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Monochromatic loose paths in multicolored k-uniform cliques

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For integers $k \geq 2$ and $\ell \geq 0$, a k-uniform hypergraph is called a *loose path of length* ℓ, and denoted by $P_\ell^{(k)}$, if it consists of ℓ edges $e_1, \ldots, e_\ell$ such that $|e_i \cap e_j| = 1$ if $|i-j| = 1$ and $e_i \cap e_j = \emptyset$ if $|i-j| \geq 2$. In other words, each pair of consecutive edges intersects on a single vertex, while all other pairs are disjoint. Let $R(P_\ell^{(k)}; r)$ be the minimum integer n such that every r-edge-coloring of the complete k-uniform hypergraph $K_n^{(k)}$ yields a monochromatic copy of $P_\ell^{(k)}$. In this paper we are mostly interested in *constructive* upper bounds on $R(P_\ell^{(k)}; r)$, meaning that on the cost of possibly enlarging the order of the complete hypergraph, we would like to efficiently find a monochromatic copy of $P_\ell^{(k)}$ in every coloring. In particular, we show that there is a constant $c > 0$ such that for all $k \geq 2$, $\ell \geq 3$, $2 \leq r \leq k-1$, and $n \geq k(\ell+1)r(1+\ln(r))$, there is an algorithm such that for every r-edge-coloring of the edges of $K_n^{(k)}$, it finds a monochromatic copy of $P_\ell^{(k)}$ in time at most cn^k. We also prove a non-constructive upper bound $R(P_\ell^{(k)}; r) \leq (k-1)\ell r$.

Keywords: multicolor Ramsey number, loose path, constructive bounds

1 Introduction

For positive integers $k \geq 2$ and $\ell \geq 0$, a k-uniform hypergraph is called a *loose path of length* ℓ, and denoted by $P_\ell^{(k)}$, if its vertex set is $\{v_1, v_2, \ldots, v_{(k-1)\ell+1}\}$ and the edge set is $\{e_i = \{v_{(i-1)(k-1)+q} : 1 \leq q \leq k\},\ i = 1, \ldots, \ell\}$, that is, for $\ell \geq 2$, each pair of consecutive edges intersects on a single vertex (see Figure 1), while for $\ell = 0$ and $\ell = 1$ it is, respectively, a single vertex and an edge. For $k = 2$ the loose path $P_\ell^{(2)}$ is just a (graph) path on $\ell + 1$ vertices.

Let H be a k-uniform hypergraph and $r \geq 2$ be an integer. The *multicolor Ramsey number* $R(H; r)$ is the minimum n such that every r-edge-coloring of the complete k-uniform hypergraph $K_n^{(k)}$ yields a monochromatic copy of H.

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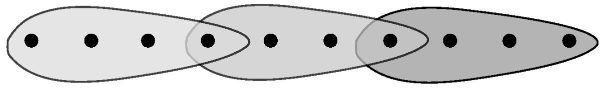

Fig. 1: A 4-uniform loose path $P_3^{(4)}$.

1.1 Known results for graphs

For graphs, determining the Ramsey number $R(P_\ell^{(2)}, r)$ is a well-known problem that attracted a lot of attention. It was shown by Gerencsér and Gyárfás (1967) that

$$R(P_\ell^{(2)}, 2) = \left\lfloor \frac{3\ell + 1}{2} \right\rfloor.$$

For three colors Figaj and Łuczak (2007) proved that $R(P_\ell^{(2)}, 3) \approx 2\ell$. Soon after, Gyárfás et al. (2007, 2008) determined this number exactly, showing that for all sufficiently large ℓ

$$R(P_\ell^{(2)}, 3) = \begin{cases} 2\ell + 1 & \text{for even } \ell, \\ 2\ell & \text{for odd } \ell, \end{cases} \tag{1}$$

as conjectured earlier by Faudree and Schelp (1975). For $r \geq 4$ much less is known. A celebrated Turán-type result of Erdős and Gallai (1959) implies that

$$R(P_\ell^{(2)}, r) \leq r\ell. \tag{2}$$

Recently, this was slightly improved by Sárközy (2016) and, subsequently, by Davies et al. (2017) who showed that for all sufficiently large ℓ,

$$R(P_\ell^{(2)}; r) \leq (r - 1/4)(\ell + 1). \tag{3}$$

1.2 Known results for hypergraphs

Let us first recall what is known about $R(P_\ell^{(k)}, r)$ for $k \geq 3$. For two colors, Gyárfás and Raeisi (2012) considered only paths of length $\ell = 2, 3, 4$ and proved that $R(P_2^{(k)}, 2) = 2k - 1$, $R(P_3^{(k)}, 2) = 3k - 1$, and $R(P_4^{(k)}, 2) = 4k - 2$. Later, for $k = 3$ or $k \geq 8$, and $\ell \geq 3$, Omidi and Shahsiah (2014, 2017) determined this number completely:

$$R(P_\ell^{(k)}, 2) = (k-1)\ell + \left\lfloor \frac{\ell + 1}{2} \right\rfloor,$$

and conjectured that the above formula is also valid for $k = 4, 5, 6, 7$.

For an arbitrary number of colors there are only few results and mainly for very short paths. The following is known.

For $\ell = 2$ and $k = 3$ (so called *bows*), Axenovich et al. (2014) determined the value of $R(P_2^{(3)}, r)$ for an infinite subsequence of integers r (including $2 \leq r \leq 10$) and for $r \to \infty$ they showed that $R(P_2^{(3)}, r) \approx \sqrt{6r}$.

