

New Results on Classical and Quantum Counter Automata*

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We show that one-way quantum one-counter automaton with zero-error is more powerful than its probabilistic counterpart on promise problems. Then, we obtain a similar separation result between Las Vegas one-way probabilistic one-counter automaton and one-way deterministic one-counter automaton.

We also obtain new results on classical counter automata regarding language recognition. It was conjectured that one-way probabilistic one blind-counter automata cannot recognize Kleene closure of equality language [A. Yakaryılmaz: Superiority of one-way and realtime quantum machines. *RAIRO - Theor. Inf. and Applic.* 46(4): 615-641 (2012)]. We show that this conjecture is false, and also show several separation results for blind/non-blind counter automata.

Keywords: quantum automata, counter automata, promise problems, blind counter, zero-error, Las-Vegas algorithms

1 Introduction

Quantum computation is a generalization of probabilistic computation which is a generalization of deterministic computation. It is natural to ask whether a quantum model is more powerful than its probabilistic counterpart and similarly whether a probabilistic model is more powerful than its deterministic counterpart. For a fair comparison between these three types of models, bounded-error models of quantum and probabilistic should be considered (as we do in this paper).

Quantum automata models are restricted models of quantum Turing machines, i.e., the type of memory and/or the direction of head movement can be restricted. We find it interesting to determine whether quantum models can have advantage in such restricted case. In this paper, we specifically focus on counter type of memory.

We have a more complete picture for constant-space models (finite state automata) when compared to models using memories (finite state automata augmented with counter(s), stack(s), tape(s), etc.). For example, one-way⁽ⁱ⁾ deterministic finite automata (1DFAs) are equivalent to one-way probabilistic finite

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⁽ⁱ⁾ The input is read as a stream from left to right and a single symbol is fed to the machine in each step. We also use two end-markers to allow the machine making some pre- and post-processing.

automata (1PFAs) and one-way quantum finite automata (1QFAs): they all define the class of regular languages [Sip06, Rab63, LQZ⁺12, AG00]. On the other hand, in the case of two-way models⁽ⁱⁱ⁾ abbreviated as 2DFA, 2PFA, and 2QFA, respectively, 2DFAs are equivalent to 1DFAs, 2PFAs are more powerful than 2DFAs, and 2QFAs are more powerful than 2PFAs. [She59, Fre81, KW97, AW02]. As a special case, one-way with ε -moves⁽ⁱⁱⁱ⁾ (1_ε) quantum finite automata (1_ε QFAs) can recognize some non-regular languages if the head is allowed to be in a superposition [AI99]. Note that ε -moves can be easily removed for the classical finite automata without increasing the number of states.

When considering finite automata using memory, there are more unanswered cases. The most challenging ones seem to be between quantum and probabilistic models. For example, 1PFAs with a counter (1PICAs) are more powerful than 1DFAs with a counter (1D1CAs) [Fre79] but we do not know whether 1QFAs with a counter (1Q1CAs) are more powerful than 1PICAs – we have only an affirmative answer for one-sided bounded-error [SY12]. For two-way models, abbreviated respectively 2D1CAs, 2P1CAs, and 2Q1CAs, only 2Q1CAs were shown to be more powerful than 2D1CAs [Yak13] and the other cases are still open. For one-way pushdown automata models, abbreviated as 1DPDAs, 1PPDAs, and 1QPDA, respectively, 1DPDAs were shown to be weaker than even Las Vegas restriction of 1PPDAs [HS10] and the question is open between quantum and probabilistic models [YFSA12].

All mentioned results above are regarding language recognition. When considering solving promise problems, the picture can dramatically change. The notion of promise problems was introduced in [ESY84]. Informally, promise problems are a generalization of language recognition such that the aim is to separate two disjoint languages that do not necessarily form the set of all strings. Promise problems have deep connection with fundamental issues in complexity theory such as unique solutions, approximation and complete problems. Readers may refer to [Gol06] for a survey on these issues. For quantum computation, the first two notable results on promise problems are the Deutsch-Jozsa algorithm [DJ92, CEMM98] and Simon's algorithm [Sim97]. The results give separation between quantum and classical computation models in the exact and bounded-error settings, respectively. Also for automata models, promise problems have been investigated intensively [MNYW05, RY14, GY15a, GY15b, GQZ15, Nak15, BMY17, ZLQG17, GY18]. The separation results can be obtained even for one-way models or the case of zero-error – a very restricted case is that unary QFAs are more powerful than unary PFAs [GY15a]. Also as pointed out in [GY15b], the effects of randomness and quantumness can be more easily shown with promise problems and some open problems defined on language recognition can be answered in the case of solving promise problems. In [MNYW05, Nak15], exact^(iv) 1_ε QPDAs are shown to be more powerful than exact 1_ε PPDAs, which are equivalent to 1_ε DPDAs.

In this paper, we show that quantum models can still be more powerful if we replace the stack with a counter: we show that exact 1Q1CAs can solve a certain promise problem that cannot be solved by exact 1PICAs, which are equivalent to 1D1CAs. As mentioned above, Las Vegas 1_ε PPDAs are more powerful than 1_ε DPDAs on language recognition. As the second separation, we obtain the same result between Las Vegas 1PICAs and 1D1CAs on promise problems. In each separation, we define a new promise problem and give an algorithm for the more powerful model, and then, we present the impossibility result for the weaker model. As far as the authors know, separation results on neither language recognition nor solving promise problems were known for those automata models. Thus, our separation results on

⁽ⁱⁱ⁾ The input is written on a single-head read-only tape between two end-markers and the head can move in both directions or stay in the same tape square in each step.

⁽ⁱⁱⁱ⁾ It is a restricted version of two-way head such that the head cannot move to the left.

^(iv) A single answer is given with probability 1.

promise problems can be regarded as an important first step toward understanding the complexities of those automata models.

Additionally, we present new results on classical counter automata. We disprove the conjecture defined by Yakaryılmaz [Yak12]: Yakaryılmaz separated IQFAs with a blind counter from 1DFAs with a blind counter by using the language EQ^* , the Kleene closure of $\text{EQ} = \{a^n b^n \mid n > 0\}$, and then, the author conjectured that the same language cannot be recognized by 1PFAs with a blind counter. However, we provide an algorithm for 1PFAs with a blind counter that recognizes EQ^* . We also show several separation results for blind/non-blind counter automata.

In the next section, we provide the required background and then we present our main results on promise problems in Section 3 and new classical results on language recognition in Section 4.

A preliminary version of the paper was presented in CIAA 2015 [NY15]. In this version, we revise the overall paper and added new results and proofs. We modify the definitions of promise problems ONE-NONE and $\text{ONE-NONE}(\tau)$ in Section 3.2 since we observe that the argument on the Las Vegas algorithm for $\text{ONE-NONE}(\tau)$ in [NY15] is not correct. After this modification, we obtain better success probability in Theorem 3 and we also give correct statement on $\text{ONE-NONE}(\tau)$ in Theorem 4. Since the promise problem ONE-NONE is modified, we provide a new impossibility proof for 1D1CAs (Theorem 5). We should remark that this new proof is more complicated (and longer) than the previous proof. Theorem 6 is a new result, which was left open in [NY15]. Additionally, we revise the second half of Section 4 and present new results regarding comparisons of classical models: Theorems 8, 9, and 10.

2 Definitions

Throughout the paper, Σ , not containing $\$$ and $\$$ (the left and the right end-markers, respectively), denotes the input alphabet; $\bar{\Sigma} = \Sigma \cup \{\$, \$\}$; Q is the set of (internal) states; $Q_a \subseteq Q$ (resp., $Q_r \subseteq Q$) is the set of accepting (resp., rejecting) states; q_0 is the initial state. For any $w \in \Sigma^*$, $w(i)$ is the i -th symbol of w , and $|w|$ is the length of w . We assume the reader knows the basics of automata theory. We denote one-way deterministic and nondeterministic finite automata as 1DFA and 1NFA, respectively.

