Discrete Mathematics and Theoretical Computer Science DMTCS vol. **21**:2, 2019, #2

A code for square permutations and convex permutominoes

Enrica Duchi

IRIF, Université Paris Diderot, France

received 30th Mar. 2019, revised 16th July 2019, accepted 10th Sep. 2019.

In this article we consider square permutations, a natural subclass of permutations that can be defined in terms of either geometric conditions or pattern avoidance, and convex permutominoes, a related subclass of polyominoes. While these two classes of objects arise independently in various contexts, they play a natural role in the description of certain random horizontal and vertical convex grid configurations.

We propose a common approach to the enumeration of these two classes of objets that allows us to explain the known common form of their generating functions, and to derive new refined formulas and linear time random generation algorithms for these objects and the associated grid configurations.

Keywords: Bijection; Encoding; Enumeration; Random sampling

1 Introduction

Square permutations and convex permutominoes are natural subclasses of permutations and polyominoes that were introduced independently in the last ten years and have been shown to enjoy remarkably simple and similar enumerative formulas: their respective generating functions $Sq(t)$ and $Cp(t)$ with respect to the size are

$$Sq(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{4t^3}{(1-4t)^{3/2}} \tag{1}$$

$$Cp(t) = \frac{t^2}{1-4t}\left(2+\frac{2t}{1-4t}\right) - \frac{t^2}{(1-4t)^{3/2}} \tag{2}$$

Equivalently, the number Sq_n of square permutations with n points and the number Cp_n of convex permutominoes with size n are respectively

$$(n+2)\,2^{2n-5} \;-\; 4(2n-5)\binom{2n-6}{n-3}$$

$$(n+2)\,2^{2n-5} \;-\; (2n-3)\binom{2n-4}{n-2}$$

ISSN 1365–8050

These results were first obtained by Mansour and Severini [9] for $Sq(t)$, then recovered by Duchi and Poulalhon [8] and by Albert *et al* [1], while for $Cp(t)$ they were first obtained by Boldi *et al* [3] and, independently, by Disanto *et al* [7]. Known proofs of these formulas rely on writing recursive decompositions resulting into linear equations with one catalytic variable that can be easily solved via the kernel method. An explicit connection between the two classes of objects was obtained by Bernini *et al* [2], resulting in a composition relation of the form $Cp(t) = ISq(t, 2)$ where $ISq(t, u)$ is the generating function of certain indecomposable square permutations counted by their size and number of free fixed points (see Section 2 for definitions and precise statements).

While relatively simple, none of these known proofs explain, as far as we know, the common shape of the formulas, nor its particular form as a difference between an asymptotically dominant rational term and a simple subdominant algebraic term.

By introducing an injective encoding of square permutations and of convex permutominoes into a same class of marked words $\mathcal{M}$, we provide here an explanation of the common form of the two formulas and generalize them to take into account natural parameters extending the Narayana refinement for Catalan numbers.

The rest of the paper is organized as follows. In Section 2 we provide an original motivation for the definitions of convex permutominoes and square permutations. In Section 3 we first present our encoding and introduce the announced class of marked words $\mathcal{M}$ containing the code words and interpreting the rational part of the formula. We then state our main results and some consequences for random generation. The remaining sections are dedicated to the proofs of the results: In Section 4 and Section 5 and Section 6 we describe the subsets of words of $\mathcal{M}$ which do not encode square permutations, indecomposable square permutations, and convex permutominoes respectively. This allows to conclude the proof of the results stated in Section 3.

2 Definitions and motivations

2.1 Square permutations

Let S be a set of points in the plane. A point of S is a *upper-right record* if there is no point in S that is both over it and to its right. In other terms,

- $(x, y) \in S$ is a upper-right record of S if for all $(x', y') \in S$, either $x' \leq x$ or $y' \leq y$ (or both).

One defines similarly *upper-left*, *bottom-left* and *bottom-right records*. Finally a point of S that is a record in one of these four directions is called a *exterior point* of S, otherwise it is an *interior point*.

