Embeddings of 3-connected 3-regular planar graphs on surfaces of non-negative Euler characteristic

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Whitney's theorem states that every 3-connected planar graph is uniquely embeddable on the sphere. On the other hand, it has many inequivalent embeddings on another surface. We shall characterize structures of a 3-connected 3-regular planar graph G embedded on the projective-plane, the torus and the Klein bottle, and give a one-to-one correspondence between inequivalent embeddings of G on each surface and some subgraphs of the dual of G embedded on the sphere. These results enable us to give explicit bounds for the number of inequivalent embeddings of G on each surface, and propose effective algorithms for enumerating and counting these embeddings.

Keywords: inequivalent embeddings, flexibility, genus distribution, planar, cubic

1 Introduction

An *embedding* of a graph G on a surface F^2, which is a compact 2-dimensional manifold without boundary, is a drawing of G on F^2 without edge crossing. Technically, we regard an embedding as an injective continuous map $f : G \to F^2$, while we often consider that G is already mapped on a surface and denote its image by G itself. The *faces* are the connected components of the open set $F^2 - f(G)$. In this paper, we focus on only finite, undirected and simple graphs. Moreover, we assume that embeddings are *cellular*, that is, each face must be homeomorphic to an open 2-cell, which contains neither handles nor crosscaps. For terminologies of topological graph theory, we refer to [1, 7].

Two embeddings $f_1, f_2 : G \to F^2$ are *equivalent* if there is a homeomorphism $h : F^2 \to F^2$ such that $hf_1 = f_2$. We say that a graph G is *uniquely embeddable* on F^2 (up to equivalence) if any two embeddings of G on F^2 are equivalent. The following two questions are important and have attracted many topological graph theorists: (1) What kind of structures generates inequivalent embeddings of a given graph? (We often call such a structure the *re-embedding structure*.) (2) How many inequivalent embeddings on a fixed surface does a graph have?

These problems were first studied by Whitney [16, 17]. He proved that one of any two embeddings of a 2-connected planar graph on the sphere can be obtained from the other by a sequence of simple local

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re-embeddings, called Whitney's 2-flipping (see [7, Sections 2 and 5] for details), and every 3-connected planar graph is uniquely embeddable on the sphere.

With regard to the uniqueness of a graph embedded on a non-spherical surface, its "face-width" plays an effective role. The *face-width* $\mathbf{fw}(G)$ of a graph G embedded on a non-spherical surface F^2 is defined by

$$\mathbf{fw}(G) = \min\{|G \cap \ell| : \ell \text{ is a noncontractible simple closed curve on } F^2\}.$$

In [6, 13, 14], it was proved that a 3-connected graph embedded on a non-spherical surface with sufficiently high face-width is uniquely embeddable on this surface. See some other studies [8, 12] for relations between the number of inequivalent embeddings of a graph on a fixed surface and its face-width.

Not only the face-width but also the connectivity has a strong relation to the number of inequivalent embeddings on surfaces with lower genera, which can be expected from Whitney's theorem. Negami [11] proved that every 6-connected toroidal graph except for three graphs is uniquely embeddable on the torus. Kitakubo and Negami [4], and Suzuki [15] studied the number of inequivalent embeddings of 5-connected and 4-connected non-planar graphs on the projective-plane, respectively. Recently, Maharry et al. [5] constructed re-embedding structures of non-planar graphs on the projective-plane completely and pointed out some mistakes in the past studies. In these papers, they analysed re-embedding structures of "non-planar" graphs.

On the other hand, Mohar et al. [9, 10] showed that 2-connected "planar" graphs embedded on non-spherical surfaces have special re-embedding structures, called "patch structures", while they have not given specific structures on each surface except for the projective-plane and not mentioned the number of inequivalent embeddings on any surface.

In Section 3, we shall construct the complete list of re-embedding structures of a planar graph G embedded on the projective-plane, the torus or the Klein bottle when G is 3-connected and 3-regular. In this argument, 3-connectivity is essential in order for us to ignore Whitney's 2-flippings, and when a 3-connected planar graph G is 3-regular, we can describe re-embedding structures of G on each surface completely. These re-embedding structures lead to the following results.

We denote the complete graph of order n by K_n and for a positive integer $k \geq 2$, a complete k-partite graph with k partite sets $V_1, V_2, \cdots, V_k$ such that $|V_i| = n_i$ for $1 \leq i \leq k$ by $K_{n_1,n_2,\ldots,n_k}$. Note that a complete k-partite graph $K_{1,1,\ldots,1}$ is isomorphic to the complete graph K_k of order k.

Theorem 1. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the projective-plane and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* K_2 *or* K_4.

Theorem 2. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the torus and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* $K_{2,2,2}$*,* $K_{2,2m}$ *or* $K_{1,1,2m-1}$ *for some positive integer* m.

Theorem 3. *There exists a one-to-one correspondence between inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the Klein bottle and subgraphs of the dual graph of the graph embedded on the sphere isomorphic to* $K_{2,2m-1}$ *or* $K_{1,1,2m}$ *for some positive integer* m*, or one of the six graphs* A_1 *to* A_6 *shown in Fig.* 1.

Based on these theorems, in Section 4, we will give explicit bounds for the number of inequivalent embeddings of a 3-connected 3-regular planar graph G on each of the projective-plane, the torus and the Klein bottle. In addition, we will propose effective algorithms for enumerating and counting these

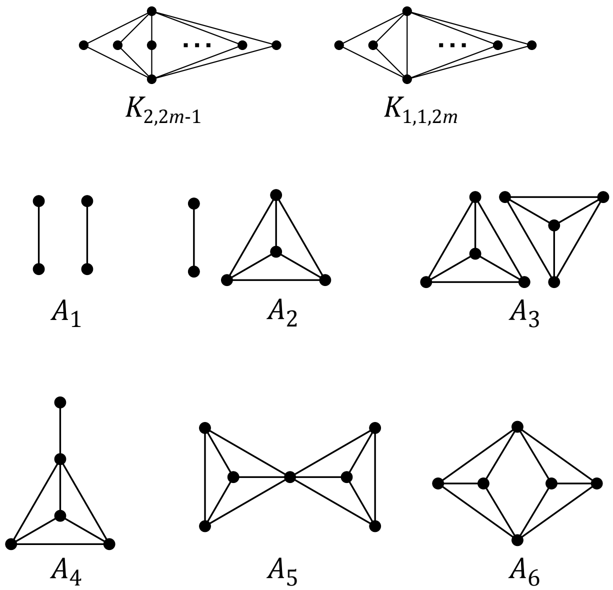

Fig. 1: The eight graphs

embeddings. In particular, even though G may have exponentially many inequivalent embeddings on the torus and the Klein bottle, we can calculate the total number of such embeddings in polynomial time.

These results allow the computation and the study of the *genus distributions* of a large family of graphs. We denote the number of inequivalent embeddings of a graph G on the orientable surface of genus k (resp. the non-orientable surface of genus h) by $g_G(k)$ (resp. $\tilde{g}_G(h)$). The genus distribution (resp. non-orientable genus distribution) of G is defined as the sequence $g_G(0), g_G(1), g_G(2), \cdots$ (resp. $\tilde{g}_G(0), \tilde{g}_G(1), \tilde{g}_G(2), \cdots$). The topic of genus distributions was introduced by Gross and Furst [2] and studied in various papers. Whether the genus distribution of every graph is log-concave is an interesting problem conjectured in [3], and still remains to be solved. From the genus distribution's point of view, we give explicit bounds for $g_G(1), \tilde{g}_G(1)$ and $\tilde{g}_G(2)$ of a 3-connected 3-regular planar graph G and algorithms for calculating them.