A promise problem $P = (P_{\text{yes}}, P_{\text{no}})$ defined on Σ is composed by two disjoint languages $P_{\text{yes}} \subseteq \Sigma^*$ and $P_{\text{no}} \subseteq \Sigma^*$, called respectively the set of yes-instances and the set of no-instances.

A promise problem $P = (P_{\text{yes}}, P_{\text{no}})$ is said to be solved by a (probabilistic or quantum) machine M with error bound $\epsilon < \frac{1}{2}$ if any yes-instance is accepted with probability at least $1 - \epsilon$ and any no-instance is rejected with probability at least $1 - \epsilon$. It is also said that P is solved by M with bounded-error. If yes-instances (resp., no-instances) are accepted (resp., rejected) exactly, then it is said that P is solved by M with negative (resp., positive) one-sided error bound ϵ . If $\epsilon = 0$, then it is said that the promise problem is solved exactly.

A promise problem $P = (P_{\text{yes}}, P_{\text{no}})$ is said to be solved by a Las Vegas machine with success probability $p > 0$ if

- any yes-instance is accepted with probability at least p and is rejected with probability 0, and,
- any no-instance is rejected with probability at least p and is accepted with probability 0.

Remark that all non-accepting or non-rejecting probabilities go to the decision of “don’t know”.

If P_{yes} is the complement of P_{no} , then conventionally it is said that the language P_{yes} is recognized by machine M instead of saying that promise problem P is solved by machine M .

For all models, the input $w \in \Sigma^*$ is placed on a read-only one-way infinite tape as $\tilde{w} = \#w\#$ between the cells indexed by 1 to $|\tilde{w}|$. At the beginning, the head is initially placed on the cell indexed by 1 and the value of the counter is set to zero. Also, in the following definitions, m denotes the maximum value by which the counter may be increased or decreased at each step.

A one-way probabilistic one-counter automaton (1P1CA) is a 5-tuple

$$M = (Q, \Sigma, \delta, q_0, Q_a),$$

where $\delta : Q \times \Sigma \times \{Z, NZ\} \times Q \times \{-m, \dots, m\} \rightarrow [0, 1]$ is a transition function such that $\delta(q, \sigma, z, q', c) = p$ means that the transition from $q \in Q$ to $q' \in Q$ increasing the counter value by $c \in \{-m, \dots, m\}$ occurs with probability $p \in [0, 1]$ if the scanned symbol is $\sigma \in \tilde{\Sigma}$ and the status of the counter value is z , where Z (resp., NZ) means zero (resp., non-zero). The transition function must satisfy the following condition since the overall probabilities must be 1 during the computation: for each triple $(q \in Q, \sigma \in \tilde{\Sigma}, z \in \{Z, NZ\})$,

$$\sum_{q' \in Q, c \in \{-m, \dots, m\}} \delta(q, \sigma, z, q', c) = 1.$$

The computation is terminated after reading the whole input ($\#w\#$) and the automaton accepts (resp., rejects) the input if the final state is in Q_a (resp., $Q \setminus Q_a$). Then, for each input, the accepting (resp., rejecting) probability can be calculated by summing up the probabilities of all the accepting (resp., rejecting) paths.

A one-way probabilistic blind one-counter automaton (1P1BCA) is a 1P1CA such that it cannot see the status of the counter during the computation and the input is automatically rejected if the value of the counter is non-zero [Gre78]. A 1P1BCA is a 5-tuple

$$M = (Q, \Sigma, \delta, q_0, Q_a),$$

where $\delta : Q \times \Sigma \times Q \times \{-m, \dots, m\} \rightarrow [0, 1]$ is a transition function such that $\delta(q, \sigma, q', c) = p$ means that the transition from $q \in Q$ to $q' \in Q$ increasing the counter value by $c \in \{-m, \dots, m\}$ occurs with probability $p \in [0, 1]$ if the scanned symbol is $\sigma \in \tilde{\Sigma}$. As described above, the transition function must satisfy the following condition: for each pair $(q \in Q, \sigma \in \tilde{\Sigma})$,

$$\sum_{q' \in Q, c \in \{-m, \dots, m\}} \delta(q, \sigma, q', c) = 1.$$

The computation is terminated after reading the whole input ($\#w\#$) and the automaton accepts the input if the counter value is zero and the state is in Q_a , otherwise it rejects the input. The accepting and rejecting probabilities for a given input are calculated as described above.

A configuration of a counter automaton (regardless of whether blind or not) is a pair (q, v) of the current state and the current counter value. Here we do not consider the head position. In our proofs, this will not lead to any confusion.

For each of the above two models, we can define its deterministic version, where the range of the transition function is restricted to $\{0, 1\}$. We abbreviate them respectively as 1D1CA and 1D1BCA.

Moreover, a one-way nondeterministic blind one-counter automaton (1N1BCA) can be defined as a 1P1BCA with a special acceptance mode such that it accepts an input if the accepting probability is non-zero and it rejects the input if the accepting probability is zero. Here, each probabilistic choice (the

probabilities are insignificant and can be removed) is called as a nondeterministic choice. Then, an input is accepted if and only if there is a path reaching an accepting condition.

Similarly, we can define a one-way universal blind one-counter automaton (1U1BCA), where the automaton accepts the input if the accepting probability is 1 and it rejects the input if the accepting probability is less than 1. In this case, each probabilistic choice (the probabilities are insignificant and can be removed) is called as a universal choice. Then, an input is accepted if and only if every computational path reaches an accepting condition.

A Las Vegas probabilistic machine is a probabilistic machine that (i) never gives a wrong answer but can give a third type of decision called “don’t know” besides “accepting” and “rejection” and (ii) both of accepting and rejecting probabilities cannot be non-zero for the same input. For one-way Las Vegas automaton model, we split the set of states into three disjoint sets: the accepting, the rejecting, and neutral states. The automaton says “don’t know” when it finishes its computation in a neutral state.

Since quantum computation is a generalization of probabilistic computation [Wat09], any quantum model is expected to simulate its classical counterpart exactly. However, the earlier quantum finite automata (QFAs) models (e.g. [KW97, MC00]) were defined in a restrictive way and they do not reflect the full power of quantum computation. Even though they were shown to be more powerful than their classical counterparts in some special cases, these QFAs models cannot simulate classical finite automata. The first quantum counter automata model was defined based on these restricted models [Kra99], and so, they were also shown not to be able to simulate its classical counterparts [YKI05]. Nowadays, we know how to define general quantum automata models that generalize probabilistic automata [Hir10, YS11]. Therefore, even a superiority result of a restricted model, as given in this paper, serves as a separation between the quantum and probabilistic model. Due to its simplicity, we give the definition of a restricted model that allows to represent our algorithm and we refer the reader to [SY12] for the definition of general quantum model. We assume the reader familiar with basics of quantum computation. We refer the reader to [NC00] for a complete reference on quantum computation, to [SY14] for a short introduction on QFAs, and to [AYar] for a comprehensive survey on QFAs.

A one-way quantum one-counter automaton (1Q1CA) is a 5-tuple

$$M = (Q, \Sigma, \delta, q_0, Q_a),$$

where $\delta : Q \times \Sigma \times \{Z, NZ\} \times Q \times \{-m, \dots, m\} \rightarrow \mathbb{C}$ is a transition function; $\delta(q, \sigma, z, q', c) = p$ means that the transition from q to q' increasing the counter value by c occurs with probability amplitude p if the scanned symbol is σ and the status of the counter value is z .

