1}, T^*_k$ and $T^*_{k+1}$ as shown in Figure 3d. If $T_{k-1}$ and $T_k$ have the structures as shown in Figure 3b, then we can obtain trees $T^*_{k-1}, T^*_k$ and $T^*_{k+1}$ as shown in Figure 3e. If $T_{k-1}$ and $T_k$ have the structures as shown in Figure 3c, then we can obtain trees $T^*_{k-1}, T^*_k$ and $T^*_{k+1}$ as shown in Figure 3f.

**Case 2:** Two of $x, y', z'$ are the same vertex in $G(v_1)$.

If $y' = z'$, since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, it is easy to construct $k+1$ internally disjoint $S$-trees. See Figure $3g$. The other cases ($x = y'$ or $x = z'$) can be proved with similar arguments.

**Case 3:** $x, y', z'$ are the same vertex in $G(v_1)$.

Since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, it is easy to construct $k+1$ internally disjoint $S$-trees. See Figure 3h.

□

We have the following observation by the argument in the proof of Proposition 3.1.

**Observation 3.1** *The $k+1$ internally disjoint $S$-trees consist of three kinds of edges — the edges of original trees (or paths), the edges corresponding the edges of original trees (or paths) and two-type edges.*

Note that $Q_n \cong P_2 \,\square\, P_2 \,\square \cdots \square\, P_2$, where $Q_n$ is the $n$-hypercube. We have the following corollary.

**Corollary 3.1** *Let $Q_n$ be the $n$-hypercube with $n \geq 2$. Then $\kappa_3(Q_n) = n-1$.*

**Proof:** Recall that $\kappa(Q_n) = n$ so that Proposition 3.1 (ii) inductively applies. It is easy to check that $\kappa_3(Q_2) = 1$. Suppose that the result holds for $Q_{n-1}$, where $n \geq 3$. We have $\kappa_3(Q_n) \geq n-1$ by Proposition 3.1. On the other hand, since $Q_n$ is $n$-regular, we have $\kappa_3(Q_n) \leq n-1$ by Theorem 1.3. Thus $\kappa_3(Q_n) = n-1$. □

**Example 3.1** *Let $K_{2n}$ be the complete graph with vertex set $V(K_{2n}) = \{u_1, u_2, \ldots, u_{2n}\}$, and let $G_n$ be the graph obtained from $K_{2n}$ by adding a new vertex $u$ and edges $uu_i$, $1 \leq i \leq n$.*

*For any $S = \{x, y, z\} \subseteq V(G)$, if $u \notin S$, then there exist $k$ internally disjoint $S$-trees in $G_n$ by Theorem 1.2. If $u \in S$, without loss of generality, assume $x = u, y = u_1, z = u_2$. Let $T_1$ be the path $u, u_1, u_{k+1}, u_2$, $T_2$ be the path $u, u_2, u_{k+2}, u_1$, and $T_i$ be the tree obtained from a path $u, u_{n+i}, u_1$ by adding an edge $u_{n+i}u_2$ for $3 \leq i \leq n$. Clearly, $T_i$, $1 \leq i \leq n$, are $n$ internally disjoint $S$-trees. So $\kappa_3(G_n) \geq n$. Since $\delta(G_n) = n$, $\kappa_3(G_n) = n$ by Theorem 1.4. By Proposition 3.1, $\kappa_3(G_n \,\square\, K_2) \geq n$. Since $G_n$ has two adjacent vertices of degree $n+1$, $\kappa_3(G_n) = n$ by Theorem 1.3. Moreover, clearly, $\kappa(G) = n$. Thus $\kappa_3(G \,\square\, K_2) = \kappa_3(G) = n$.*