First of all, we focus on facial cycles in a 3-connected 3-regular planar graph embedded on the sphere in Section 2.

2 Re-embeddings of planar graphs with twisted edges

2.1 Rotation systems and embedding schemes

First, we introduce two combinatorial ways of describing embeddings of a graph; "rotation systems" and "embedding schemes". A general description on the rotation system and the embedding scheme can be found in [7].

Suppose that a connected graph G is embedded on an orientable surface. A *rotation* ρ_v around a vertex v of G is a cyclic permutation of edges incident with a vertex v such that $\rho_v(e)$ is the successor of e in the clockwise ordering around v. A *rotation system* for the embedded graph G is the collection of ρ_v, denoted by $\rho = \{\rho_v : v \in V(G)\}$. It is well-known that every embedding of a connected graph on an orientable surface is uniquely determined up to equivalence by its rotation system. Moreover, there are no rotation systems representing this embedding other than this rotation system and its inverse.

On the other hand, in order to include embeddings on nonorientable surfaces, we have to add the following concept. Let $f(G)$ be another embedding of G on a surface, which is not necessarily orientable. There are two possible cyclic ordering of edges incident with each vertex v of $f(G)$. Choose one of them and denote it by ρ_v. A closed walk C in an embedded graph is *facial* if C bounds a face of the graph. A *signature* of $E(G)$ is a map outputting 1 or -1 from each edge of G, denoted by λ, such that for an edge $e = uv$ with its endvertices u and v, $\lambda(e) = 1$ if a subwalk induced by the three edges $\rho_u(e)$, e and $\rho_v^{-1}(e)$ is included in a facial walk, otherwise $\lambda(e) = -1$. It can be shown that this definition of the signature λ is consistent, that is, $\lambda(uv) = \lambda(vu)$ for every edge uv. The pair (ρ, λ), where $\rho = \{\rho_v : v \in V(G)\}$ is obtained by the above procedure, is called an *embedding scheme* for $f(G)$. An embedding scheme determines exactly one embedding of G. Unfortunately, an embedding scheme representing a given embedding of G is not uniquely determined, unlike rotation systems for the orientable case. For an embedded graph G associated with a given embedding scheme (ρ, λ), an edge e with $\lambda(e) = -1$ is called *twisted*. If there are no twisted edges then G is embedded on an orientable surface obtained by the rotation system ρ.

Hereafter, suppose that G is 3-connected, 3-regular and planar. In addition, we assume that G is already embedded on the sphere with its rotation system $\rho = \{\rho_v : v \in V(G)\}$. (By Whitney's theorem, G is uniquely embeddable on the sphere.) Since G is 3-regular, there are only two possible rotations around each vertex of G, and one of them is the inverse of the other. This implies that for any embedding $f(G)$ of G on any surface, we can choose ρ_v as the local rotation around each vertex v. Thus, $f(G)$ can be determined by an embedding scheme (ρ, λ) with a suitable signature λ. We denote the set of twisted edges associated with this embedding scheme by X. In this situation, we regard this embedding as a re-embedding of G obtained by twisting all edges of X and denote it by $f_X(G)$. In addition, let F_X^2 be the surface where $f_X(G)$ is embedded.

2.2 Facial cycles in planar graphs

Choose two distinct subsets X_1 and X_2 of $E(G)$. Then, there is an edge e belonging to only one of either X_1 or X_2. We may assume that $e \in X_2$. It is easy to check that every facial walk of a 3-connected planar graph embedded on the sphere is a cycle. Thus, there are exactly two facial cycles containing e of G, denoted by C and C'. Let e_1 and e_2 be the edges of C' adjacent to e. Fig. 2 presents local neighbourhoods around $f_{X_1}(e)$ and $f_{X_2}(e)$. Note that $f_{X_1}(C)$ and $f_{X_2}(C)$ are drawn by bold lines in Fig. 2.

For the walk $W = e_1 e e_2$ of G, $f_{X_1}(W)$ constructs a consecutive part of a facial walk in $f_{X_1}(G)$ on $F_{X_1}^2$ but $f_{X_2}(W)$ are not so on $F_{X_2}^2$. Thus, $F_{X_1}^2 \neq F_{X_2}^2$, or $f_{X_1}(G)$ and $f_{X_2}(G)$ are not equivalent. It implies that the choice of a subset X of $E(G)$ uniquely induces the re-embedding $f_X(G)$ of G up to equivalence. Moreover, the total number of inequivalent embeddings of G is $2^{|E(G)|}$, and F_X^2 is homeomorphic to the sphere if and only if X is empty.

In the situation shown in the right of Fig. 2, we say that two cycles $f_{X_2}(C)$ and $f_{X_2}(C')$ *cross* along an edge e. Since G is 3-connected and planar, there are no edges and vertices contained in both C and C' other than e and its endvertices. Thus, $f_{X_2}(C)$ and $f_{X_2}(C')$ cross exactly once. Note that any two cycles in $f_X(G)$ for a given subset X of $E(G)$ do not cross at a single vertex since G is 3-regular.

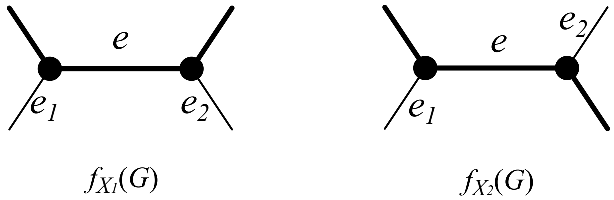

Fig. 2: The neighbourhoods around e in $f_{X_1}(G)$ and $f_{X_2}(G)$

Let H be a subgraph of a graph G'. An H-*bridge* in G' is a subgraph of G' induced by either an edge not in H but with both ends in H, or a component of $G' - V(H)$ together with all edges joining it to H. Note that any two H-bridges are edge-disjoint. It has been known that for a facial cycle C of a 3-connected planar graph G, there is only one C-bridge in G (e.g., see [7, p.39–40]).

Lemma 4. *Let C be a facial cycle in G and let $f_X(G)$ be a re-embedding of G with a given subset X of $E(G)$. Then, $f_X(C)$ is a non-separating cycle on F_X^2 if and only if $f_X(C)$ has a twisted edge.*

Proof: It is easy to see that if $f_X(C)$ has no twisted edges then it is facial in $f_X(G)$ and hence it separates F_X^2 into two regions.

Suppose that $f_X(C)$ has a twisted edge $f_X(e)$, that is, an edge e of G is in X. Let C' be the other facial cycle of G containing e. As shown in the right of Fig. 2, $f_X(C)$ and $f_X(C')$ cross along e and hence two edges of $f_X(C')$ adjacent to $f_X(e)$ are located separately in opposite sides of $f_X(C)$. However, both of these edges are contained in the unique $f_X(C)$-bridge. This implies that $f_X(C)$ does not separate F_X^2. □

3 Characterizations of re-embedding structures

In this section, we shall characterize the structures of $f_X(G)$ when F_X^2 is homeomorphic to the projective-plane, the torus or the Klein bottle, and show the following theorems.