$|q, v\rangle$ (resp., $\langle q, v|$), called a ket (resp., bra), denotes the column (resp., row) vector where the entry corresponding to (q, v) is one and the remaining entries are zeros. That is, $\{|q, v\rangle\}$ is an orthonormal basis of $l_2(Q \times \mathbb{Z})$. For each $\sigma \in \tilde{\Sigma}$, we define a time evolution operator U_σ as follows:

$$U_\sigma |q, v\rangle = \sum_{(q', c) \in Q \times \{-m, \dots, m\}} \delta(q, \sigma, z(v), q', c) |q', v + c\rangle,$$

where $z(v) = Z$ (resp., $z(v) = NZ$) if $v = 0$ (resp., $v \neq 0$). In order to be a well-formed automaton, U_σ must be unitary. The computation of a 1Q1CA is described by $|\Psi\rangle = U_{\tilde{w}(|\tilde{w}|)} U_{\tilde{w}(|\tilde{w}|-1)} \cdots U_{\tilde{w}(1)} |q_0, 0\rangle$. The following projective measurement P is applied to $|\Psi\rangle$ at the end of the computation:

$$P = \{P_a = \sum_{q \in Q_a, v \in \mathbb{Z}} |q, v\rangle \langle q, v|, P_r = \sum_{q \notin Q_a, v \in \mathbb{Z}} |q, v\rangle \langle q, v|\}.$$

Then, we have “ a ” (resp., “ r ”) with probability $\langle \Psi | P_a | \Psi \rangle$ (resp., $\langle \Psi | P_r | \Psi \rangle$). The automaton accepts (resp., rejects) the input if we have “ a ” (resp., “ r ”) as the outcome.

3 New separation results on promise problems

We start with the separation of exact quantum model from deterministic one and then we give the separation of Las Vegas probabilistic model from deterministic one.

3.1 Separation of exact 1Q1CAs and 1D1CAs

We show that there exists a promise problem that can be solved by 1Q1CAs exactly but not by any 1D1CAs. For our purpose, we evaluate XOR value of two comparisons. Let a, b, c , and d be four even positive numbers. Our first comparison is whether $a = c$ and the second one is whether $b = d$, and, our aim is to decide whether

$$((a = c) \text{ XOR } (b = d))$$

is true or false. Remark that this expression takes the value of true if and only if exactly one of the comparisons fails.

In order to implement this decision procedure by 1Q1CAs, we give the numbers as $0^a \# 0^b \# 0^c \# 0^d$. However, due to some technical difficulties, we also append four more numbers as $\# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2}$, which will help the automaton to set the counter to zero at the end of the computation so that an appropriate quantum interference can be done between the different configurations, i.e., two configurations having different counter values do not interfere.

Formally, we define our promise problem as follows. Let XOR-EQ be the set of strings of the form $0^a \# 0^b \# 0^c \# 0^d \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2}$ such that a, b, c , and d are even and satisfy the following:

$$a - c + (-1)^{\delta_{a,c}}(k_1 - k_2) = b - d + (-1)^{\delta_{b,d}}(l_1 - l_2),$$

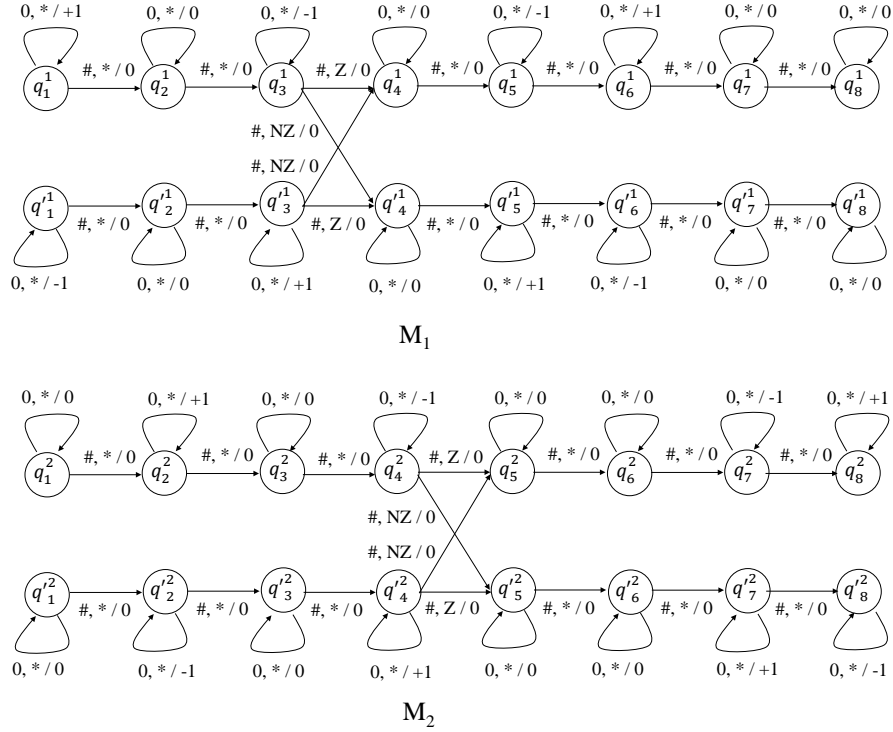
where $\delta_{u,v} = 1$ if $u = v$, and $\delta_{u,v} = 0$ otherwise. Then, the set XOR-EQ is our promise. We define yes-instances (XOR-EQ_{yes}) as the set of strings in XOR-EQ such that $((a = c) \text{ xor } (b = d))$ takes the value of true. Then, no-instances (XOR-EQ_{no}) are the ones taking the value of false, or equivalently XOR-EQ \setminus XOR-EQ_{yes}.

Theorem 1. *The promise problem XOR-EQ can be solved by 1Q1CAs exactly.*

Proof: We can construct a one-way deterministic reversible one-counter automaton M_1 , which is a special case of the 1Q1CA model,^(v) that decides whether $a = c$ as follows.

1. M_1 reads the first block 0^a and increases the counter by one at each transition.
2. M_1 skips the second block 0^b .
3. M_1 reads the third block 0^c and decreases the counter by one at each transition. At the end of this block, M_1 decides whether $a = c$ or not.
4. M_1 skips the fourth block 0^d .

^(v) A classical reversible operation defined on the set of configurations is a unitary operator containing only 0s and 1s.


Fig. 1: Subautomata M_1 and M_2

5. M_1 reads the fifth block 0^{k_1} and increases the counter by one if $a \neq c$ (decreases the counter by one if $a = c$) at each transition.
6. M_1 reads the sixth block 0^{k_2} and decreases the counter by one if $a \neq c$ (increases the counter by one if $a = c$) at each transition.
7. M_1 skips the seventh and the eighth blocks.

Similarly, we can construct a 1Q1CA M_2 that decides whether $b = d$ by comparing b with d using the counter and then the counter is set to zero after reading 0^{l_1} and 0^{l_2} . We illustrate M_1 and M_2 in Figure 1.

In the figure, each label of the edges is of the form $(\sigma, z/c)$, where $\sigma \in \Sigma$, $z \in \{Z, NZ\}$, and $c \in \{-1, 0, +1\}$. A label $(\sigma, z/c)$ means that the transition occurs when the input symbol is σ and the status of the counter value is z ($*$ denotes a wild card which matches any of Z and NZ), and the counter value is updated by $c \in \{-1, 0, +1\}$. The initial state is q_1^1/q_1^2 for M_1/M_2 , respectively. The set of accepting states is $\{q_8^1\}/\{q_8^2\}$ for M_1/M_2 , respectively. Also the set of rejecting states is $\{q_8^1\}/\{q_8^2\}$ for M_1/M_2 , respectively. It is easy to see that if we set the initial state to q_1^1 for M_1 (q_1^2 for M_2), the output is inverted.

We use the algorithm in [CEMM98] (the improved version of Deutsch-Jozsa algorithm [DJ92]) to compute the exclusive-or exactly using the two sub-automata as the oracle for Deutsch's problem [Deu85].