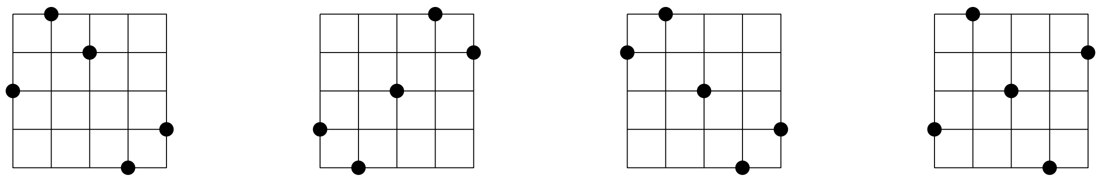
In the rest of the paper we identify a permutation σ of the set $\{1, \dots, n\}$ with its geometric representation, the set of points $R(\sigma) = \{(i, \sigma(i)) \mid i = 1, \dots, n\}$. In particular for simplicity, *the point* $\sigma(i)$ *of the permutation* σ stands for *the point* $(i, \sigma(i))$ *of the geometric representation of the permutation* σ.

Observe that the records of $R(\sigma)$ correspond to the records of σ in the usual combinatorial sense: upper-right records of $R(\sigma)$ are right-left maxima of σ, bottom-right records are right-left minima, and so on for other types of records.

Then we have the following definition (see Figure 1):

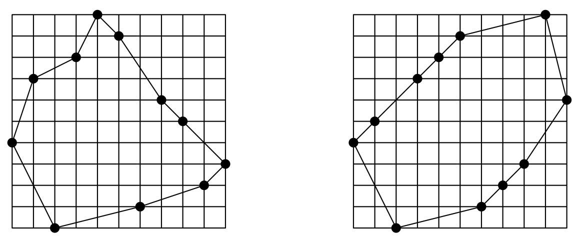
- A permutation is a *square permutation* if its geometric representation has no interior point.

The terminology square permutation arises from the following path representation: given a permutation, connect the successive upper-left records from left to right to form the *left-upper path* of σ, and define



(a) (b) (c) (d)

Fig. 1: (a) The square permutation $3, 5, 4, 1, 2$. (b) A square permutation with an upper-left record which is also a bottom-right record. (c) A square permutation with an upper-right record which is also a bottom-left record. (d) A permutation that is not square.



(a) (b)

Fig. 2: (a) A triangular permutation. (b) A parallel permutation.

accordingly the right-upper path, right-lower path and left-lower path. Then an equivalent definition is the following:

- A *square permutation* is a permutation such that each point belongs to the left-upper, right-upper, right-lower or right-upper path.

The following two subclasses of square permutations play an important role in our approach:

- A *triangular permutation* is a square permutation such that each point belongs either to the left-upper path, the right-upper path, or the right-lower path (see Figure 2 (a)).

- A *parallel permutation* is a square permutation such that all points belong to the left-upper or the right-lower path (see Figure 2 (b)).

The *s*ize of a permutation is its number of points. We let $\mathcal{S}_q$ denote the class of square permutations and $\mathcal{S}_{q_n}$ denote the number of square permutations of size n. We let $\mathcal{T}$ denote the class of triangular permutations and $\mathcal{T}_n$ denote the number of triangular permutations of size n. We let $\mathcal{P}$ denote the class of parallel permutations and $\mathcal{P}_n$ denote the number of parallel permutations of size n.

Remark 1 *We would like to point out that:*

- *Square permutations can be characterized in terms of pattern avoidance, in particular Waton proved [10] that they are all the permutations avoiding the following sixteen patterns of length five:*

$$\left\{ \begin{array}{cccccccc} 14325 & 14352 & 15324 & 15342 & 24315 & 24351 & 25314 & 25341 \\ 41325 & 41352 & 42315 & 42351 & 51324 & 51342 & 52314 & 52341 \end{array} \right\}$$

- *Triangular permutations are all the permutations avoiding the following four patterns of length four:*

$$\left\{ \begin{array}{cccc} 3214 & 3241 & 4213 & 4231 \end{array} \right\}$$

- *Parallel permutations are permutations avoiding* 321*, indeed they can be seen as two interleaved increasing sequences of points.*

2.2 Convex permutominoes

Like for square permutations, we use an approach in terms of points in the plane to introduce convexity in the context of self-avoiding polygons:

- A closed self-avoiding walk on the square lattice is a *convex polygon* if, as set of points, it has no interior point.