Theorem 5. *A 3-connected 3-regular graph embedded on the projective-plane is planar if and only if it has one of the two structures $(P1)$ and $(P2)$ shown in Fig. 3.*

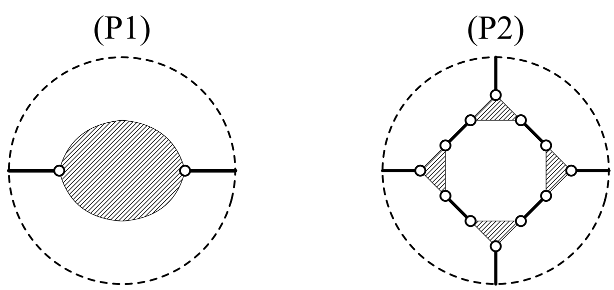

Fig. 3: Re-embedding structures on the projective-plane

Theorem 6. *A* 3*-connected* 3*-regular graph embedded on the torus is planar if and only if it has one of the two structures* $(T1)$*,* $(T2)$ *and* $(T3)$ *shown in Fig.* 4*.*

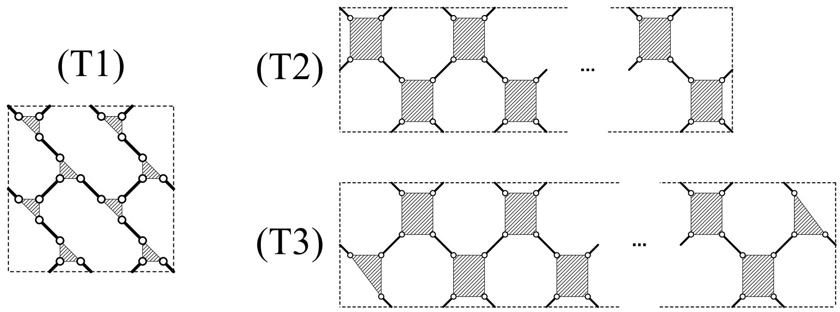

Fig. 4: Re-embedding structures on the torus

Theorem 7. *A* 3*-connected* 3*-regular graph embedded on the Klein bottle is planar if and only if it has one of the eight structures* $(K1)$ *to* $(K8)$ *shown in Fig.* 5*.*

In Fig. 3, each pair of antipodal points on the dashed circle should be identified to recover the projective-plane. Similarly, in Fig. 4, to recover the torus, both pairs of opposite sides of dashed rectangle should be identified in the same direction, and in Fig. 5, to recover the Klein bottle, the top and bottom sides of the dashed rectangle should be identified in the same direction while the left and right sides should be identified in the opposite direction. In these figures, each of shaded areas corresponds to a component of the graph obtained form the original graph by deleting all edges drawn by bold lines. Some vertices on the boundary of such an area may not be different from each other, that is, the edges drawn by bold lines may not be disjoint. We omit a series of shaded rectangles from $(T2)$, $(T3)$, $(K1)$ and $(K2)$. Both $(T2)$ and $(T3)$ have an even number of shaded rectangles ($(T3)$ may have no shaded rectangle), while both $(K1)$ and $(K2)$ have an odd number of shaded rectangles.

In [10], the re-embedding structure of 2-connected planar graphs on the projective-plane was analysed in detail (see Theorem 3.2 in [10]), while we focus on 3-connected 3-regular planar graphs. Then, our re-embedding structure on the projective-plane, shown in Theorem 5, is a special case in [10], and follows from it. However, our proof is very simple and important for us to understand other Theorems (e.g. Theorem 1). We thus provide a full proof of Theorem 5. Moreover, we also construct the re-embedding structures on the torus and the Klein bottle, which is not characterized completely in [9, 10]. One may think that the case of the Klein bottle can be easily obtained from the case of projective-plane, but this is not true. Some structures in Theorem 7 (e.g. $(K3)$, $(K4)$ and $(K5)$) can be regarded as simple combinations of the structure in Theorem 5, but some are not (e.g. $(K1)$ and $(K2)$).

Let H_X be the subgraph of the dual of G (embedded on the sphere) induced by all edges dual to edges of the given subset X of $E(G)$. Then, there is a vertex of H_X located in the inside of each face of G whose facial cycle has an edge in X. We shall specify what H_X is isomorphic to when F_X^2 is homeomorphic to the projective-plane, the torus or the Klein bottle, which are essential ideas to prove not only Theorems 5, 6 and 7 but also Theorems 1, 2 and 3.

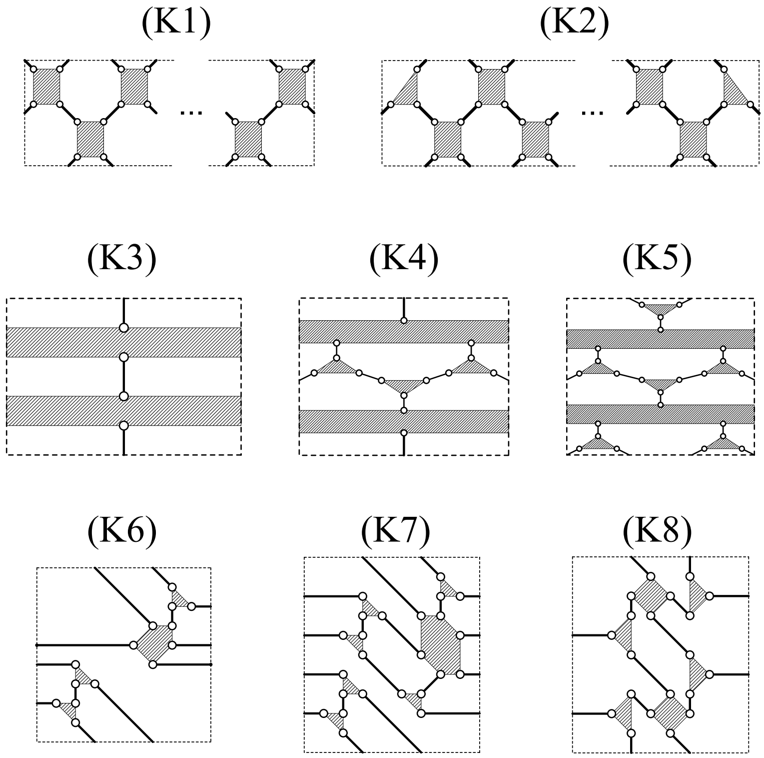

Fig. 5: Re-embedding structures on the Klein bottle

First of all, we give a simple condition of H_X when F_X^2 is homeomorphic to a nonorientable surface. It has been known that an embedding scheme defines an embedding of a given graph on nonorientable surface if and only if there is a cycle containing an odd number of twisted edges (see [7, p.24–25]). It implies the following lemma.

Lemma 8. *For a given subset X of $E(G)$, F_X^2 is nonorientable if and only if there is a vertex of odd degree in H_X.*

Proof: If there is a vertex of odd degree in H_X then the facial cycle, denoted by C, corresponding to the vertex contains an odd number of edges in X. Thus, $f_X(C)$ contains an odd number of twisted edges.

Suppose that F_X^2 is nonorientable. Then, there is a cycle in $f_X(G)$ containing an odd number of twisted edges, that is, there is a cycle in G containing an odd number of edges in X, denoted by C'. Since C' separates the sphere into two regions, the edges dual to $X \cap E(C')$ form an edge-cut of H_X, whose cardinality is odd. Thus, there is a vertex of odd degree in H_X by the handshaking lemma. □

3.1 On the projective-plane

Lemma 9. *For a given subset X of $E(G)$, F_X^2 is homeomorphic to the projective-plane if and only if H_X is isomorphic to K_2 or K_4.*

Proof: Suppose that F_X^2 is homeomorphic to the projective-plane. Any two non-separating simple closed curves on the projective-plane cross at least once.