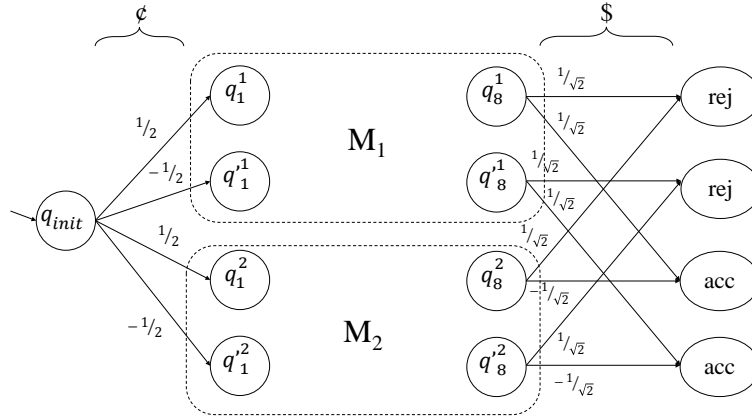


Fig. 2: Simulation of the Deutsch-Jozsa algorithm

Note that the counter values are the same between M_1 and M_2 at the moment of reading the last input symbol. Thus, we can construct a 1QICA that solves XOR-EQ by simulating the improved Deutsch-Jozsa algorithm [CEMM98] on it by running M_1 and M_2 in a superposition, which is illustrated in Figure 2. In the figure, the value on each edge represents the amplitude associated with the transition. The first and the last transitions occur when it reads the left and the right end-markers, respectively. It is straightforward to see that the time evolution operators can be extended to unitary operators by adding dummy states and/or transitions. \square

Theorem 2. *No 1DICA can solve XOR-EQ.*

Proof: We assume that there exists a 1DICA M that solves XOR-EQ. Note that M can have at most $O(n)$ possible configurations for a string whose length is less than n , i.e., a constant number of possible states with $O(n)$ possible counter values. Also note that there are $\Theta(n^2)$ possible partial inputs of the form $0^a \# 0^b \#$ whose length is less than n . Thus, there exist two distinct partial inputs

$$0^a \# 0^b \# \text{ and } 0^{a'} \# 0^{b'} \#$$

such that the configurations after reading them are the same. We will show that there exists a suffix string,

$$0^c \# 0^d \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2},$$

such that either

$$u_1 = 0^a \# 0^b \# 0^c \# 0^d \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2}$$

is a yes-instance and

$$u_2 = 0^{a'} \# 0^{b'} \# 0^c \# 0^d \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2}$$

is a no-instance, or, vice versa. However, M cannot distinguish u_1 and u_2 since the two configurations after reading $0^a \# 0^b \#$ and $0^{a'} \# 0^{b'} \#$, respectively, are the same. This is a contradiction.

Now, we show how to obtain u_1 and u_2 as desired.

We start with the case of $a \neq a'$. We set l_1 and l_2 to some values providing that

$$d = \frac{b + b' + a - a'}{2} + (l_1 - l_2)$$

is even (this is possible since a, b, a' , and b' are even) and $b \neq d$ and $b' \neq d$. Then, we set k_1 and k_2 to such values providing that

$$-(k_1 - k_2) = b - d + (l_1 - l_2).$$

Thus, both u_1 and u_2 , i.e.

$$u_1 = 0^a \# 0^b \# 0^a \# 0^d \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2}$$

and

$$u_2 = 0^{a'} \# 0^{b'} \# 0^a \# 0^d \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2},$$

become promised input strings since

$$-(k_1 - k_2) = b - d + (l_1 - l_2) \text{ and } a' - a + (k_1 - k_2) = b' - d + (l_1 - l_2).$$

In this setting, the former one is a yes-instance and the latter one is a no-instance.

In the following, we show how to obtain the desired u_1 and u_2 when $a = a'$. Note that, in this case, $b \neq b'$.

We set k_1 and k_2 to some values providing that

$$c = \frac{a + a' + b - b'}{2} + (k_1 - k_2)$$

is even (this is possible since a, b, a' , and b' are even) and $a \neq c$ and $a' \neq c$. Then, we set l_1 and l_2 providing that

$$a - c + (k_1 - k_2) = -(l_1 - l_2).$$

Thus, both u_1 and u_2 , i.e.,

$$u_1 = 0^a \# 0^b \# 0^c \# 0^b \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2}$$

and

$$u_2 = 0^{a'} \# 0^{b'} \# 0^c \# 0^b \# 0^{k_1} \# 0^{k_2} \# 0^{l_1} \# 0^{l_2},$$

become promised input strings since

$$a - c + (k_1 - k_2) = -(l_1 - l_2) \text{ and } a' - c + (k_1 - k_2) = b' - b + (l_1 - l_2).$$

In this setting, again the former one is a yes-instance and the latter one is a no-instance. \square

3.2 Separation of Las Vegas 1P1CAs and 1D1CAs

We show that there exists a promise problem that Las Vegas 1P1CAs can solve but 1D1CAs cannot. Our idea is inspired from [RY14].

Let $|w|_a$ be the number of occurrences of the symbol a in the string w . We define the sets ONE and NONE as follows. All strings from $\text{ONE} \cup \text{NONE}$ have the form uy , where $u \in \{a, b, c\}^*$, $y \in \{d\}^*$, and $|y| \geq |u|$. Moreover, for any string $uy \in \text{ONE}$, the number of symbols is equal for exactly one pair of (a, b) , (b, c) , or (c, a) , i.e., $|u|_\alpha = |u|_\beta$ for exactly one pair $(\alpha, \beta) \in \{(a, b), (b, c), (c, a)\}$. Also, for any string $uy \in \text{NONE}$, the number of symbols is equal for none of the pair of (a, b) , (b, c) , or (c, a) , i.e., $|u|_\alpha \neq |u|_\beta$ for any pair $(\alpha, \beta) \in \{(a, b), (b, c), (c, a)\}$.

We define a promise problem ONE-NONE, where $\text{ONE-NONE}_{\text{yes}}$ (composed by yes instances) is formed by the concatenation $\text{ONE} \cdot \text{NONE}$ and $\text{ONE-NONE}_{\text{no}}$ (composed by no instances) is formed by the concatenation $\text{NONE} \cdot \text{ONE}$.

Theorem 3. *The promise problem ONE-NONE can be solved by a Las Vegas 1P1CA P with success probability $p = \frac{1}{3}$.*

Proof: Let $u_1 y_1 u_2 y_2$ be a promised input, where $u_1 \subseteq \{a, b, c\}^*$, $y_1 \subseteq \{d\}^*$, $u_2 \subseteq \{a, b, c\}^*$, and $y_2 \subseteq \{d\}^*$. The details of P are as follows. At the beginning, the computation splits into 3 different paths with equal probabilities and each path compares a pair $((a, b), (b, c), \text{ or } (c, a))$ in u_1 . If one of them is succeeded, the input is accepted in that path. All non-accepting paths set their counter to zero by reading y_1 , and then immediately each of them splits into three new different paths with equal probability. Each subpath compares a pair $((a, b), (b, c), \text{ or } (c, a))$ in u_2 . If one of them is succeeded, the input is rejected. Otherwise, P says “don’t know”.

If the input is a yes instance, then the numbers of symbols are equal only for a single pair of u_1 . Then the input is accepted with probability $\frac{1}{3}$ in one of the first three paths, and the computation ends in a neutral state in all the other cases. Similarly, if the input is a no instance, then it is rejected with probability $\frac{1}{3}$ and the automaton says “don’t know” with probability $\frac{2}{3}$. \square

To get a better error bound, we can use the promise problem $\text{ONE-NONE}(t)$, where yes-instances ($\text{ONE-NONE}_{\text{yes}}(t)$) are formed by $(\text{ONE-NONE}_{\text{yes}})^t$ and no-instances ($\text{ONE-NONE}_{\text{no}}(t)$) are formed by $(\text{ONE-NONE}_{\text{no}})^t$. That is, the error bound can be reduced to $(\frac{2}{3})^t$ for 1P1CAs, where $t > 1$.