This notion of convexity for polygon is equivalent to the one used in the context of polyomino enumeration (see [5], and more precisely [6] for convex polygons).

The *turnpoints* of a polygon on the square lattice are the vertices of the lattice at which the underlying walk changes direction. The *sides* of a polygon are the sequences of steps between turnpoints: any polygon on the square lattice has the same number of sides and turnpoints and they are both even. The *upper walk* of a convex polygon is the sub-walk of the polygon starting from the highest of its leftmost points with a horizontal step and ending at the highest of its rightmost points with a horizontal step. An *upper side* of a polygon is a horizontal side that belongs to the upper walk of the polygon. One defines similarly the *left walk* and the *left sides* of a polygon.

We are interested in the subclass of polygons that are "generic" in the sense that they do not have two sides in the same column or line, and "reduced" in the sense that each line or column that they intersect contains a side:

- A polygon P on the square lattice is a *permutomino* of size n if each line of abscissa or ordinate i contains exactly one side of P for $i = 0, \dots, n$.

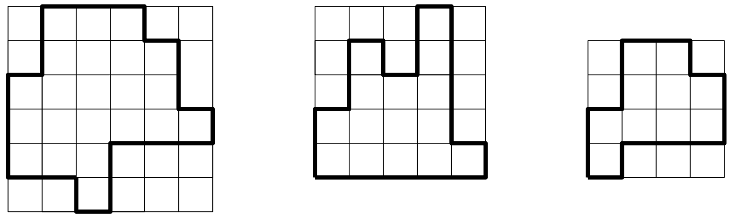
Finally a *convex permutomino* is a permutomino which is convex (see Figure 3). Equivalently:

- A convex permutomino is a permutomino such that each point of the underlying walk is a record.

In fact it is clearly enough to check that each turnpoint is a record. As for square permutations, two subclasses of convex polygons will be of particular interest:

- A *directed convex permutomino* is a permutomino such that each turnpoint is either a left-upper, a right-upper, or a right-lower record (see Figure 4 (a)).

- A *parallelogram permutomino* is a permutomino such that each turnpoint is either a left-upper or a right-lower record (see Figure 4 (b)).



(a) (b) (c)

Fig. 3: (a) A convex permutomino of size 7. (b) A permutomino that is not convex. (c) A convex polygon that is not a permutomino.

(a) (b)

Fig. 4: (a) a directed convex permutomino; (b) a parallelogram permutomino.

Remark 2 *We would like to point out that permutominoes can be viewed as collections of unit squares lattice cells glued together such that the interior is connected, that is, they are a subclass of polyominoes.*

We let $\mathcal{C}_p$ denote the class of convex permutominoes and $\mathcal{C}_{p_n}$ the number of convex permutominoes of size n. We let $\mathcal{D}_p$ denote the class of directed convex permutominoes and $\mathcal{D}_{p_n}$ denote the number of directed convex permutominoes of size n. We let $\mathcal{P}_p$ be the class of parallelogram permutominoes and $\mathcal{P}_{p_n}$ the number of parallelogram permutominoes of size n. Finally, we let $D_p(t)$ and $P_p(t)$ denote respectively the generating functions of triangular and parallelogram permutominoes.

Square permutations versus convex permutominoes It is known that, besides the similarities between Formulas (1) and (2) for square permutations and convex permutominoes, the parallel between these two classes extends also to their subclasses. In particular, we have that:

$$T(t) = B(t) \quad \text{and} \quad D_p(t) = \frac{B(t)}{2} \qquad \text{where} \quad B(t) = \frac{1}{\sqrt{1-4t}}$$

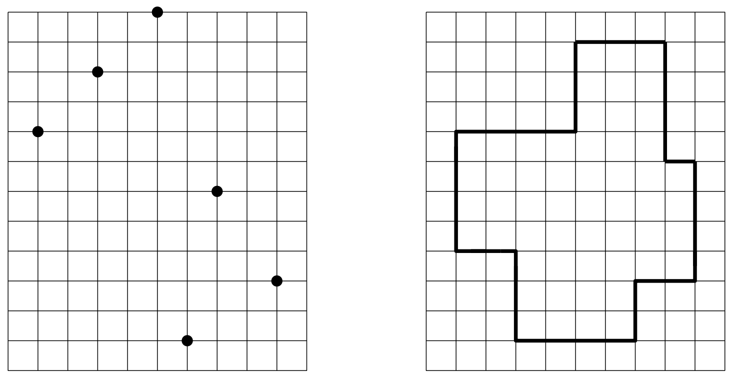
is the generating function of central binomial coefficients, and

$$P(t) = P_p(t) = C(t) \qquad \text{where} \quad C(t) = \frac{1-2t-\sqrt{1-4t}}{2t}$$

is the generating function of Catalan numbers.