Let C and C' be any two facial cycles in G each of which has an edge of X. By Lemma 4, $f_X(C)$ and $f_X(C')$ are non-separating cycles on F_X^2 and cross at most once. Thus, $f_X(C)$ and $f_X(C')$ cross exactly once and hence C and C' have exactly one common edge in X. It implies that any two vertices in H_X are adjacent to each other, that is, H_X must be a complete graph. Since H_X is planar and induced by edges, H_X must be isomorphic to K_2, K_3 or K_4. However, H_X is not isomorphic to K_3 by Lemma 8.

If H_X is isomorphic to K_2 or K_4 then G must have one of the structures shown in Fig. 6 (H_X is drawn by squares and dashed lines). Note that K_4 is uniquely embeddable on the sphere and hence the structure is determined uniquely. In Fig. 6, we represent edges in X by bold lines and each component of $G - X$ by shaded area together with some vertices, each of which is an end vertex of an edge in X, on its boundary. Note that X does not have to be a matching, that is, two edges in X may have a common end vertex.

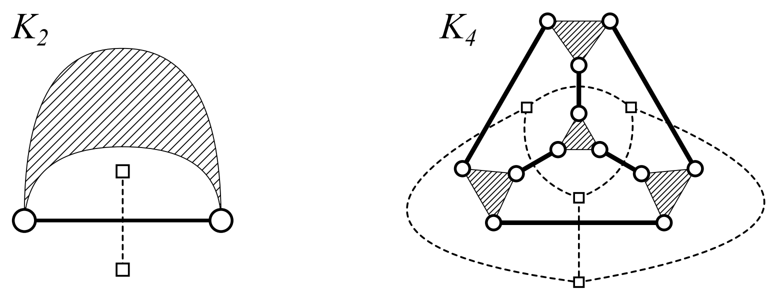

Fig. 6: Two structures of G with H_X

In the situation shown in Fig. 6, by twisting all edges of X, we obtain the re-embedding $f_X(G)$ into the projective-plane shown in Fig. 3. □

Proof of Theorem 5: Let G be a 3-connected 3-regular planar graph. Any embedding of G on a non-spherical surface can be represented by $f_X(G)$ with a suitable non-empty subset X of $E(G)$. By lemma 9, if $f_X(G)$ is a re-embedding of G into the projective-plane then it has one of the structures shown in Fig. 3.

Conversely, it is easy to see that if a 3-connected 3-regular graph has one of the structures shown in Fig. 3, then it can be embedded on the sphere so that it has one of the structures shown in Fig. 6. □

3.2 On the torus

Lemma 10. *For a given subset X of $E(G)$, F_X^2 is homeomorphic to the torus if and only if H_X is isomorphic to $K_{2,2,2}$, $K_{2,2m}$, or $K_{1,1,2m-1}$ for some positive integer m.*

Proof: Suppose that F_X^2 is homeomorphic to the torus. For two simple closed curves crossing at most once on the torus, they cross if and only if they are not homotopic.

Let C and C' be any two facial cycles in G each of which has an edge of X. By Lemma 4, $f_X(C)$ and $f_X(C')$ are non-separating cycles on F_X^2 and cross at most once. Then, $f_X(C)$ and $f_X(C')$ are homotopic if and only if they do not cross, that is, C and C' have no common edge in X, and hence two vertices in H_X corresponding to them are not adjacent. It implies that H_X must be a complete multipartite graph and each partite set corresponds to a non-null homotopy class on the torus.

It is easy to check that any planar complete miltipartite graph is isomorphic to one of the 7 graphs $K_{1,1,1,2}$, $K_{1,1,1,1} = K_4$, $K_{2,2,2}, K_{1,2,2}$, $K_{1,1,n}$, $K_{2,n}$ and $K_{1,n}$ for some natural number n. By Lemma 8, any vertex of H_X has even degree. Then, as H_X is planar, H_X is isomorphic to $K_{2,2,2}$, or $K_{2,2m}$ or $K_{1,1,2m-1}$ for some positive integer m.

Conversely, if H_X is isomorphic to $K_{2,2,2}, K_{2,2m}$ or $K_{1,1,2m-1}$, then G must have one of the structure shown in Fig. 7. Note that all of $K_{2,2,2}, K_{2,2m}$ and $K_{1,1,2m-1}$ is uniquely embeddable on the sphere if we neglect the labels of their vertices.

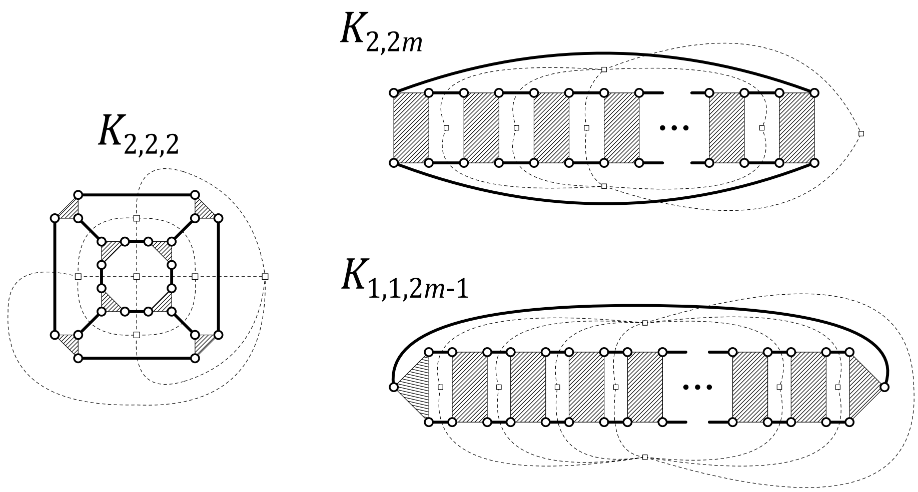

Fig. 7: Three structures of G with H_X

In the situation shown in Fig. 7, by twisting all edges of X, we obtain the re-embedding $f_X(G)$ into the torus shown in Fig. 4.

□

Proof of Theorem 6: Like Theorem 5, this theorem follows immediately from the key lemma; Lemma 10. □

3.3 On the Klein bottle

A simple closed curve on a surface is said to be 2-*sided* if it divides its annular neighbourhood into two parts, and to be 1-*sided* otherwise.

Lemma 11. *For a given subset X of $E(G)$, F_X^2 is homeomorphic to the Klein bottle if and only if H_X is isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$ for some positive integer m, or one of the six graphs A_1 to A_6 shown in Fig. 1.*

Proof: Suppose that F_X^2 is homeomorphic to the Klein bottle. There are exactly two mutually disjoint non-separating simple closed 1-sided curves and exactly one non-separating 2-sided curve on the Klein bottle up to homotopy.

Let C and C' be any two facial cycles in G each of which has an edge of X. Then, $f_X(C)$ and $f_X(C')$ are non-separating cycles on F_X^2 and cross at most once. We first assume that both $f_X(C)$ and $f_X(C')$ are 1-sided. Then, $f_X(C)$ and $f_X(C')$ cross if and only if they are homotopic. Second, we assume that one of $f_X(C)$ and $f_X(C')$ is 1-sided and the other is 2-sided. Then, they cross. Third, we assume that both $f_X(C)$ and $f_X(C')$ are 2-sided. Then they are homotopic and hence do not cross.