Theorem 4. *The promise problem $\text{ONE-NONE}(t)$ can be solved by a Las Vegas 1P1CA P with success probability $p = 1 - (\frac{2}{3})^t$.*

Proof: The details of P are the following. Let $w = w_1 \cdots w_{2t}$ be a promised input, where for all $i = 1, \dots, t$, either $w_{2i-1} \in \text{ONE}$ and $w_{2i} \in \text{NONE}$ or $w_{2i-1} \in \text{NONE}$ and $w_{2i} \in \text{ONE}$. For each part $w_{2i-1} w_{2i} = u_{2i-1} y_{2i-1} u_{2i} y_{2i}$ ($i = 1, \dots, t$) of the input, the automaton applies the same strategy as in the previous theorem, where $u_{2i-1} \subseteq \{a, b, c\}^*$, $y_{2i-1} \subseteq \{d\}^*$, $u_{2i} \subseteq \{a, b, c\}^*$, and $y_{2i} \subseteq \{d\}^*$. If the automaton comes to decision “don’t know”, it continues with the next pair until the end. \square

Now, we show that neither ONE-NONE nor $\text{ONE-NONE}(t)$ can be solved by 1D1CAs. We start with the proof for ONE-NONE, which forms the base for the proof of $\text{ONE-NONE}(t)$.

Theorem 5. *There is no 1D1CA that can solve promise problem ONE-NONE.*

Proof: Let us prove by contradiction and assume that there exists a 1DICA M that solves ONE-NONE. We call two promised inputs w and w' non-equivalent with respect to ONE-NONE if either $w \in \text{ONE-NONE}_{\text{yes}}$ and $w' \in \text{ONE-NONE}_{\text{no}}$ or $w \in \text{ONE-NONE}_{\text{no}}$ and $w' \in \text{ONE-NONE}_{\text{yes}}$.

We will have a contradiction with our assumption if we show that there exist two strings w and w' that are non-equivalent with respect to ONE-NONE such that M finishes reading w and w' with the same state and with the same status of counter.

In the proof, we use the following notation. We denote by $c(w)$, $q(w)$, and $v(w)$ the configuration, the state, and the value of the counter of M after reading the partial input w , respectively, and by σ an arbitrary symbol of input alphabet. If the automaton reaches the configuration c' from the configuration c when reading w , we denote it $c \xrightarrow{w} c'$. Let m be the maximum value by which M may change the value of counter in one step.

Lemma 1. *Let $c(w) = (q, v)$ be the configuration after reading a string w . If M starts from $c(w)$ and, for $n \geq |Q|$, all $v(w), v(w\sigma), v(w\sigma^2), \dots, v(w\sigma^n)$ are non-zero, then the following is true:*

1. *there exist n_1 and n_2 ($0 \leq n_1 < n_2 \leq n$ and $n_1 < |Q|$) such that $q(w\sigma^{n_1}) = q(w\sigma^{n_2})$;*
2. *there exist numbers t and r ($0 < t \leq |Q|$ and $0 \leq |r| \leq m \cdot |Q|$) such that M moves cyclically through some states q_{i_1}, \dots, q_{i_t} returning to the same state after every t steps, and the value of the counter is changed by the same number r after every t steps as long as the value of the counter is not zero.*

Proof: Both statements follow from the Pigeon-hole principle. If the status of counter is the same, then M , reading only σ 's, is simply a unary automaton and so it always enters a cycle of states after reading more than $|Q|$ symbols. Thus the first statement is immediate.

We pick the smallest n_1 and n_2 ($0 \leq n_1 < n_2 \leq n$) such that all $q(w), q(w\sigma), \dots, q(w\sigma^{n_1}), \dots, q(w\sigma^{n_2-1})$ are different and $q(w\sigma^{n_1}) = q(w\sigma^{n_2})$. Then the number $t = n_2 - n_1$ is the period of cycle (M moves cyclically from one state to the next reading σ and returns to the same state after every t steps as long as the value of the counter is non-zero). Since after each t steps M is in the same state, the counter is changed by the same value after every t steps as long as the value of the counter is not zero. \square

Let $c = (q, v)$ be the configuration as given in the lemma. Now we focus on a computation on M reading only σ 's before the counter hits zero. We call t and r the period and the difference of the cycle, respectively. Without loss of generality, we assume that $v > 0$. The set Q of states can be divided into disjoint subsets $Q_1^\sigma, \dots, Q_k^\sigma$ ($Q_1^\sigma \cup \dots \cup Q_k^\sigma = Q$), where two states q and q' belong to the same subset Q_j^σ iff M moves from q and q' to the same cycle reading σ 's. We call such partition $\mathcal{Q}^\sigma = \{Q_1^\sigma, \dots, Q_k^\sigma\}$ of the set Q as σ -partition. From Lemma 1, we have that each cycle (and hence each subset from σ -partition) has two characteristics: its period t and its difference r .

Let $c_1 = (q_1, v_1)$ and $c_2 = (q_2, v_2)$ be two different configurations. We will say that c_1 and c_2 are σ -synchronized if there exists some configuration c and the numbers n_1 and $n_2 \geq 0$ such that $c_1 \xrightarrow{\sigma^{n_1}} c$ and $c_2 \xrightarrow{\sigma^{n_2}} c$.

Lemma 2. *Let $c = (q, v)$ and $c' = (q, v')$ be two different configurations with the same state such that v and v' have the same sign, $|v|, |v'| > m \cdot |Q|$ (m is the maximum value by which the counter can be increased during one step), and $|v - v'|$ is a multiple of r_σ , where r_σ is the difference of the subset from σ -partition \mathcal{Q}^σ that contains q . Then c and c' are σ -synchronized.*

Proof: Let Q_j^σ be the subset from σ -partition Q^σ that contains q . From the definition of σ -partition, it follows that if M starts its computation in q and reads only σ 's, then it enters its cycle in k ($k \leq |Q|$) steps and stays there until the counter hits zero. Let t_σ and r_σ be the period and the difference of this cycle respectively, q' be the first visited state in the cycle after the first k steps, and r' be the value added to the counter in these k steps. Then we have $c \xrightarrow{\sigma^k} (q', v + r')$, $c' \xrightarrow{\sigma^k} (q', v' + r')$, and $|v - v'|$ is a multiple of r_σ .

After reading σ^k , M will return to the state q' and will increase the counter by the same value r_σ after every t_σ steps as long as the value of the counter is not zero. It is clear that there exists some non-negative integer l such that either $c \xrightarrow{\sigma^{k+t_\sigma \cdot l}} (q', v + r' + l \cdot r_\sigma) = (q', v' + r')$ or $c' \xrightarrow{\sigma^{k+t_\sigma \cdot l}} (q', v' + r' + l \cdot r_\sigma) = (q', v + r')$. Thus, c and c' are σ -synchronized. \square

Lemma 3. *There exists at least one pair of strings (u_1, u_2) such that $u_1 \in \text{ONE}$, $u_2 \in \text{NONE}$, and*

- (A) $c(u_1) = c(u_2)$ or
- (B) $c(u_1) = (q(u_1), v(u_1)) \neq c(u_2) = (q(u_2), v(u_2))$ but $q(u_1) = q(u_2)$, $|v(u_1)|, |v(u_2)| \in \omega(n)$, where n is a sufficiently long length.

Proof: We have two cases.

Case 1: There exists a symbol $\sigma \in \{a, b, c\}$ such that $|v(\sigma^n)| \in O(1)$ holds. Without loss of generality, we pick $\sigma = a$. Then, for all inputs a^n , we have a constant number of all possible configurations, since the number of states is constant and the possible different values of counter is bounded by $O(1)$. So there exist n_1 and n_2 ($n_1 < n_2$) such that $c(a^{n_1}) = c(a^{n_2})$.