For $k \geq 4$, and large r, the Ramsey number $R(P_2^{(k)}, r)$ can be easily upper bounded by a standard application of Turán numbers (by counting the average number of edges per color). Recall that for a given k-uniform hypergraph H, the *Turán number*, $\mathrm{ex}_k(n; H)$, is the maximum number of edges in an n vertex k-uniform hypergraph with no copy of H. It was proved by Frankl (1977) that $\mathrm{ex}_k(n; P_2^{(k)}) = \binom{n-2}{k-2}$ for n sufficiently large, from which it follows quickly that $R(P_2^{(k)}, r) \leq \sqrt{k(k-1)r}$.

For $\ell \geq 3$, a similar approach via Turán numbers $\mathrm{ex}_k(n; P_\ell^{(k)})$, determined for large n by Füredi et al. (2014), yields for large r,

$$R(P_\ell^{(k)}; r) \leq k\ell r/2, \tag{4}$$

and, slightly better for $\ell = 3$,

$$R(P_3^{(k)}; r) \leq kr. \tag{5}$$

In the smallest instance $k = \ell = 3$, owing to the validity of formula $\mathrm{ex}_3(n; P_3^{(3)}) = \binom{n-1}{2}$ for all $n \geq 8$ (see Jackowska et al. (2016)), the above bound holds for all $r \geq 3$:

$$R(P_3^{(3)}, r) \leq 3r. \tag{6}$$

Recently, Łuczak and Polcyn twice improved (6) significantly. First, in Łuczak and Polcyn (2017), they showed that $R(P_3^{(3)}, r) \leq 2r + O(\sqrt{r})$, then, in Łuczak and Polcyn (2018), they broke the barrier of $2r$ by proving the bound $R(P_3^{(3)}, r) < 1.98r$, both results for large r. This still seems to be far from the true value which is conjectured to be equal to $r + 6$, the current best lower bound. In a series of papers Jackowska, Polcyn, and Ruciński (Jackowska et al. (2016), Polcyn (2017), Polcyn and Ruciński (2017)) confirmed this conjecture for $r \leq 10$. Finally, for $\ell = 3$, k arbitrary, and r large, Łuczak et al. (2018) showed an upper bound $R(P_3^{(k)}, r) \leq 250r$ which is independent of k.

In the next section we show a general upper bound, obtained iteratively for all $k \geq 2$, starting from the Erdős-Gallai bound (2) $R(P_\ell^{(2)}, r) \leq r\ell$.

Theorem 1.1 *For all $k \geq 2$, $\ell \geq 3$, and $r \geq 2$ we have $R(P_\ell^{(k)}; r) \leq (k-1)\ell r$.*

Theorem 1.1 can be easily improved for $r \geq 3$ provided ℓ is large. Using (1) instead of (2), we obtain for three colors that

$$R(P_\ell^{(k)}; 3) \leq (3k-4)\ell,$$

and for $r \geq 4$, by (3),

$$R(P_\ell^{(k)}; r) \leq (k-1)\ell r - \ell/4.$$

On the other hand, for large r, the bound (4) is roughly twice better than the one in Theorem 1.1.

1.3 Constructive bounds

In this paper we are mostly interested in *constructive* bounds which means that on the cost of possibly enlarging the order of the complete hypergraph, we would like to efficiently find a monochromatic copy of a target hypergraph F in every coloring. Clearly, by examining all copies of F in $K_n^{(k)}$ for $n \geq R(F; r)$, we can always find a monochromatic one in time $O(n^{|V(F)|})$. Hence, we are interested in complexity not depending on F, preferably $O(n^k)$. Given a k-graph F, a constant $c > 0$ and integers r and n, we say that a property $\mathcal{R}(F, r, c, n)$ holds if there is an algorithm such that for every r-edge-coloring of the edges of $K_n^{(k)}$, it finds a monochromatic copy of F in time at most cn^k. For graphs, a constructive result of this type can be deduced from the proof of Lemma 3.5 in Dudek and Prałat (2017).

Theorem 1.2 (Dudek and Prałat (2017)) *There is a constant $c > 0$ such that for all $\ell \geq 3$, $r \geq 2$, and $n \geq 2^{r+1}\ell$, property $\mathcal{R}(P_\ell^{(2)}, r, c, n)$ holds.*

Our goal is to obtain similar constructive results for loose hyperpaths. In Section 2, we show that, by replacing the Erdős-Gallai bound (2) with the assumption on n given in Theorem 1.2, the proof of Theorem 1.1 can be easily adapted to yield a constructive result.

Theorem 1.3 *There is a constant $c > 0$ such that for all $k \geq 2$, $\ell \geq 3$, $r \geq 2$, and $n \geq 2^{r+1}\ell + (k-2)\ell r$, property $\mathcal{R}(P_\ell^{(k)}, r, c, n)$ holds.*

Our second constructive bound (valid only for $r \leq k$) utilizes a more sophisticated algorithm.

Theorem 1.4 *There is a constant $c > 0$ such that for all $k \geq 2$, $\ell \geq 3$, $2 \leq r \leq k$, and $n \geq k(\ell + 1)r\left(1 + \frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right)$, property $\mathcal{R}(P_\ell^{(k)}, r, c, n)$ holds. For $r = 2$, the bound on n can be improved to $n \geq (2k-2)\ell + k$.*

Note that for $r = 2$ the lower bound on n in Theorem 1.4 is very close to that in Theorem 1.1. For $r = k$ the bound in Theorem 1.4 assumes a simple form

$$n \geq k^2(\ell + 1)(2 + \ln(k-1)).$$

Furthermore, when $r \leq k-1$, one can show (see Claim 4.2) that

$$\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right) \leq \ln\left(1 + \frac{r-1}{k-r}\right)$$

yielding the following corollary.