The vertex of H_X corresponding to C has odd degree if and only if $f_X(C)$ is 1-sided. Thus, the facts mentioned in the last paragraph imply that H_X has the following conditions. (1) The vertices of odd degree in H_X induce a graph having at most two components each of which is isomorphic to a complete graph. (2) Any vertex of even degree and any vertex of odd degree are adjacent. (3) The vertices of even degree in H_X are independent, that is, any pair of such vertices are not adjacent.

Let V_{odd} (resp. V_{even}) be the set of vertices of odd (resp. even) degree in H_X. Since V_{even} is a independent set and any vertex of V_{even} is adjacent to each vertex of V_{odd}, $|V_{odd}|$ is even.

Case 1: V_{odd} induces a complete graph K_m. As H_X is planar, $m = 2$ or 4.

Subcase 1a: $m = 2$. It is easy to see that $|V_{even}|$ is even. Then, H_X is isomorphic to $K_{1,1,2k}$ with some non-negative integer k. However, if H_X is isomorphic to $K_{1,1,0} = K_2$ then F_X^2 is homeomorphic to the projective-plane by Lemma 9. Then, $k \geq 1$.

Subcase 1b: $m = 4$. If there is at least one vertex in V_{even} then H_X is not planar since it contains K_5 as a subgraph. Moreover, if V_{even} is empty, then H_X is isomorphic to K_4 and hence F_X^2 is homeomorphic to the projective-plane by Lemma 9. Therefore, $m \neq 4$.

Case 2: V_{odd} induces two disjoint complete graphs K_m and K_n. Then, we have $m + n = 2, 4, 6, 8$.

Subcase 2a: $m+n = 2$, that is, $m = n = 1$. In this situation, H_X is isomorphic to $K_{2,2k-1}$ with some positive integer k.

Subcase 2b: $m + n = 4$, that is, $m = n = 2$ or $m = 1, n = 3$. Suppose that $m = n = 2$. If $|V_{even}| \geq 3$ then H_X is not planar since it contains $K_{3,3}$ as a subgraph. If $|V_{even}| = 1$ then each vertex of H_X has even degree, which contradicts Lemma 8. If $|V_{even}| = 0$ or 2 then H_X corresponds to A_1 or A_6, respectively.

Suppose that $m = 1$ and $n = 3$. If $|V_{even}| \geq 3$ then H_X is not planar since it contains $K_{3,3}$ as a subgraph. If $|V_{even}| = 0, 2$ then each vertex of H_X has even degree, which contradicts Lemma 8. If $|V_{even}| = 1$ then H_X corresponds to A_4.

Subcase 2c: $m+n = 6$, that is, $m = n = 3$ or $m = 2, n = 4$. Suppose that $m = n = 3$. If $|V_{even}| \geq 3$ then H_X is not planar since it contains $K_{3,3}$ as a subgraph. If $|V_{even}| = 0, 2$ then each vertex of H_X has even degree, which contradicts Lemma 8. If $|V_{even}| = 1$ then H_X corresponds to A_5.

Suppose that $m = 2, n = 4$. If $|V_{even}| \geq 1$ then H_X is not planar since it contains K_5 as a subgraph. If $|V_{even}| = 0$ then H_X corresponds to A_2.

Subcase 2d: $m + n = 8$, that is, $m = n = 4$. If $|V_{even}| \geq 1$ then H_X is not planar since it contains K_5 as a subgraph. If $|V_{even}| = 0$ then H_X corresponds to A_3.

According to the above results, H_X is isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$ for some positive integer m, or H_X is isomorphic to one of the six graphs A_1 to A_6.

Conversely, if H_X is isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$ for some positive integer m, or H_X is isomorphic to one of the six graphs A_1 to A_6, then G must have one of the structure shown in Fig. 8. Note that all graphs shown in Fig. 1 are uniquely embeddable on the sphere if we neglect the labels of their vertices.

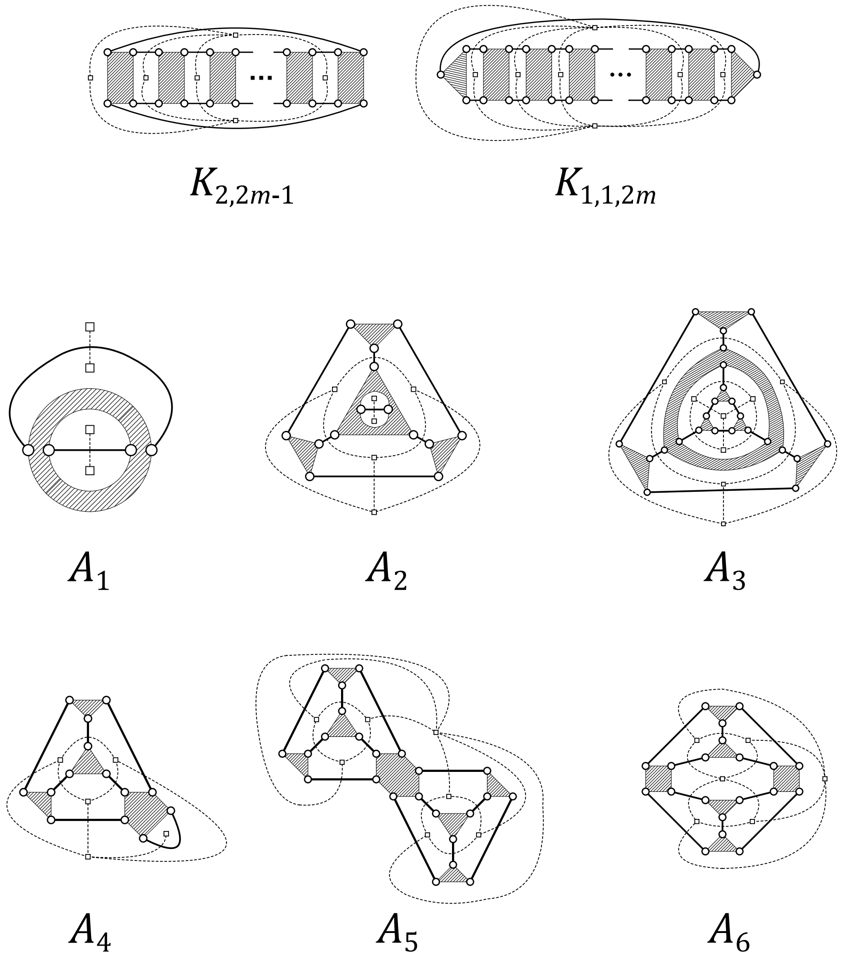

Fig. 8: Eight structures of G with H_X

In the situation shown in Fig. 8, by twisting all edges of X, we obtain the re-embedding $f_X(G)$ into the Klein bottle shown in Fig. 5. □

Proof of Theorem 7: Like Theorem 5, this theorem follows immediately from the key lemma; Lemma 11. □

3.4 Proof of Theorems

Theorems 1, 2 and 3 immediately follow from Lemmas 9, 10 and 11, respectively.

Proof Proof of Theorems 1, 2 and 3: Let G be a 3-connected 3-regular planar graph. Any embedding of G on any surface is equivalent to an embedding $f_X(G)$ associated with a suitable subset X of $E(G)$. Moreover, such X is unique. Thus, Lemmas 9, 10 and 11 imply Theorems 1, 2 and 3, respectively. $\square$

4 Inequivalent embeddings

In this section, we first give explicit bounds for the number of inequivalent embeddings of G on each of the projective-plane, the torus and the Klein bottle. After that, we propose algorithms for enumerating and counting these embeddings.