We take $u = b^{n_1} d^{n_1+n_2}$. It is clear that $c(a^{n_1}u) = c(a^{n_2}u)$ therefore $q(a^{n_1}u) = q(a^{n_2}u)$ and $v(a^{n_1}u) = v(a^{n_2}u)$. But $a^{n_1}u \in \text{ONE}$ and $a^{n_2}u \in \text{NONE}$.

Case 2: For every symbol $\sigma \in \{a, b, c\}$, $|v(\sigma^n)| \in \omega(1)$ holds. We will construct u_1 and u_2 in four steps and in each step we define a part of them, i.e.,

$$u_1 = x_1 x_2 x_3 x_4 \text{ and } u_2 = y_1 y_2 y_3 y_4,$$

where $x_1, y_1 \in \{a\}^*$, $x_2, y_2 \in \{b\}^*$, $x_3, y_3 \in \{c\}^*$, and $x_4, y_4 \in \{d\}^*$.

Step 1. We pick $r = r(n)$ such that $|v(a^r)| \in \omega(n)$ and we set $x_1 = y_1 = a^r$. Since $|v(a^n)| \in \omega(1)$, we can always find such an r depending on n . Moreover, at each step of a computation, M can increase or decrease the value of the counter by constant amount, and so, for any string z ($|z| \in O(n)$), we always have $|v(a^r z)| \in \omega(n)$.

Step 2. For $k > |Q|$, we consider the following sequence of states $q(a^r b)$, $q(a^r b^2)$, \dots , $q(a^r b^k)$. Then, there must exist two distinct non-negative integers k_1 and k_2 ($k_1 < k_2 < k$ and $k_1 \leq |Q|$) such that all $q(a^r b), \dots, q(a^r b^{k_1}), \dots, q(a^r b^{k_2-1})$ are different and $q(a^r b^{k_2}) = q(a^r b^{k_1})$. By Lemma 1, the number $t_b = k_2 - k_1$ is the period of cycle and r_b is the difference of cycle.

We set $x_2 = b^{k_1}$ and $y_2 = b^{k_2}$. Let $N_a = v(a^r)$, $N_{b_1} = v(a^r b^{k_1}) - v(a^r)$, and $N_{b_2} = v(a^r b^{k_2}) - v(a^r)$. Then we have

$$\begin{aligned} q(a^r b^{k_1}) &= q(a^r b^{k_2}), \\ v(a^r b^{k_1}) &= N_a + N_{b_1}, \\ v(a^r b^{k_2}) &= N_a + N_{b_2}, \text{ and} \\ N_{b_2} - N_{b_1} &= r_b. \end{aligned}$$

Step 3. We set $x_3 = y_3 = c^{k_1}$. Let $N_c = v(a^r b^{k_1} c^{k_1}) - v(a^r b^{k_1})$. Then, we have the followings:

$$\begin{aligned} q(a^r b^{k_1} c^{k_1}) &= q(a^r b^{k_2} c^{k_1}), \\ v(a^r b^{k_1} c^{k_1}) &= N_a + N_{b_1} + N_c, \\ v(a^r b^{k_2} c^{k_1}) &= N_a + N_{b_2} + N_c, \text{ and} \\ v(a^r b^{k_2} c^{k_1}) - v(a^r b^{k_1} c^{k_1}) &= r_b. \end{aligned} \tag{1}$$

Step 4. Let Q_j^d be the subset from d -partition \mathcal{Q}^d that contains $q(x_1x_2x_3) = q(y_1y_2y_3)$ and then t_d and r_d be the period and the difference of the cycle, respectively. Depending on the values r_b (from the step 2) and r_d , we can have different cases. We denote $x_1x_2x_3$ by x_{123} and $y_1y_2y_3$ by y_{123} .

The case of $r_b = 0$: $v(x_{123}) = v(y_{123})$ and so $c(x_{123}) = c(y_{123})$. We set $x_4 = y_4 = d^l$ such that $|x_4| \geq |y_{123}| > |x_{123}|$. Then $c(x_{123}x_4) = c(y_{123}y_4)$ but $x_1x_2x_3x_4 \in \text{ONE}$ and $y_1y_2y_3y_4 \in \text{NONE}$.

The case of $r_b \neq 0$ and $r_d \neq 0$: If $|v(x_{123}) - v(y_{123})| = r_b$ is a multiple of r_d , then due to Lemma 2, we can conclude that $c(x_{123})$ and $c(y_{123})$ are d -synchronized. We set $x_4 = d^{l_1+l}$ and $y_4 = d^{l_2+l}$, where l_1 and l_2 are the numbers of steps that are needed to synchronize the configurations $c(x_{123})$ and $c(y_{123})$ respectively, and l is the value providing that $|x_4| \geq |x_{123}|$ and $|y_4| \geq |y_{123}|$. Thus, we can follow that $c(x_{123}x_4) = c(y_{123}y_4)$ but $x_1x_2x_3x_4 \in \text{ONE}$ and $y_1y_2y_3y_4 \in \text{NONE}$.

If r_b is not multiple of r_d , then we can re-define y_2 as $b^{k_1+t_b \cdot r_d}$ by setting $k_2 = k_1 + t_b \cdot r_d$. Then, Equations 1 can be rewritten as

$$\begin{aligned} v(a^r b^{k_1} c^{k_1}) &= N_a + N_{b_1} + N_c, \\ v(a^r b^{k_2} c^{k_1}) &= N_a + N_{b_1} + r_b \cdot r_d + N_c, \\ v(a^r b^{k_2} c^{k_1}) - v(a^r b^{k_1} c^{k_1}) &= r_b \cdot r_d. \end{aligned}$$

This update on y_2 concludes that $c(x_{123})$ and $c(y_{123})$ are d -synchronized as described above, and so $c(x_{123}x_4) = c(y_{123}y_4)$ but $x_1x_2x_3x_4 \in \text{ONE}$ and $y_1y_2y_3y_4 \in \text{NONE}$.

The case of $r_b \neq 0$ and $r_d = 0$: We set $x_4 = y_4 = d^{l_1+l}$ such that l_1 is the minimum numbers of steps that is sufficient to enter the cycle and l is the minimum value providing that $|y_4| \geq |y_{123}| > |x_{123}|$. Thus, we can follow that $q(x_{123}x_4) = q(y_{123}y_4)$ and $|v(x_{123}x_4)| \in \omega(n)$ and $|v(y_{123}y_4)| \in \omega(n)$, but $x_1x_2x_3x_4 \in \text{ONE}$ and $y_1y_2y_3y_4 \in \text{NONE}$. \square

Now, we construct the pair of strings w and w' that are non-equivalent with respect to ONE-NONE. Due to Lemma 3, there exist two strings u_1 and u_2 such that $u_1 \in \text{ONE}$ and $u_2 \in \text{NONE}$, and

- (A) $c(u_1) = c(u_2)$ or
- (B) $c(u_1) \neq c(u_2)$ but $q(u_1) = q(u_2)$ and $|v(u_1)|, |v(u_2)| \in \omega(n)$.

For simplicity, we call the pair u_1, u_2 as A-type if it satisfies Condition A and we call it as B-type if the Condition B is satisfied.

If the pair u_1 and u_2 is A-type, then, by assuming $c(u_1)$ is the initial configuration, we can construct two new strings u'_1 and u'_2 as described above such that $u'_1 \in \text{NONE}$ and $u'_2 \in \text{ONE}$, and then, the pair $w = u_1u'_1$ and $w' = u_2u'_2$ is either A-type or B-type. Thus, M gives the same decisions for w and w' but $w \in \text{ONE-NONE}_{\text{yes}}$ and $w' \in \text{ONE-NONE}_{\text{no}}$.