Corollary 1.5 *There is a constant $c > 0$ such that for all $k \geq 3$, $\ell \geq 3$, $2 \leq r \leq k-1$, and $n \geq k(\ell+1)r\left(1 + \ln\left(1 + \frac{r-1}{k-r}\right)\right)$, property $\mathcal{R}(P_\ell^{(k)}, r, c, n)$ holds.*

We can further replace the lower bound on n in Corollary 1.5 by (slightly weaker but simpler)

$$n \geq k(\ell + 1)r(1 + \ln r).$$

Observe that in several instances the lower bound in Theorem 1.4 (and also in Corollary 1.5) is significantly better (that means smaller) than the one in Theorem 1.3 (for example for large k and $k/2 \leq r \leq k$). On the other hand, for some instances bounds in Theorems 1.3 and 1.4 are basically the same. For example, for fixed r, large k and $\ell \geq k$ the lower bound is $k\ell r + o(k\ell)$. This also matches the bound from Theorem 1.1.

2 Proof of Theorems 1.1 and 1.3

For completeness, we begin with proving bounds (4)-(6).

Proposition 2.1 *For all $k \geq 3$ and $\ell \geq 3$, inequalities (4) and (5) hold for large r, while inequality (5) holds for all r.*

Proof: It has been proved in Füredi et al. (2014) and Kostochka et al. (2017) that for all $k \geq 3$ and $\ell \geq 3$, except for $k = \ell = 3$ (but see Acknowledgements in Jackowska et al. (2016)), and for n sufficiently large

$$\mathrm{ex}_k(n; P_\ell^{(k)}) = \binom{n}{k} - \binom{n-t}{k} + \delta_\ell \binom{n-k-t}{k-2},$$

where $\delta_\ell = 0$ if ℓ is odd and, otherwise, $\delta_\ell = 1$, while $t = \lfloor \frac{\ell+1}{2} \rfloor - 1$. Regardless of the parity of ℓ, for every $\varepsilon > 0$ and sufficiently large n, this Turán number is smaller than $(1+\varepsilon)tn^{k-1}/(k-1)!$. With some foresight, we require that $\varepsilon \leq (2\ell-1)^{-1}$. Thus, for fixed $\varepsilon > 0$, $k \geq 2$ and $\ell \geq 3$ and all sufficiently large r, the average number of edges per color in an r-coloring of the complete k-graph $K_n^{(k)}$ with $n \geq \ell kr/2$ is

$$\frac{\binom{n}{k}}{r} \geq (1-\varepsilon)\frac{n^k}{rk!} \geq (1-\varepsilon)\frac{\ell n^{k-1}}{2(k-1)!} \geq (1+\varepsilon)\frac{(\ell-1)n^{k-1}}{2(k-1)!} > \mathrm{ex}_k(n; P_\ell^{(k)}),$$

which proves (4).

For $\ell = 3$, the formula for $\mathrm{ex}_k(n; P_\ell^{(k)})$ simplifies to $\mathrm{ex}_k(n; P_3^{(k)}) = \binom{n-1}{k-1}$ and we have

$$\frac{\binom{n}{k}}{r} \geq \binom{n-1}{k-1}$$

already for $n \geq kr$. Since the only extremal k-graph in this case is the full star and it is impossible that all colors are stars, we get (5).

Finally, for $\ell = k = 3$ it was proved in Jackowska et al. (2016) that $\mathrm{ex}_3(n; P_3^{(3)}) = \binom{n-1}{2}$ for all $n \geq 8$ and the same argument as above applies to all $r \geq 3$. □

Preparing for the proof of Theorem 1.1, recall that Erdős and Gallai (1959) showed that the Turán number for a graph path $P_\ell^{(2)}$ satisfies the bound $\mathrm{ex}_2(n; P_\ell^{(2)}) \leq \frac{1}{2}(\ell-1)n$. This immediately yields, by the same argument as in the above proof, that the majority color in $K_{r\ell}$ contains a copy of $P_\ell^{(2)}$, and consequently $R(P_\ell^{(2)}; r) \leq r\ell$. We are going to use this result by blowing up the edges of a graph to obtain a 3-graph, then blowing the edges of a 3-graph to obtain a 4-graph, and so on. Formally, we call an edge of a hypergraph *selfish* if it contains a vertex of degree one, that is, a vertex which belongs exclusively to this edge. We call a hypergraph H *selfish* if every edge of H is selfish. Clearly, for $k \geq 3$ and $\ell \geq 1$, the loose path $P_\ell^{(k)}$ is selfish.

A selfish k-graph H can be reduced to a $(k-1)$-graph G_H by removing one vertex of degree one from each edge of H. Inversely, every $(k-1)$-graph G can be turned into a selfish k-graph H, called *a selfish extension of* G, such that $G = G_H$, by adding $|E(G)|$ vertices, one to each edge of G. Note that $|E(H)| = |E(G_H)|$.