**Remark 3.1** *We know that the bounds of $(i)$ and $(ii)$ in Theorem 3.1 are sharp by Example 3.1 and Corollary 3.1.*

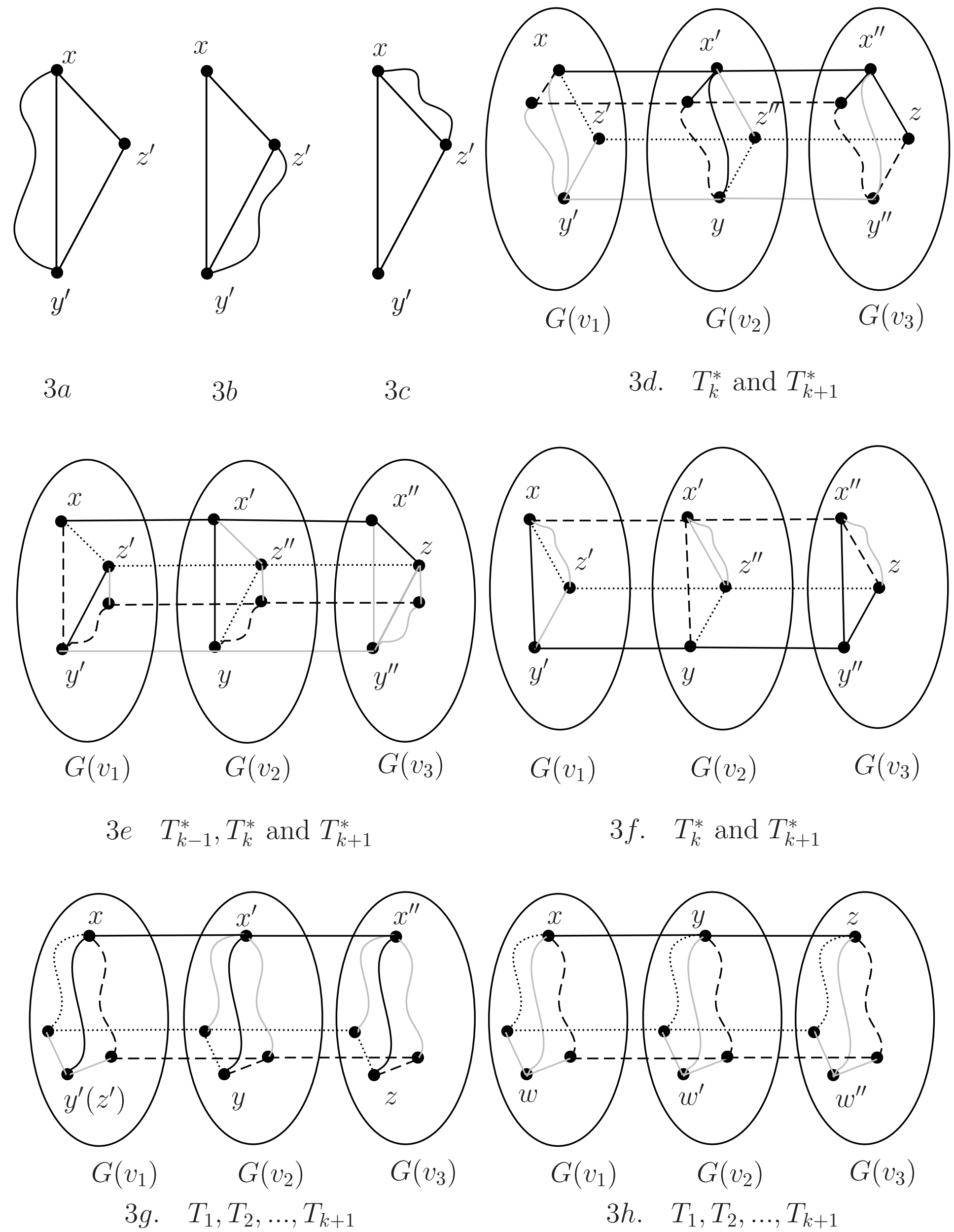


**Fig. 3:** The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in $T_i^*$.

**Proposition 3.2** *Let $G$ be a connected graph and $T$ be a tree. The following assertions hold:*
$(i)$ *If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G \,\Box\, T) \geq \kappa_3(G)$. Moreover, the bound is sharp;*
$(ii)$ *If $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G \,\Box\, T) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp.*

**Proof:** Since the proofs of $(i)$ and $(ii)$ are similar, we only show $(ii)$. It suffices to show that for any $S = \{x, y, z\} \subseteq V(G \,\Box\, H)$, there exist $k + 1$ internally disjoint $S$-trees. Set $\kappa_3(G) = k$, $V(G) = \{u_1, u_2, \ldots, u_n\}$, and $V(T) = \{v_1, v_2, \ldots, v_m\}$.

Let $x \in V(G(v_i)), y \in V(G(v_j)), z \in V(G(v_k))$ be three distinct vertices. If there exists a path in $T$ containing $v_i, v_j$ and $v_k$, then we are done from Proposition 3.1. If $i, j$ and $k$ are not distinct integers, such a path must exist. Thus, suppose that $i, j$ and $k$ are distinct integers, and that there exists no path containing $v_i, v_j$ and $v_k$. By Observation 2.1, there exists a tree $T'$ in $T$ such that $d_{T'}(v_i) = d_{T'}(v_j) = d_{T'}(v_k) = 1$ and all the vertices of $V(T) \setminus \{v_i, v_j, v_k\}$ have degree 2 except for one vertex, say $v_4$ with $d_T(v_4) = 3$. Without loss of generality, we set $i = 1, j = 2, k = 3$. Furthermore, we assume $v_iv_4 \in E(T')$, where $1 \leq i \leq 3$. In the following argument, we can see that this assumption has no influence on the correctness of our proof.