4.1 The number of inequivalent embeddings

Based on Theorems 1, 2 and 3, we show the following three results.

Theorem 12. *A 3-connected 3-regular planar graph with n vertices has at least $\frac{3}{2}n$ and at most $2n-1$ inequivalent embeddings on the projective-plane.*

Theorem 13. *A 3-connected 3-regular planar graph with $n \geq 5$ vertices has at least $\frac{5}{2}n$ inequivalent embeddings on the torus.*

Theorem 14. *A 3-connected 3-regular planar graph with n vertices has at least $\frac{3}{8}n(3n+2)$ inequivalent embeddings on the Klein bottle.*

Before we prove these theorems, we consider a situation where the dual of G embedded on the sphere has many subgraphs isomorphic to K_4, which is useful for showing the upper bound of Theorem 12 and characterizing graphs attaining this bound.

A *triangulation* on a surface is an embedding of a graph on the surface such that each face is bounded by a cycle of order 3 and any two faces are incident with at most one common edge. A graph embedded on the sphere is 3-connected and 3-regular if and only if the dual is a triangulation on the sphere. For a triangulation T, a 3-*vertex addition* is an operation of adding a vertex into a face Δ of T and joining the new vertex to the vertices on the boundary of Δ.

Lemma 15. *Every triangulation T on the sphere has at most $(|V(T)|-3)$ subgraphs isomorphic to K_4. In particular, T attains the upper bound if and only if T is obtained from K_4 embedded on the sphere by a sequence of 3-vertex additions.*

Proof: The proof is by induction on the number of vertices.

If $|V(T)| = 4$ then T is K_4 itself and hence the result clearly holds. Thus, we assume $|V(T)| \geq 5$.

If T has no separating cycle of order 3 then T has no subgraph isomorphic to K_4. Thus, we may assume that T has a separating cycle C of order 3, which separates the sphere into two regions, denoted by R_1 and R_2. Let T_1 (resp. T_2) be the subgraph of T induced by the vertices lying on R_1 (resp. R_2) with its boundary. Then, both of T_1 and T_2 is also a triangulation on the sphere. Note that $T_1 \cap T_2 = C$ and $T_1 \cup T_2 = G$. For any vertices $x \in V(T_1)\backslash V(C)$ and $y \in V(T_2)\backslash V(C)$, there is no edge whose endvertices are x and y, and hence there are no subgraphs of T isomorphic to K_4 having both x and y. Thus, the number of subgraphs of T isomorphic to K_4 is at most

$$(|V(T_1)|-3)+(|V(T_2)|-3) = (|V(T)|+3)-6 = |V(T)|-3.$$

Next, we characterize triangulations attaining this upper bounds. Let $\tilde{T}$ be a triangulation on the sphere obtained from T by one operation of a 3-vertex addition and $\tilde{v}$ be the additional vertex of $\tilde{T}$. There is exactly one subgraph of $\tilde{T}$ isomorphic to K_4 including $\tilde{v}$. If T is obtained from K_4 by a sequence of 3-vertex addition then T has has exactly $|V(T)| - 3$ subgraphs isomorphic to K_4, and hence $\tilde{T}$ has exactly $|V(T)| - 2 = |V(\tilde{T})| - 3$ subgraphs isomorphic to K_4.

Conversely, suppose that T has exactly $|V(T)| - 3$ subgraphs isomorphic to K_4. We may assume that $|V(T)| \geq 5$ and T has a separating cycle C of order 3. Then, T_1 and T_2, which are defined in the same way as above, must have exactly $|V(T_1)| - 3$ and $|V(T_2)| - 3$ subgraphs isomorphic to K_4, respectively, and hence both are obtained from K_4 by a sequence of 3-vertex additions.

Let T' be a triangulation on the sphere obtained from K_4 by a sequence of 3-vertex addition but not K_4. It is easy to check that any two vertices of degree 3 are not adjacent in T'. Thus, an operation of a 3-vertex addition from T' will not decrease the number of vertices of order 3, and hence T' has at least two vertices of degree 3.

The above facts imply that we can obtain K_4 from T_2 by deleting a vertex of degree 3 without deleting the vertices on C. By applying these operations to T and deleting the last vertex from R_2, we have just obtained T_1. Therefore, T is also obtained from K_4 by a sequence of 3-vertex additions. □

Proof of Theorem 12: Let G be a 3-connected 3-regular planar graph embedded on the sphere with n vertices and G^* be its dual. Choose an edge e of G and put $X = \{e\}$. Then, H_X is isomorphic to K_2 and hence $f_X(G)$ is embedded on the projective-plane by Lemma 9. It implies that G has at least $|E(G)|$ inequivalent embeddings on the projective-plane. Since G is 3-regular, we have $|E(G)| = \frac{3}{2}n$.

By Lemma 15, there are at most $|V(G^*)| - 3$ subgraphs of G^* isomorphic to K_4. By Euler's formula, $|V(G^*)| = (|V(G)| + 4)/2 = \frac{n+4}{2}$ and hence G^* has at most $\left(\frac{n+4}{2} - 3\right)$ subgraphs isomorphic to K_4. Thus, by Theorem 1, G has at most $\frac{3}{2}n + \left(\frac{n}{2} - 1\right) = 2n - 1$ inequivalent embeddings on the projective-plane. □

Not only 3-connected 3-regular planar graphs, any 2-connected graph G has $|E(G)|$ inequivalent embeddings on the projective-plane by twisting each edge of G. Then, the assumptions on 3-connectivity and 3-regularity are not necessary in the lower bound in Theorem 12. However, these assumptions are clearly necessary in the upper bound in Theorem 12. In fact, we can easily construct non-3-connected or non-3-regular 2-connected planar graphs having exponentially many inequivalent embeddings on the projective-plane.

Proof of Theorem 13: Let G be a 3-connected 3-regular planar graph embedded on the sphere with at least 5 vertices and G^* be its dual. Every face of G^* is bounded by a cycle of order 3 ($= K_{1,1,1}$) and every edge of G^* forms a chord of a cycle of order 4 ($= K_{2,2}$) since it is incident with just two triangle faces. If G is not isomorphic to K_4, then there are no other chords in this cycle. Thus, G^* has at least $|V(G)|$ cycles of order 3 and at least $|E(G)|$ cycles of order 4 as subgraphs. As G is 3-regular, $|E(G)| = 3n/2$. Then, by Theorem 2, G has at least $n + 3n/2 = 5n/2$ inequivalent embeddings on the torus. □

Note that a 3-connected 3-regular planar graph with at most 4 vertices must be isomorphic to K_4, which has exactly $7 \leq 5 \cdot 4/2$ inequivalent embeddings on the torus.

Proof of Theorem 14: Let G be a 3-connected 3-regular planar graph embedded on the sphere with n vertices and G^* be its dual. Choose two distinct edges e_1 and e_2 of G and put $X = \{e_1, e_2\}$. Then, H_X is isomorphic to $K_{2,1}$ or A_1. By Lemma 11, $f_X(G)$ is embedded on the Klein bottle.

In addition, we try to find subgraphs isomorphic to $K_{1,1,2}$ in G^*. Since G^* is a triangulation on the sphere, every edge e^* is incident with just two triangle faces. The five edges bounding these faces induce subgraphs of G^* isomorphic to $K_{1,1,2}$. Then, G^* has $|E(G^*)|$ subgraphs isomorphic to $K_{1,1,2}$.