Lemma 2.2 *For a given integer $k \geq 3$, let H be a selfish k-graph with $G = G_H$. Then*

$$R(H; r) \leq R(G; r) + r(|E(H)| - 1) + 1.$$

Proof: Let $n = R(G; r) + r(|E(H)| - 1) + 1$, $V = U \cup W$, $U \cap W = \emptyset$, $|V| = n$, $|U| = R(G; r)$, and $|W| = r(|E(H)| - 1) + 1$. Consider an r-coloring of the edges of $K_n^{(k)}$. For every $(k-1)$-tuple e of vertices in U, we choose the most frequent color on all the k-tuples $e \cup \{w\}$, $w \in W$ (see Figure 2). This

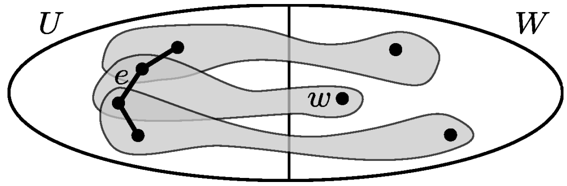

Fig. 2: Proving Lemma 2.2.

induces an r-coloring of the edges of the clique $K_{|U|}^{(k-1)}$ on vertex set U. By the definition of $R(G;r)$, this yields a monochromatic (say, *red*) copy G' of G (in the induced coloring). Note that for each edge e of G' the *red* color appears on at least $|E(H)|$ k-tuples containing e and, thus, one can find a *red* selfish extension H' of G' which is isomorphic to H. □

Proof of Theorem 1.1.: We use induction on k. For $k = 2$ the theorem coincides with the Erdős-Gallai result (2). Assume that for some $k \geq 3$ we have $R(P_\ell^{(k-1)}; r) \leq (k-2)\ell r$, observe that $P_\ell^{(k-1)} = G_{P_\ell^{(k)}}$, and apply Lemma 2.2 obtaining

$$R(P_\ell^{(k)}; r) \leq R(P_\ell^{(k-1)}; r) + r(\ell - 1) + 1 \leq (k-2)\ell r + \ell r = (k-1)\ell r.$$

□

Proof of Theorem 1.3: To get the desired lower bound on n it suffices to replace in the base step of induction the Erdős-Gallai bound (2) by the one from Theorem 1.2 which yields

$$R(P_\ell^{(k)}; r) \leq R(P_\ell^{(k-1)}; r) + r(\ell - 1) + 1 \leq 2^{r+1}\ell + (k-3)\ell r + \ell r = 2^{r+1}\ell + (k-2)\ell r.$$

It remains to show that the performance time does not exceed cn^k for some $c > 0$, which, by Theorem 1.2, is the case when $k = 2$. Without loss of generality, assume that $c \geq 1$. Suppose that for some $k \geq 3$ it holds for $(k-1)$-uniform hypergraphs. Similarly as in Lemma 2.2, we arbitrarily partition $V = U \cup W$ with $|U| \geq 2^{r+1}\ell + (k-3)\ell r$ and $|W| = r(\ell - 1) + 1$. Next we color each $(k-1)$-tuple e in U by the most frequent color on the k-tuples $e \cup \{w\}$, $w \in W$. This requires no more than

$$\binom{|U|}{k-1} \times |W| \leq n^k/(k-1)!$$

steps. Finally, by inductive assumption, in time at most cn^{k-1} we find a monochromatic copy of $P_\ell^{(k-1)}$ in U which can be extended to a monochromatic $P_\ell^{(k)}$ in no more than

$$\ell|W| \leq r\ell^2 \leq 2^{r-1}\ell^2 \leq 2^{r-1}(2^{-r-1}n)^2 = 2^{-r-3}n^2$$

steps. Altogether, the performance time, using bounds $r \geq 2$, $k \geq 3$, $\ell \geq 3$, and so $n \geq 30$, is

$$n^k/(k-1)! + cn^{k-1} + 2^{-r-3}n^2 \leq (1/2 + c/30 + 1/960)n^k \leq cn^k,$$

as required. □

3 Proof of Theorem 1.4

The proof is based on the depth first search (DFS) algorithm. Such approach for graphs was first successfully applied by Ben-Eliezer et al. (2012b,a) and for Ramsey-type problems by Dudek and Prałat (2015, 2017).

Given integers k and $2 \le m \le k$, and disjoint sets of vertices $W_1, \dots, W_{m-1}, V_m$, *an m-partite complete k-graph* $K^{(k)}(W_1, \dots, W_{m-1}, V_m)$ consists of all k-tuples of vertices with exactly one element in each W_i, $i = 1, \dots, m-1$, and $k - m + 1$ elements in V_m. Note that if $|W_i| \ge \ell$, $i = 1, \dots, m-1$, and $|V_m| \ge \ell(k-m) + 1$ for $m \le k-1$ (or $|V_m| \ge \ell$ for $m = k$), then $K^{(k)}(W_1, \dots, W_{m-1}, V_m)$ contains $P_\ell^{(k)}$. Indeed, if $m \le k-1$, then we inductively find a copy of $P_\ell^{(k)}$ in $K^{(k)}(W_1, \dots, W_{m-1}, V_m)$, edge by edge, by making sure that for each edge e, $|e \cap W_i| = 1$ (for $i = 1, \dots, m-1$) and $|e \cap V_m| = m - k + 1$ and the consecutive edges of $P_\ell^{(k)}$ intersect in V_m. In the remaining case, when $m = k$, the consecutive edges of $P_\ell^{(k)}$ intersect either in W_1 or V_k by alternating between these two sets.