Let $P$ be the unique path in $T'$ connecting $v_1$ and $v_2$. By Proposition 3.1, we can construct $k + 1$ internally disjoint $\{x, y, z'\}$-trees $T_i$, $1 \leq k + 1$, in $G \,\Box\, P$, where $z'$ is the vertex corresponding to $z$ in $G(v_4)$. By a similar method of Proposition 3.1, we can construct $k+1$ internally disjoint $S$-trees in $G \,\Box\, T$ on the basis of these trees. □

**Remark 3.2** *We know that the bounds of $(i)$ and $(ii)$ in Proposition 3.2 are sharp by Example 3.1 and Corollary 3.1.*

**Observation 3.2** *The $k + 1$ internally disjoint $S$-trees consist of three kinds of edges — the edges of original trees (or paths), the edges corresponding the edges of original trees (or paths) and two-type edges.*

# 4 The Cartesian product of two general graphs

**Observation 4.1** *Let $G$ and $H$ be two connected graphs, $x, y, z$ be three distinct vertices in $H$, and $T_1, T_2, \ldots, T_k$ be $k$ internally disjoint $\{x, y, z\}$-trees in $H$. Then $G \,\Box\, \bigcup_{i=1}^{k} T_i = \bigcup_{i=1}^{k} (G \,\Box\, T_i)$ has the structure as shown in Figure 4. Moreover, $(G \,\Box\, T_i) \cap (G \,\Box\, T_j) = G(x) \cup G(y) \cup G(z)$ for $i \neq j$. In order to show the structure of $G \,\Box\, \bigcup_{i=1}^{k} T_i$ clearly, we take $k$ copies of $G(y)$, and $k$ copies of $G(z)$. Note that, these $k$ copes of $G(y)$ (resp. $G(z)$) represent the same graph.*

**Example 4.1** *Let $H$ be the complete graph of order 4. The structure of $G \,\Box\, (T_1 \cup T_2)$ is shown in Figure 5.*

Now we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5:** Since the proofs of $(i)$ and $(ii)$ are similar, we only show $(ii)$. Without loss of generality, we set $\kappa_3(G) := k, \kappa_3(H) := \ell$. It suffices to show that for any $S = \{x, y, z\} \subseteq V(G \,\Box\, H)$, there exist $k + \ell$ internally disjoint $S$-trees. Assume $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(T) = \{v_1, v_2, \ldots, v_m\}$.

Let $x \in V(G(v_i)), y \in V(G(v_j)), z \in V(G(v_k))$ be three distinct vertices in $G \,\Box\, H$. We will do only the case that $i, j, k$ are distinct integers. Other two possibilities are similar. Without loss of generality, set $i = 1, j = 2, k = 3$. Since $\kappa_3(H) = \ell$, there exist $\ell$ internally disjoint $\{v_1, v_2, v_3\}$-trees $T_i$, $1 \leq i \leq \ell$, in $H$. We use $G_i$ to denote $G \,\Box\, T_i$. By Observation 4.1, we know that $G \,\Box\, \bigcup_{i=1}^{\ell} T_i = \bigcup_{i=1}^{\ell} G_i$ and

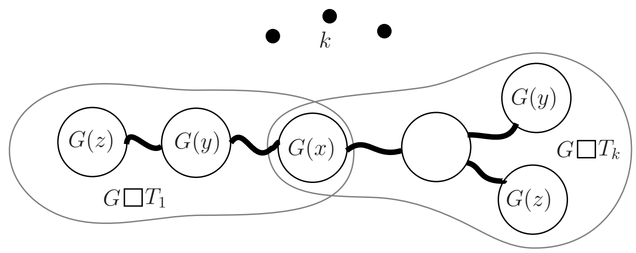


**Fig. 4:** The structure of $G \,\square \bigcup_{i=1}^{k} T_i$.

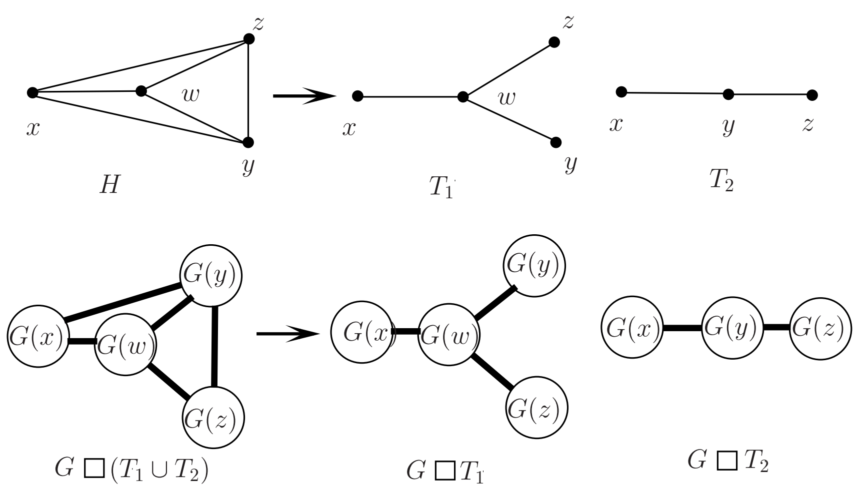


**Fig. 5:** The structure of $G \,\square\, (T_1 \cup T_2)$.

$G_i \cap G_j = G(v_1) \cup G(v_2) \cup G(v_3)$ for $i \neq j$. Let $y', z'$ be the vertices corresponding to $y, z$ in $G(v_1)$, respectively.