If the pair u_1 and u_2 is B-type, then we can define u'_1 and u'_2 as follows. Since the value of counter is superlinear in n , there exist two minimal non-negative integers k_1 and k_2 such that $k_1 < k_2$, $k_1 \leq |Q|$ and

$q(u_1 a^{k_1}) = q(u_2 a^{k_2})$. We set $u'_1 = a^{k_2} b^{k_1} d^{k_1+k_2}$ and $u'_2 = a^{k_1} b^{k_1} d^{k_1+k_2}$. It is clear that (1) $u'_1 \in \text{NONE}$ and $u'_2 \in \text{ONE}$, and (2) $q(u_1 u'_1) = q(u_2 u'_2)$ and the values of counter are superlinear in n ($\in w(n)$) for $w = u_1 u'_1$ and $w' = u_2 u'_2$. Thus, M gives the same decisions for w and w' but $w \in \text{ONE-NONE}_{\text{yes}}$ and $w' \in \text{ONE-NONE}_{\text{no}}$.

With the contradiction in each possible case, we can conclude that there is no 1DICA solving ONE-NONE. \square

We can easily extend the above impossibility proof for the promise problem ONE-NONE(τ) ($\tau > 1$).

Theorem 6. *There is no 1DICA that can solve promise problem ONE-NONE(τ) for any $\tau \in \mathbb{Z}^+$.*

Proof: Let M be a 1DICA solving ONE-NONE(τ). In the previous proof, when starting in some configuration, say $c(u)$ for some strings u , we show how to construct two different strings, say u_1 and u_2 , such that $u_1 \in \text{ONE}$, $u_2 \in \text{NONE}$, and M , after reading u_1 and u_2 , ends with either the same configuration ($c(uu_1) = c(uu_2)$) or with the same state ($q(uu_1) = q(uu_2)$) and with some values of counter that are superlinear for some sufficiently big n (i.e., $|v(uu_1)|, |v(uu_2)| \in \omega(n)$). We call the former pair as A-type and the latter pair as B-type. For B-type pairs, we assume here that the values of counter are not allowed to be less than a value quadratic in n and t , i.e., $|v(uu_1)|, |v(uu_2)| \in \Omega(t^2 n^2)$.

Based on these facts, we can construct the following two strings

$$w = w_1 w_2 \cdots w_t \text{ and } w' = w'_1 w'_2 \cdots w'_t$$

such that each $w_j \in \text{ONE-NONE}_{\text{yes}}$, each $w'_j \in \text{ONE-NONE}_{\text{no}}$ ($1 \leq j \leq t$), but M , after reading w and w' , finishes its computation in the same state and in the same status of counter. Thus, M gives the same decision for a yes-instance and for a no-instance, which leads us to conclude that M cannot solve ONE-NONE(τ).

We start from the initial configuration. Each w_j (w'_j) is composed by two strings $u_j y_j$ ($u'_j y'_j$) such that $u_j, y'_j \in \text{ONE}$ and $y_j, u'_j \in \text{NONE}$. For $j = 1, \dots, t$, we first construct u_j and u'_j , and then y_j and y'_j . If all pairs are A-type and then the construction is straightforward since M ends in the same configurations after reading A-type pairs and from the same configuration we can always construct two new pairs as desired.

If we obtain a B-type pairs at some point of the construction, we can define the remaining parts of w and w' as we do at the end of the previous proof. First, we can be sure that the values of counter are quadratic in t and n ($|v(uu_1)|, |v(uu_2)| \in \Omega(t^2 n^2)$). Then, the length each new obtained pair can be easily bounded by $O(n)$. That means the status of the counter will be the same for the remaining of the computation and so M behaves like a deterministic finite automaton. Thus, it is very easy to fool M when constructing the remaining pairs that requires equality checks. \square

4 New results on classical counter automata

In this section, we show the results that separate the expressive power of several models of blind/non-blind counter automata. For this purpose, we denote the class of languages recognized by a *Model* as $\mathcal{L}(\text{Model})$.

First of all, we present a 1P1BCA algorithm for the Kleene closure of equality language:

$$\text{EQ}^* = \{\varepsilon\} \cup \{a^{n_1} b^{n_1} \cdots a^{n_k} b^{n_k} \mid n_i > 0 (1 \leq i \leq k), k \geq 1\},$$

which was shown not to be recognized by any one-way deterministic finite automaton with multi blind counters [Gre78]. Recently, Yakaryılmaz presented a negative one-sided error 1Q1BCA algorithm for this

language and he conjectured that it cannot be recognized by 1P1BCAs [Yak12]. Now, we show that this conjecture is false. It is also surprising that our new algorithm is kind of a probabilistic adaptation of the quantum algorithm given by Yakaryılmaz.

Theorem 7. *The language EQ^* can be recognized by a 1P1BCA M with negative one-sided error bound $\frac{1}{3}$.*

Proof: We assume that the input is of the form $a^{n_1}b^{m_1} \dots a^{n_k}b^{m_k}$. Otherwise, M rejects the input deterministically (exactly). At the beginning of each block $a^{n_l}b^{m_l}$ ($1 \leq l \leq k$), M selects one of the following three paths (Path_i 's) with equal probability:

$$\text{Path}_i(1 \leq i \leq 3) : \quad M \text{ increases (resp., decreases) the counter by } i \\ \text{each time reading an } a \text{ (resp., } b \text{) of the block.}$$

The computation always ends in an accepting state (except the deterministic check mentioned at the beginning). Thus, the input is accepted if and only if the value of counter is zero. It is obvious that M accepts any member of EQ^* with certainty. We consider the case that the input $w \notin \text{EQ}^*$. Let i_{max} be the greatest index satisfying $n_{i_{max}} \neq m_{i_{max}}$, i.e., $a^{n_{i_{max}}}b^{m_{i_{max}}}$ is the last block satisfying $n_{i_{max}} \neq m_{i_{max}}$. Let path' be a probabilistic path before reading the i_{max} -th block having the counter value c . This path will split into three sub-paths $\text{subpath}'_1$, $\text{subpath}'_2$, and $\text{subpath}'_3$ and each subpath reads the block as described above. Let c_1 , c_2 , and c_3 be the counter values of these sub-paths, respectively, after reading the block. Any computation starts from $\text{subpath}'_i$ will have the same counter value of c_i at the end of the computation since the remaining blocks have the same numbers of a 's and b 's, where $1 \leq i \leq 3$. Assume that $\text{subpath}'_i$ leads to a decision of acceptance. This is possible only if $c_i = 0$. Let $d = n_{i_{max}} - m_{i_{max}} \neq 0$. Then the values of c_1 , c_2 , and c_3 are $c + d$, $c + 2d$, and $c + 3d$, respectively. Therefore, only one of them can be zero. That is, the maximum accepting probability that path' can contribute is $\frac{1}{3}$. This is the case also for all other probabilistic paths that exist just before reading the i_{max} -th block. Therefore the overall accepting path can be bounded by $\frac{1}{3}$. \square

It is clear from the analysis given in the proof that the error bound can be reduced to $\frac{1}{k}$ for any k by spiting into k probabilistic paths on each block instead of 3.

Corollary 1. *The language EQ^* can be recognized by a 1P1BCA M with any negative one-sided error bound $\epsilon \leq \frac{1}{2}$.*

Remark that since any language recognized by a 1P1BCA with negative one-sided error is recognized by 1U1BCA, we can also conclude that EQ^* can be recognized by 1U1BCAs.

Even though any number of blind counters is useless for a 1DFA (or a 1NFA) [Gre78], a single non-blind counter is enough in order to recognize EQ^* , i.e., 1D1CAs can recognize EQ^* . These facts together with Theorem 7 imply that $\mathcal{L}(1D1BCA) \subsetneq \mathcal{L}(1P1BCA) \cap \mathcal{L}(1D1CA)$. Another related result is that Freivalds [Fre79] proved that $\text{EQ}^3 = \{c^n d^n e^n \mid n \geq 0\}$ can be recognized by 1P1BCAs with arbitrary small negative one-sided error bound and this non-context free language, of course, cannot be recognized by a 1D1CA. We represent our result with the known facts in Figure 3. We still do not know whether bounded-error 1Q1BCAs are more powerful than bounded-error 1P1BCAs.