We now give a description of the algorithm. As an input there is an r-coloring of the edges of the complete k-graph $K_n^{(k)}$. The algorithm consists of $r-1$ implementations of DFS subroutine, each round exploring the edges of one color only and either finding a monochromatic copy of $P_\ell^{(k)}$ or decreasing the number of colors present on a large subset of vertices, until after the $(r-1)$st round we end up with a monochromatic complete r-partite subgraph, large enough to contain a copy of $P_\ell^{(k)}$.

During the ith round, while trying to build a copy of the path $P_\ell^{(k)}$ in the ith color, the algorithm selects a subset $W_{i,i}$ from a set of still available vertices $V_i \subseteq V$ and, by the end of the round, creates trash bins S_i and T_i. The search for $P_\ell^{(k)}$ is realized by a DFS process which maintains a working path P (in the form of a sequence of vertices) whose endpoints (the first or the last $k-1$ vertices on the sequence) are either extended to a longer path or otherwise put into $W_{i,i}$. The round is terminated whenever P becomes a copy of $P_\ell^{(k)}$ or the size of $W_{i,i}$ reaches certain threshold, whatever comes first. In the latter case we set $S_i = V(P)$.

To better depict the extension process, we introduce the following terminology. An edge of $P_\ell^{(k)}$ is called *pendant* if it contains at most one vertex of degree two. The vertices of degree one, belonging to the pendant edges of $P_\ell^{(k)}$ are called *pendant*. In particular, in $P_1^{(k)}$ all its k vertices are pendant. For convenience, the unique vertex of the path $P_0^{(k)}$ is also considered to be pendant. Observe that for $t \ge 0$, to extend a copy P of $P_t^{(k)}$ to a copy of $P_{t+1}^{(k)}$ one needs to add a new edge which shares exactly one vertex with P and that vertex has to be pendant in P. Our algorithm may also come across a situation when $P = \emptyset$, that is, P has no vertices at all. Then by an extension of P we mean any edge whatsoever.

The sets $W_{i,i}$ have a double subscript, because they are updated in the later rounds to $W_{i,i+1}$, $W_{i,i+2}$, and so on, until at the end of the $(r-1)$st round (unless a monochromatic $P_\ell^{(k)}$ has been found) one obtains sets $W_i := W_{i,r-1}$, $i = 1, \dots, r-1$, a final trash set $T = \bigcup_{i=1}^{r-1} T_i \cup \bigcup_{i=1}^{r-1} S_i$ and the remainder set $V_r = V \setminus (\bigcup_{i=1}^{r-1} W_i \cup T)$ such that all k-tuples of vertices in $K^{(k)}(W_1, \dots, W_{r-1}, V_r)$ are of color r. As an input of the ith round we take sets $W_{j,i-1}$, $j = 1, \dots, i-1$, and V_{i-1}, inherited from the previous round, and rename them to $W_{j,i}$, $j = 1, \dots, i-1$, and V_i. We also set $T_i = \emptyset$ and $P = \emptyset$, and update all these sets dynamically until the round ends.

Now come the details. For $1 \leq i \leq r-1$, let

$$\tau_i = \begin{cases} (i-1)\left(\frac{\ell}{k-r+1} + \frac{\ell+1}{k-r+2} + \cdots + \frac{\ell+1}{k-i}\right) & \text{if } 1 \leq i \leq r-2, \\ (r-2)\frac{\ell}{k-r+1} & \text{if } i = r-1, \end{cases} \tag{7}$$

and

$$t_i = \tau_i + 2(i-1).$$

Note that τ_i is generally not an integer. It can be easily shown (see Claim 4.1) that for all $2 \leq r \leq k$ and $1 \leq i \leq r-1$

$$\tau_i \leq (i-1)(\ell+1)\left(\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right). \tag{8}$$

Before giving a general description of the ith round, we deal separately with the 1st and 2nd round.

Round 1

Set $V_1 = V$, $W_{1,1} = \emptyset$, and $P = \emptyset$. Select an arbitrary edge e of color one (say, *red*), add its vertices to P (in any order), reset $V_1 := V_1 \setminus e$, and try to extend P to a red copy of $P_2^{(k)}$. If successful, we appropriately enlarge P, diminish V_1, and try to further extend P to a red copy of $P_3^{(k)}$. This procedure is repeated until finally we either find a red copy of $P_\ell^{(k)}$ or, otherwise, end up with a red copy P of $P_t^{(k)}$, for some $1 \leq t \leq \ell-1$, which cannot be extended any more. In the latter case we shorten P by moving all its pendant vertices to $W_{1,1}$ and try to extend the remaining red path again. When $t \geq 2$, the new path has $t-2$ edges. If $t = 2$, P becomes a single vertex path $P_0^{(k)}$, while if $t = 1$, it becomes empty.

Let us first consider the simplest but instructive case $r = 2$ in which only one round is performed. We terminate Round 1 as soon as

$$|W_{1,1}| \geq \ell. \tag{9}$$

If at some point $P = \emptyset$ and cannot be extended (which means there are no red edges within V_1), but (9) fails to hold, then we move $\ell - |W_{1,1}|$ arbitrary vertices from $V_1 = V \setminus W_{1,1}$ to $W_{1,1}$ and stop. At that moment, no edge of $K^{(k)}(W_{1,1}, V_1)$ is red (so, all of them must be, say, blue). Moreover, since the size of $W_{1,1}$ increases by increments of at most $2(k-1)$, we have

$$\ell \leq |W_{1,1}| \leq \ell + 2(k-1) - 1,$$

and, consequently,

$$|V_1| = n - |W_{1,1}| - |V(P)| \geq n - \ell - 2(k-1) + 1 - |V(P_{\ell-1}^{(k)})| \geq \ell(k-2) + 1$$

by our bound on n (see Theorem 1.4, case $r = 2$). This means that the completely blue copy of $K^{(k)}(W_{1,1}, V_1)$ is large enough to contain a copy of $P_\ell^{(k)}$.