If $x, y', z'$ are distinct vertices in $G(v_1)$. Since $\kappa_3(G(v_1)) = k$, there exist $k$ internally disjoint $\{x, y', z'\}$-trees $T'_j$, $1 \leq j \leq k$, in $G(v_1)$. Let $k_0, k_1, \ldots, k_\ell$ be integers such that $0 = k_0 < k_1 < \cdots < k_\ell = k$. Similar to the proofs of Proposition 3.1, we can construct $k_i - k_{i-1} + 1$ internally disjoint $S$-trees $T_{i,j_i}, 1 \leq j_i \leq k_i - k_{i-1} + 1$, in $(\bigcup_{j=k_{i-1}+1}^{k_i} T'_j) \,\square\, T_i$ for each $i$, where $1 \leq i \leq \ell$. By Observations 3.1 and 3.2, $T_{i,j_i}$ and $T_{r,j_r}$ are internally disjoint for $i \neq r$. Thus $T_{i,j_i}, 1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$ are $k + \ell$ internally disjoint $S$-trees.

If exactly two of $x, y', z'$ are the same vertex in $G(v_i)$. Without loss of generality, assume $y' = z'$. Since $\kappa(G(v_1)) > k$, there exist $k + 1$ internally disjoint $x - y'$ paths $P_i$, $1 \leq i \leq k + 1$, in $G(v_1)$ by Menger's Theorem. Note that at most one of them is a path of length 1. Let $P_{k+1}$ be such a path if $xy' \in E(G(v_1))$, and let $k_0, k_1, \ldots, k_\ell$ be integers such that $0 = k_0 < k_1 < \cdots < k_\ell = k + 1$. Similar to the proofs of Proposition 3.1, we can construct $k_i - k_{i-1} + 1$ internally disjoint $S$-trees $T_{i,j_i}, 1 \leq j_i \leq k_i - k_{i-1} + 1$, in $(\bigcup_{j=k_{i-1}+1}^{k_i} P_j) \,\square\, T_i$ for each $i$, where $1 \leq i \leq \ell - 1$, and $k_\ell - k_{\ell-1}$ internally disjoint $S$-trees $T_{\ell,j_\ell}, 1 \leq j_i \leq k_\ell - k_{\ell-1}$, in $(\bigcup_{j=k_{\ell-1}+1}^{k_\ell} P_j) \,\square\, T_i$. By Observation 3.1 and 3.2, $T_{i,j_i}$ and $T_{r,j_r}$ are internally disjoint for $i \neq r$. Thus $T_{i,j_i}, 1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$ are $k + \ell$ internally disjoint $S$-trees.

If all of $x, y', z'$ are the same vertex in $G(v_i)$. Since $\delta(G(v_1)) \geq \kappa(G(v_1)) > k$, $x$ has $k$ neighbors, say $x_1, x_2, \ldots, x_k$, in $G(v_1))$. Let $P_i$ be the path $xx_i$, and let $k_0, k_1, \ldots, k_\ell$ be integers such that $0 = k_0 < k_1 < \cdots < k_\ell = k$. Similar to the proofs of Proposition 3.1, we can construct $k_i - k_{i-1} + 1$ internally disjoint $S$-trees $T_{i,j_i}, 1 \leq j_i \leq k_i - k_{i-1} + 1$, in $(\bigcup_{j=k_{i-1}+1}^{k_i} P_j) \,\square\, T_i$ for each $i$, where $1 \leq i \leq \ell$. By Observation 3.1 and 3.2, $T_{i,j_i}$ and $T_{r,j_r}$ are internally disjoint for $i \neq r$. Thus $T_{i,j_i}, 1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$ are $k + \ell$ internally disjoint $S$-trees.

We now show that bounds of Theorem 1.5 are sharp. For $(i)$, Example 3.1 is a sharp example. Let $K_n$ be a complete graph with $n$ vertices, and $P_m$ be a path with $m$ vertices, where $m \geq 2$. We have $\kappa_3(P_m) = 1$, and $\kappa_3(K_n) = n - 2$ by Theorem 1.2. It is easy to check that $\kappa_3(K_n \,\square\, P_m) = n - 2 + 1 = n - 1$. Thus, $K_n \,\square\, P_m$ is a sharp example for $(ii)$. □

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