These results imply that, by Theorem 3, G has at least $\binom{|E(G)|}{2} + |E(G^*)| = \frac{3}{8}n(3n+2)$ inequivalent embeddings on the Klein bottle. □

4.2 Examples

First, we characterize the graphs attaining the lower bound of Theorem 12. By Theorem 12, the following clearly holds.

Corollary 16. *A 3-connected 3-regular planar graph G with n vertices has exactly $\frac{3}{2}n$ inequivalent embeddings on the projective-plane if and only if the dual of G embedded on the sphere has no subgraph isomorphic to K_4.* □

By this corollary, we show the following two families of graphs attaining the lower bound of Theorem 12.

A graph is *cyclically k-edge-connected* if there is no set of at most $k-1$ edges such that the graph obtained by deleting these edges has at least two components having a cycle.

Proposition 17. *A 3-connected 3-regular planar graph G with $n \geq 5$ vertices has exactly $\frac{3}{2}n$ inequivalent embeddings on the projective-plane if G is bipartite or cyclically 4-edge-connected.*

Proof: We only have to show that the dual G^* of G embedded on the sphere has no subgraph isomorphic to K_4.

If G is bipartite then degree of each vertex of G^* is even, that is, G^* is a *even* triangulation. It is well-known that every even triangulation on the sphere is (vertex) 3-colorable and hence has no subgraph isomorphic to K_4.

If G is cyclically 4-edge-connected and $n \geq 5$, then G^* has no separating cycle of order 3 and hence has no subgraph isomorphic to K_4. □

Second, we characterize the graphs attaining the upper bound of Theorem 12. Towards this goal, we introduce a transforming operation of G, which corresponds to a 3-vertex addition in the dual of G.

Let v be a vertex of G, and u_1, u_2 and u_3 be vertices adjacent to v. A *truncation* of a vertex v in G is an operation of replacing a small part around v with a cycle of order 3 shown in Fig. 9; delete v and add new vertices v_1, v_2 and v_3 together with six edges $u_1v_1, u_2v_2, u_3v_3, v_1v_2, v_2v_3$ and v_3v_1.

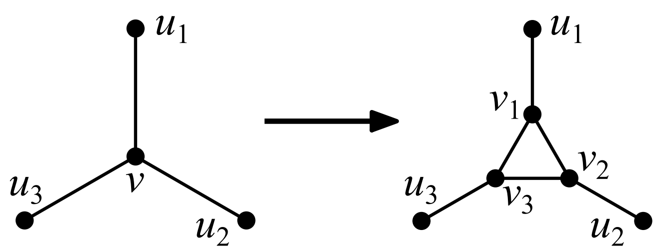

Fig. 9: Truncation of a vertex

The resulting graph, denoted by G', is also 3-connected, 3-regular and planar. The dual $(G')^*$ is obtained from the dual G^* of G by a 3-vertex addition. Then, the following clearly holds by Lemma 15.

Corollary 18. *A* 3*-connected* 3*-regular planar graph G with n vertices has exactly* $2n - 1$ *inequivalent embeddings on the projective-plane if and only if G is obtained from K_4 by a sequence of truncations.* □

Third, we provided graphs attaining the lower bounds of Theorems 13 and 14.

Corollary 19. *A* 3*-connected* 3*-regular planar graph G with $n \geq 5$ vertices has exactly* $\frac{5}{2}n$ *inequivalent embeddings on the torus if and only if G is cyclically* 5*-edge-connected.*

Corollary 20. *A* 3*-connected* 3*-regular planar graph G with n vertices has exactly* $\frac{3}{8}n(3n + 2)$ *inequivalent embeddings on the Klein bottle if G is cyclically* 5*-edge-connected.*

Proof of Corollaries 19 and 20: Suppose that G is cyclically 5-edge-connected. In each of the six structures shown in Fig. 7 and Fig. 8 corresponding to $K_{2,2,2}, A_2, A_3, A_4, A_5$ and A_6, we can easily find a set of at most four edges (drawn by bold lines) such that the graph obtained from G by deleting these edges has at least two components having a cycle. (The shaded annular area in A_2 must have a cycle.) Then, G has none of these six structures and hence G^* has no subgraph isomorphic to one of the six graphs $K_{2,2,2}, A_2, A_3, A_4, A_5$ and A_6.

Suppose that G^* has a subgraph isomorphic to $K_{2,n}$ or $K_{1,1,n}$ with $n \geq 3$. Then, G has one of the four structures shown in Fig. 7 and Fig. 8 corresponding $K_{2,2m}, K_{1,1,2m-1}, K_{2,2m-1}$ and $K_{1,1,2m}$. In both case, G has at least three shaded areas, and since G is cyclically 5-edge-connected, all shaded areas except for at most one have no cycle. Hence, there are two consecutive shaded areas having no cycle in G, one of which is rectangle. These shaded areas together with edges joining them form one of the three subgraphs shown in Fig. 10.

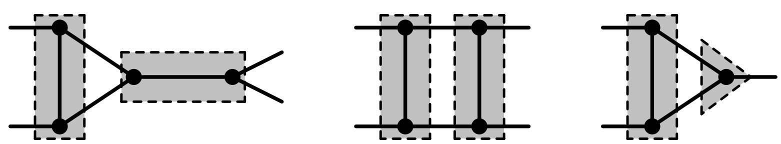

Fig. 10: Subgraphs formed by two consecutive shaded areas

These subgraphs have a cycle and can be separated from G by deleting at most four edges. This implies that G has at most one more shaded rectangle and that it contains no cycle. Hence it must be one of the three graphs shown in Fig. 11. These graphs have distinct structures but each graph is the same as the others. However, this graph is not cyclically 5-edge-connected, a contradiction.

Therefore, G^* has no subgraph isomorphic to $K_{2,n}$ or $K_{1,1,n}$ with $n \geq 3$, and hence G has only $\frac{5}{2}n$ inequivalent embeddings mentioned in the proof of Theorem 13 on the torus, and only $\frac{3}{8}n(3n + 2)$ inequivalent embeddings mentioned in the proof of Theorem 14 on the Klein bottle.

Suppose G is not cyclically 5-edge-conneted. Since G is 3-connected, there are three or four edges of G whose removal results in a disconnected graph having exactly two components, both of which contain a cycle. Let X be such edges. Then, H_X is isomorphic to $K_{1,1,1} = K_3$ or $K_{2,2}$, and hence $f_X(G)$ is embedded on the torus. However, this re-embedding conforms to none of re-embeddings mentioned in

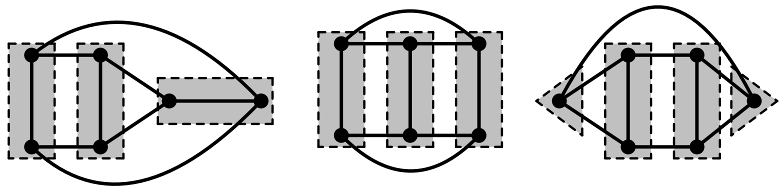

Fig. 11: Three graphs

the proof of Theorem 13. Thus, the number of inequivalent embeddings of G on the torus is more than $\frac{5n}{2}$. □

A graph attaining the lower bound of Theorem 14 is not necessarily cyclically 5-edge-connected. For example, the following graph shown in Fig. 12 is such a graph. We can construct infinitely many such graphs but we omit this here.

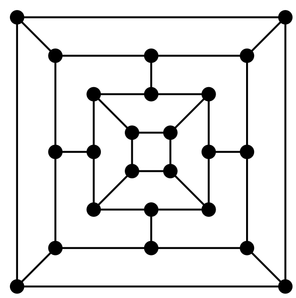

Fig. 12: A graph attaining the lower bound of Theorem 14

Finally, we show graphs having exponentially many inequivalent embeddings on the torus and the Klein bottle.