When $r \geq 3$, there are still more rounds ahead during which the set $W_{1,1}$ will be cut down, so we need to ensure it is large enough to survive the entire process. We terminate Round 1 as soon as

$$|W_{1,1}| \geq (k-1)\tau_2 + \ell + 1. \tag{10}$$

If at some point $P = \emptyset$ and cannot be extended and (10) fails to hold, we move $\lceil (k-1)\tau_2 \rceil + \ell + 1 - |W_{1,1}|$ arbitrary vertices from $V_1 = V \setminus W_{1,1}$ to $W_{1,1}$ and stop.

Since the size of $W_{1,1}$ increases by increments of at most $2(k-1)$ and the right-hand side of (10) is not necessarily integer, we also have

$$|W_{1,1}| \leq (k-1)\tau_2 + \ell + 1 + 2(k-1). \tag{11}$$

Finally, we set $S_1 := P$, $T_1 = \emptyset$ for mere convenience, and $V_1 := V \setminus (W_{1,1} \cup S_1 \cup T_1)$. Note that $|S_1| \leq |V(P^{(k)}_{\ell-1})| = (\ell-1)(k-1)+1$. Also, it is important to realize that no edge of $K^{(k)}(W_{1,1}, V_1)$ is colored red.

Round 2

We begin with resetting $W_{1,2} := W_{1,1}$ and $V_2 := V_1$, and setting $P := \emptyset$, $W_{2,2} = \emptyset$, and $T_2 := \emptyset$. In this round only the edges of color two (say, blue) belonging to $K^{(k)}(W_{1,2}, V_2)$ are considered. Let us denote the set of these edges by E_2. We choose an arbitrary edge $e \in E_2$, set $P = e$, and try to extend P to a copy of $P^{(k)}_2$ in E_2 but only in such a way that the vertex of e belonging to $W_{1,2}$ remains of degree one on the path. Then, we try to extend P to a copy of $P^{(k)}_3$ in E_2, etc., always making sure that the vertices in $W_{1,2}$ are of degree one. Eventually, either we find a blue copy of $P^{(k)}_\ell$ or end up with a blue copy P of $P^{(k)}_t$, for some $1 \leq t \leq \ell - 1$, which cannot be further extended. We move the pendant vertices of P belonging to $W_{1,2}$ to the trash set T_2, while the remaining pendant vertices of P go to $W_{2,2}$. Then we try to extend the shortened path again. By moving the pendant vertices of P in $W_{1,2}$ to T_2 we make sure that in the next iterations there will be no blue edge e with exactly one vertex in $W_{1,2}$, one vertex in $W_{2,2}$ and $(k-2)$ vertices in $V_2 \setminus W_{2,2}$. We terminate Round 2 as soon as

$$|W_{2,2}| \geq (k-2)\tau_2.$$

If at some point $P = \emptyset$ and cannot be extended and $|W_{2,2}| < (k-2)\tau_2$, then we move $\lceil (k-2)\tau_2 \rceil - |W_{2,2}|$ arbitrary vertices from V_2 to $W_{2,2}$ and stop. Note that at the end of this round

$$|W_{2,2}| \leq (k-2)\tau_2 + 2(k-2). \tag{12}$$

We set $S_2 := V(P)$ and $V_2 := V \setminus (W_{1,2} \cup W_{2,2} \cup S_1 \cup S_2 \cup T_2)$. Observe that no edge of $K^{(k)}(W_{1,2}, W_{2,2}, V_2)$ is red or blue. We will now show that

$$|T_2| \leq t_2 \qquad \text{and} \qquad |W_{1,2}| \geq (k-2)\tau_2. \tag{13}$$

First observe that the size of $W_{1,1}$ (the set obtained in Round 1) satisfies

$$|W_{1,1}| \leq |W_{1,2}| + |T_2| + \ell - 1. \tag{14}$$

Indeed, at the end of this round $W_{1,1}$ is the union of $W_{1,2} \cup T_2$ and the vertices in $V(P) \cap W_{1,2}$ that were moved to S_2. Since $|V(P) \cap W_{1,2}| \leq \ell - 1$, (14) holds.

Also note that each vertex in T_2 can be matched with a set of $k-2$ or $k-1$ vertices in $W_{2,2}$, and all these sets are disjoint. Consequently,

$$|W_{2,2}| \geq (k-2)|T_2|. \tag{15}$$

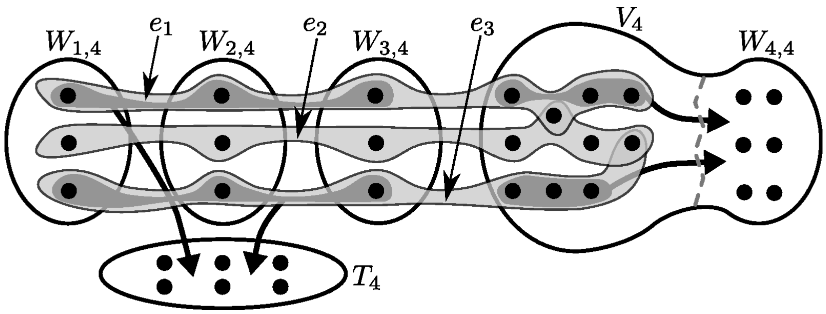

Fig. 3: Applying the algorithm to a 7-uniform hypergraph. Here $i = 4$ and path P, which consists of edges e_1, e_2, and e_3, cannot be extended. Therefore, the vertices in $V(P) \cap (W_{1,4} \cup W_{2,4} \cup W_{3,4})$ are moved to the trash bin T_4 and the pendant vertices in $V_4 \cap (e_1 \cup e_3)$ are moved to $W_{4,4}$.