Our next result is on incomparability of 1U1BCAs and 1D1CAs. In order to show it, we consider the complement of EQ^* :

$$\overline{\text{EQ}^*} = \overline{\{a^{n_1}b^{n_1} \dots a^{n_m}b^{n_m} \mid m \geq 0\}}.$$

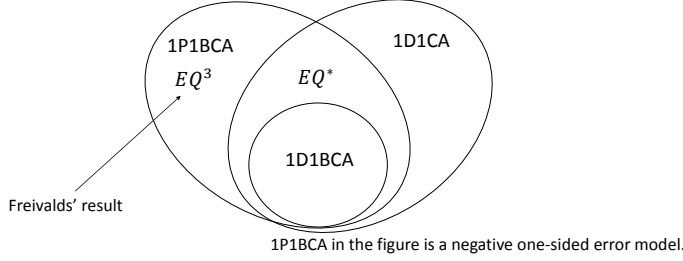


Fig. 3: Separation between $\mathcal{L}(1D1BCA)$ and $\mathcal{L}(1P1BCA) \cap \mathcal{L}(1D1CA)$

Theorem 8. *1U1BCAs cannot recognize $\overline{EQ^*}$.*

Proof: We assume that there exists a 1U1BCA M that recognizes $\overline{EQ^*}$. Let Q be the set of states of M . Let $w = a^{n_1}b^{n_1} \dots a^{n_m}b^{n_m}$ be a string not in $\overline{EQ^*}$ where $n_1 > |Q|$ ($w \in EQ^*$). Then, there exists a rejecting path for w , say p . We consider the computation along the path p .

Since $n_1 > |Q|$, there exists a state $s \in Q$ such that M enters s at least twice when reading a^{n_1} . We assume that M enters the state s just after reading a^t and $a^{t'}$ ($0 < t < t' < n_1$) in the first block. In other words, M enters the state s after the t -th step and the t' -th step. We divide the path p into three subpaths $p = p_1 \cdot p_2 \cdot p_3$ where p_2 starts from $(t+1)$ -th step and finishes at the t' -th step. Then both of $p' = p_1 \cdot p_2 \cdot p_2 \cdot p_3$ and $p'' = p_1 \cdot p_2 \cdot p_2 \cdot p_2 \cdot p_3$ are valid computation paths for input strings $w_1 = a^{n_1+(t'-t)}b^{n_1} \dots a^{n_m}b^{n_m}$ and $w_2 = a^{n_1+2(t'-t)}b^{n_1} \dots a^{n_m}b^{n_m}$, respectively. Note that $w_1, w_2 \in \overline{EQ^*}$. Also, note that both of p' and p'' lead to the same final state as p . Then, at least one of $\{p', p''\}$ has non-zero counter value or both of them has the same counter value as p at the end of the computation. This is because if the counter value increases by $d (\neq 0)$ along with p_2 , then the final counter values are different for p' and p'' . If $d = 0$, then the counter values are the same for p, p' , and p'' . Therefore, at least one of $\{p', p''\}$ is a rejecting path. (Remember that, by the definition of blind counter automata, computation that ends with a non-zero counter value is always rejected.) However, both of w_1 and w_2 are in $\overline{EQ^*}$. This is a contradiction. \square

Figure 4 summarizes the incomparability of 1U1BCAs and 1D1CAs. Note that 1D1CA can recognize $\overline{EQ^*}$ and a negative one-sided bounded-error 1P1BCA algorithm is also a 1U1BCA algorithm since all members accepted with probability 1.

Our last result is the separation between $\mathcal{L}(1P1CA)$ and $\mathcal{L}(1U1BCA) \cup \mathcal{L}(1D1CA)$. In order to show it, we define language L as follows:

$$L = \overline{EQ^*} \cup EQ^3 \text{ where } EQ^3 = \{c^n d^n e^n \mid n \geq 0\}.$$

Then, we have the following theorems.

Theorem 9. *L can be recognized by a 1P1CA with negative one-sided bounded error.*

Proof: Let $M_{\overline{EQ^*}}$ be a 1D1CA that recognizes $\overline{EQ^*}$. Also, let M_{EQ^3} be a 1P1CA that recognizes EQ^3 with negative one-sided bounded error. In order to recognize L , we use $M_{\overline{EQ^*}}$ and M_{EQ^3} as subautomata. Note that $\overline{EQ^*} \subset \{a, b\}^*$ and $EQ^3 \subset \{c, d, e\}^*$. Thus, the following 1P1CA M recognizes L :

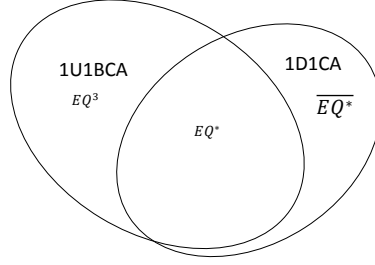
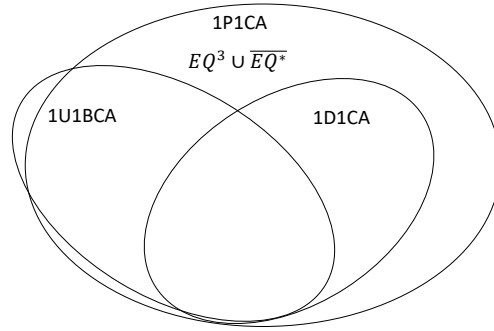


Fig. 4: Incomparability of 1U1BCAs and 1D1CAs



1P1CA in the figure is a negative one-sided error model.

Fig. 5: Separation between $\mathcal{L}(1P1CA)$ and $\mathcal{L}(1U1BCA) \cup \mathcal{L}(1D1CA)$

- if the input is an empty string, M accepts it.
- if the first symbol is a or b , M executes $M_{\overline{EQ^*}}$.
- otherwise, M executes M_{EQ^3} .

□

Theorem 10. *Neither 1D1CAs nor 1U1BCAs can recognize L .*

Proof: If there exists a 1D1CA that recognizes L , then it can be regarded as a 1D1CA that recognizes EQ^3 by ignoring transitions for the symbols a and b . This is a contradiction.

Similarly, if there exists a 1U1BCA that recognizes L , then it can be regarded as a 1U1BCA that recognizes $\overline{EQ^*}$ by ignoring transitions for the symbols c , d , and e . This is a contradiction. □

By Theorems 9 and 10, we can separate $\mathcal{L}(1P1CA)$ and $\mathcal{L}(1U1BCA) \cup \mathcal{L}(1D1CA)$ as illustrated in Figure 5.

As pointed before languages recognized by 1P1BCAs are also recognizable by 1U1BCAs. Thus, combining all the above results, we have the hierarchy of the models as illustrated in Figure 6.

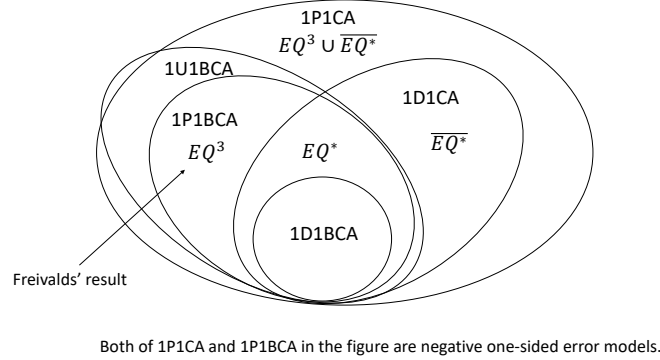


Fig. 6: Hierarchy of various models of counter automata

As a future work, we find interesting to identify whether there is an alternation hierarchy for one-way blind-counter automata with and without ε -moves.

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