Proposition 21. *For a* 3*-connected* 3*-regular planar graph G with n vertices, if the dual G^* of G embedded on the sphere has a subgraph isomorphic to $K_{1,1,m}$ with a positive integer m then G has at least $2^m - 1$ inequivalent embeddings on each of the torus and the Klein bottle.*

Proof: For the complete tripartite graph $K_{1,1,m}$ with partite sets $V_1 = \{u\}, V_2 = \{v\}$ and $V_3 = \{a_1, a_2, \ldots, a_m\}$, let S be any non-empty subset of V_3. A subgraph induced by $V_1 \cup V_2 \cup S$ is isomorphic to $K_{1,1,|S|}$. Deleting an edge uv from this subgraph, we obtain a subgraph isomorphic to $K_{2,|S|}$.

It implies that if G^* has a subgraph isomorphic to $K_{1,1,m}$, then we can find $2(2^m - 1)$ subgraphs in G^* isomorphic to $K_{1,1,k}$ or $K_{2,k}$ for some positive integer k. Moreover, in these subgraphs, the number of subgraphs isomorphic to $K_{1,1,k}$ is the same as the one isomorphic to $K_{2,k}$ for any $1 \leq k \leq m$. Then, by Theorems 2 and 3, G has at least $2^m - 1$ inequivalent embeddings on each of the torus and the Klein bottle. □

4.3 Algorithms

By Theorem 12, the number of inequivalent embeddings of G on the projective-plane is $O(n)$ with respect to the number n of vertices of G. In fact, we can easily enumerate these embeddings in polynomial-time with respect to n. Note that we regard an enumeration of embedding schemes of a graph as one of the embeddings of the graph.

Theorem 22. *There is a polynomial time algorithm for enumerating inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on the projective-plane.*

Proof: Let G be a 3-connected 3-regular planar graph. The embedding of G on the sphere, its dual G^* and another embedding $f_X(G)$ with a given subset X of $E(G)$ can be obtained in polynomial time. Then, we only have to find subgraphs isomorphic to K_2 or K_4 in G^* by Lemma 9, which can be done in polynomial time. □

On the other hand, there are 3-connected 3-regular planar graphs having exponentially many inequivalent embeddings on the torus and the Klein bottle by Proposition 21. Then, we cannot enumerate inequivalent embeddings of such a graph on the torus or the Klein bottle in polynomial time. However, we shall give a "polynomial delay" algorithm for enumerating them. An enumeration algorithm is said to be *polynomial delay* if the maximum computation time between two consecutive outputs is polynomial in the input size.

For the complete miltipartite graphs $K_{2,m+2}$ and $K_{1,1,m+1}$ with any positive integer m, there are exactly two vertices whose degree is not two. We call them *apex vertices*.

Theorem 23. *There is a polynomial delay algorithm for enumerating inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on each of the torus and the Klein bottle.*

Proof: Let G be a 3-connected 3-regular planar graph. We enumerate inequivalent embeddings of G on the torus and the Klein bottle simultaneously. Like the proof of Theorem 22, we only have to find subgraphs isomorphic to $K_{2,m+1}$ or $K_{1,1,m}$ for any positive integer m, or one of the seven graphs $A_1, \cdots, A_6$ and $K_{2,2,2}$. (Every time we find such a subgraph, output a embedding corresponding to it)

First, we find subgraphs isomorphic to one of the nine graphs $A_1, \cdots, A_6$, $K_{1,1,1} = K_3, K_{2,2}$ and $K_{2,2,2}$ in G^*, which can be done in polynomial time. Second, for a pair of vertices u and v of G^*, enumerate vertices adjacent to both of them. Third, enumerate subgraphs isomorphic to $K_{2,m+2}$ or $K_{1,1,m+1}$ whose apex vertices are u and v. In such a subgraph, all non-apex vertices are already enumerated in the second step. Thus, the third step can be done in polynomial delay time.

To repeat the second and third step for any pair of vertices, in the end, we have just enumerated all subgraphs isomorphic to $K_{2,m+2}$ or $K_{1,1,m+1}$ in G^*. □

We can calculate the total number of inequivalent embeddings of G on each of the projective-plane, the torus and the Klein bottle in polynomial time by a simple improvement of the above algorithms.

Corollary 24. *There is a polynomial time algorithm for counting the number of inequivalent embeddings of a* 3*-connected* 3*-regular planar graph on each of the projective-plane, the torus and the Klein bottle.*

Proof: The projective-planar case clearly holds. We only have to count all the embeddings enumerated in the algorithm of Theorem 22. Thus, we may consider embeddings on the torus and the Klein bottle.

Let G be a 3-connected 3-regular planar graph. On the basis of the algorithm in Theorem 23, we shall count subgraphs of G^* isomorphic to $K_{2,2,2}, K_{2,2m}$ or $K_{1,1,2m-1}$, which correspond to embeddings on the torus, and count subgraphs isomorphic to $K_{2,2m-1}$ or $K_{1,1,2m}$, or one of the six graphs A_1 to A_6, which correspond to embeddings on the Klein bottle.

Like the proof of Theorem 23, we first count the number of subgraphs of G^* isomorphic to one of the three graphs $K_{2,2,2}, K_{2,2}$ and $K_{1,1,1} = K_3$, and denote it by N_T. Similarly, we count the number of subgraphs isomorphic to one of the six graphs A_1 to A_6, and denote it by N_K.

Second, for the pair of vertices u and v in the proof of Theorem 23, assume that exactly k vertices are adjacent to both u and v. Let $f_T(u, v)$ (resp. $f_K(u, v)$) be the number of subgraphs isomorphic to $K_{2,2m+2}$ or $K_{1,1,2m+1}$ (resp. $K_{2,2m-1}$ or $K_{1,1,2m}$) for any positive integer m whose apex vertices are u and v. If u and v are adjacent to each other in G^*, then we have

$$f_T(u, v) = \sum_{i=3}^{k} \binom{k}{i} = 2^k - \frac{k(k-1)}{2} - k - 1 = 2^k - \frac{k^2 + k + 2}{2},$$

$$f_K(u, v) = \sum_{i=1}^{k} \binom{k}{i} = 2^k - 1.$$

Otherwise,

$$f_T(u, v) = \sum_{i=2}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i} = 2^{k-1} - \frac{k(k-1)}{2} - 1 = 2^{k-1} - \frac{k^2 - k + 2}{2},$$

$$f_K(u, v) = \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k}{2i-1} = 2^{k-1}.$$

Add the sum of $f_T(u, v)$ (resp. $f_K(u, v)$) taken over all pairs of vertices u, v to N_T (resp. N_K). These are the total numbers of inequivalent embeddings of G on the torus and the Klein bottle. Thus, we can obtain this number in polynomial time. □

5 Concluding remarks

In this paper, we have shown the re-embedding structures of a 3-connected 3-regular planar graph G on the projective-plane, the torus and the Klein bottle. These structures enable us to count inequivalent embeddings of G on each surface easily.

In order to extend our result to surfaces with higher genera, we should show the complete lists of re-embedding structures of G on these surfaces like Theorems 5, 6 and 7. However, we think that there are a large number of re-embedding types even on an orientable surface with genus 2 or a nonorientable surface with genus 3. Then, it seems to be difficult to give such complete lists without additional assumptions.

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