Inequality (15) immediately implies that

$$|T_2| \overset{(15)}{\leq} \frac{1}{k-2}|W_{2,2}| \overset{(12)}{\leq} \tau_2 + 2 = t_2.$$

Furthermore,

$$(k-1)\tau_2 + \ell + 1 \overset{(10)}{\leq} |W_{1,1}| \overset{(14)}{\leq} |W_{1,2}| + |T_2| + \ell - 1 \leq |W_{1,2}| + \tau_2 + \ell + 1,$$

completing the proof of (13).

From now on we proceed inductively. Assume that $i \geq 3$ and we have just finished round $i-1$ constructing so far, for each $1 \leq j \leq i-1$, sets S_j, T_j, and $W_{j,i-1}$, satisfying

$$|W_{j,i-1}| \geq \frac{k-i+1}{i-2}\tau_{i-1}, \tag{16}$$

$|S_{i-1}| \leq |V(P^{(k)}_{\ell-1})|$, and $|T_{i-1}| \leq t_{i-1}$, and the residual set

$$V_{i-1} = V \setminus \bigcup_{j=1}^{i-1}(W_{j,i-1} \cup S_j \cup T_j)$$

such that $K^{(k)}(W_{1,i-1}, \ldots, W_{i-1,i-1}, V_{i-1})$ contains no edge of color $1, 2, \ldots,$ or $i-1$.

Round i, $3 \leq i \leq r-1$

We begin the ith round by resetting $W_{1,i} := W_{1,i-1}, \ldots, W_{i-1,i} := W_{i-1,i-1}$, and $V_i := V_{i-1}$, and setting $P := \emptyset$, $W_{i,i} := \emptyset$, and $T_i := \emptyset$. We consider only edges of color i in $K^{(k)}(W_{1,i}, \ldots, W_{i-1,i}, V_i)$. Let us denote the set of such edges by E_i.

As in the previous steps we are trying to extend the current path P using the edges of E_i, but only in such a way that the vertices from P that are in $W_{1,i} \cup \cdots \cup W_{i-1,i}$ have degree one in P and the vertices of degree two in P belong to V_i. When an extension is no longer possible and $P \neq \emptyset$, we move the pendant vertices of P belonging to $\bigcup_{j=1}^{i-1} W_{j,i}$ to the trash set T_i, while the remaining pendant vertices of P go to $W_{i,i}$ (see Figure 3). Then we try to extend the shortened path. We terminate the ith round as soon as

$$|W_{i,i}| \geq \frac{k-i}{i-1}\tau_i.$$

If $P = \emptyset$ and cannot be extended and $|W_{i,i}| < \frac{k-i}{i-1}\tau_i$, then we move $\lceil \frac{k-i}{i-1}\tau_i \rceil - |W_{i,i}|$ vertices from V_i to $W_{i,i}$ and stop. This yields that

$$|W_{i,i}| \leq \frac{k-i}{i-1}\tau_i + 2(k-i). \tag{17}$$

Similarly as in (14) and (15) notice that for all $1 \leq j \leq i-1$

$$|W_{j,i-1}| \leq |W_{j,i}| + \frac{|T_i|}{i-1} + \ell - 1 \tag{18}$$

and

$$|T_i| \leq \frac{i-1}{k-i}|W_{i,i}| \leq \tau_i + 2(i-1) = t_i. \tag{19}$$

Thus,

$$\frac{k-i+1}{i-2}\tau_{i-1} \overset{(16)}{\leq} |W_{j,i-1}| \overset{(18),(19)}{\leq} |W_{j,i}| + \frac{\tau_i}{i-1} + 2 + \ell - 1 = |W_{j,i}| + \frac{\tau_i}{i-1} + \ell + 1$$

and, since also

$$\frac{k-i+1}{i-2}\tau_{i-1} \overset{(7)}{=} \frac{k-i+1}{i-1}\tau_i + \ell + 1,$$

we get

$$|W_{j,i}| \geq \frac{k-i}{i-1}\tau_i. \tag{20}$$

Finally we set $S_i := V(P)$. Consequently, when the ith round ends, we have (20) for all $1 \leq j \leq i$. We also have $|S_i| \leq |V(P_{\ell-1}^{(k)})|$, $|T_i| \leq t_i$, and $V_i = V \setminus \bigcup_{j=1}^{i}(W_{j,i} \cup S_j \cup T_j)$ such that $K^{(k)}(W_{1,i}, \ldots, W_{i-1,i}, W_{i,i}, V_i)$ has no edges of color $1, 2, \ldots,$ or i.

In particular, when the $(r-1)$st round is finished, we have, for each $1 \leq j \leq r-1$,

$$|W_{j,r-1}| \geq \frac{k-r+1}{r-2}\tau_{r-1}, \tag{21}$$

$|S_{r-1}| \leq |V(P_{\ell-1}^{(k)})|$ and $|T_{r-1}| \leq t_{r-1}$. Set $W_j := W_{j,r-1}$, $j = 1, \ldots, r-1$, and $V_r := V \setminus \bigcup_{j=1}^{r-1}(W_j \cup S_j \cup T_j)$ and observe that $K^{(k)}(W_1, \ldots, W_{r-1}, V_r)$ has only edges of color r.

By (21), for each $1 \leq j \leq r-1$

$$|W_j| \overset{(21)}{\geq} \frac{k-r+1}{r-2}\tau_{r-1} \overset{(7)}{=} \ell.$$

Now we are going to show that $|V_r| \geq \ell(k-r+1)$ which will complete the proof as this bound yields a monochromatic copy of $P_\ell^{(k)}$ inside $K^{(k)}(W_1, \ldots, W_{r-1}, V_r)$. (Actually for $r \leq k-1$ it suffices to show that $|V_r| \geq \ell(k-r)+1$.)

First observe that

$$|W_{1,1}| + \cdots + |W_{r-2,r-2}| \geq |W_1| + \cdots + |W_{r-2}| + |T_1| + \cdots + |T_{r-1}|. \tag{22}$$

This is easy to see, since during the process

$$W_{i,i} \supseteq W_{i,r-1} \cup (W_{i,i} \cap (T_{i+1} \cup \cdots \cup T_{r-1}))\,.$$

Also,

$$\begin{aligned}|W_{1,1}| &\overset{(11)}{\leq} (k-1)\tau_2 + 2(k-1) + \ell + 1 \\ &\overset{(8)}{\leq} (k-1)(\ell+1)\left(\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right) + 2(k-1) + \ell + 1\end{aligned}$$

and, for $2 \leq i \leq r-1$,

$$\begin{aligned}|W_{i,i}| &\overset{(17)}{\leq} \frac{k-i}{i-1}\tau_i + 2(k-i) \\ &\overset{(8)}{\leq} (k-i)(\ell+1)\left(\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right) + 2(k-i).\end{aligned}$$

Since

$$\sum_{i=1}^{r-1}(k-i) = (k-r/2)(r-1),$$

we have by (22) that

$$\begin{aligned}&|W_1| + \ldots + |W_{r-1}| + |T_2| + \cdots + |T_{r-1}| \\ &\qquad \leq (\ell+1)\left(\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right)(k-r/2)(r-1) \\ &\qquad\qquad + (2k-r)(r-1) + \ell + 1 \\ &\qquad \leq k(\ell+1)r\left(\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right) \\ &\qquad\qquad + (2k-r)(r-1) + \ell + 1.\end{aligned}$$

As also $|S_i| \leq |V(P^{(k)}_{\ell-1})| = (k-1)(\ell-1) + 1$ for each $1 \leq i \leq r-1$ and

$$|V_r| = |V| - \sum_{i=1}^{r-1}(|W_i| + |T_i| + |S_i|),$$

we finally obtain, using the lower bound on $n = |V|$, that

$$\begin{aligned}|V_r| &\geq k(\ell+1)r - (2k-r)(r-1) - \ell - 1 - (r-1)\left[(k-1)(\ell-1) + 1\right] \\ &= \ell(2r-3) + (r-1)(r-2) + (k-1) + \ell(k-r+1) \geq \ell(k-r+1),\end{aligned}$$

since the first three terms in the last line are nonnegative.

To check the $O(n^k)$ complexity time, observe that in the worst-case scenario we need to go over all edges colored by the first $r-1$ colors and no edge is visited more than once.

4 Auxiliary inequalities

For the sake of completeness we prove here two straightforward inequalities.

Claim 4.1 *Let* $2 \le r \le k$, $1 \le i \le r-1$ *and*

$$\tau_i = \begin{cases} (i-1)\left(\frac{\ell}{k-r+1} + \frac{\ell+1}{k-r+2} + \cdots + \frac{\ell+1}{k-i}\right) & \text{if } 1 \le i \le r-2, \\ (r-2)\frac{\ell}{k-r+1} & \text{if } i = r-1. \end{cases}$$

Then,

$$\tau_i \le (i-1)(\ell+1)\left(\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right)\right).$$

Proof: It suffices to observe that

$$\begin{aligned} \frac{1}{k-r+1} + \frac{1}{k-r+2} + \cdots + \frac{1}{k-i} &\le \frac{1}{k-r+1} + \int_{k-r+1}^{k-i} \frac{dx}{x} \\ &= \frac{1}{k-r+1} + \ln\left(\frac{k-i}{k-r+1}\right) \\ &\le \frac{1}{k-r+1} + \ln\left(\frac{k-1}{k-r+1}\right) \\ &= \frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right). \end{aligned}$$

□

Claim 4.2 *For all* $2 \le r \le k-1$ *we have*

$$\frac{1}{k-r+1} + \ln\left(1 + \frac{r-2}{k-r+1}\right) \le \ln\left(1 + \frac{r-1}{k-r}\right). \tag{23}$$

Proof: Let $f(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}$ and observe that $f'(x) = \frac{-1}{x(x+1)^2}$. Hence, $f(x)$ is decreasing for $x > 0$ and so $f(x) \ge \lim_{x\to\infty} f(x) = 0$. Consequently, for $x = k-r$ (by assumption $k-r \ge 1$) we get that

$$\frac{1}{k-r+1} \le \ln\left(1 + \frac{1}{k-r}\right) = \ln\left(\frac{k-r+1}{k-r}\right) = \ln\left(\frac{k-1}{k-r}\right) - \ln\left(\frac{k-1}{k-r+1}\right),$$

which is equivalent to (23). □

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