2.3 A link with random grid configurations.

The relevance of the number Sq_n of square permutations and Cp_n of convex permutominoes is highlighted by the following propositions (see Figure 5):



(a) (b)

Fig. 5: (a) A set of 6 exterior points on a grid of size $M \times N$. (b) A convex polygon with 12 turnpoints on a grid of size $M \times N$.

Proposition 3 *The number $E_{M,N}(n)$ of configurations of n exterior points on a $M \times N$ grid satisfies*

$$E_{M,N}(n) \underset{M,N\to\infty}{\sim} Sq_n \cdot \binom{M}{n}\binom{N}{n}.$$

Proof: Observe that the probability to have a non generic configuration, that is a configuration with two or more points belonging to the same column or row, goes to 0. Indeed M and N tends to infinity unlike the number of points n. Given a generic configuration of n exterior points on a grid $M \times N$ it can be uniquely decomposed into a square permutation giving its reduced shape and a selection of n of the N columns and n of the M rows that are used by the points. □

A similar result holds for convex polygons on the square lattice with a fixed number of turnpoints.

Proposition 4 *The number $C_{M,N}(n)$ of convex polygons with $2n$ turnpoints on a $M \times N$ grid satisfies*

$$C_{M,N}(n) \underset{M,N\to\infty}{\sim} Cp_n \cdot \binom{M}{n}\binom{N}{n}.$$

Proof: Observe that the probability to have a non generic polygon, with two or more turnpoints belonging to the same column or row, goes to 0. Indeed M and N tends to infinity unlike the number of turnpoints n. Given a generic convex polygon it can be uniquely decomposed into a convex permutomino giving its reduced shape and a selection of n of the N columns and n of the M rows that are used by the turnpoints. □

2.4 Convex permutominoes and colored square permutations

A permutation σ of $\{1,\ldots,n\}$ is *decomposable* if there exist two permutations π of $\{1,\ldots,k\}$ and π' of $\{1,\ldots,\ell\}$ with $k+\ell=n$ such that σ in vector notation is the permutation $\pi_1,\pi_2,\ldots,\pi_k,\pi'_1+k,\pi'_2+$

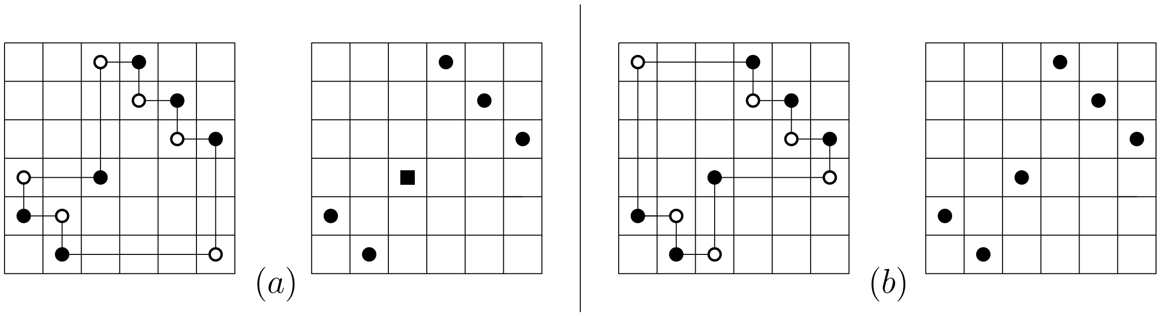


Fig. 6: (a) A permutomino and its corresponding square permutation with a colored fixed point (represented with a little square). (b) A permutomino and its corresponding square permutation with a non-colored fixed point.

$k, \ldots, \pi'_\ell + k$. It is *co-decomposable* if there exist two permutations π of $\{1, \ldots, k\}$ and π' of $\{1, \ldots, \ell\}$ with $k + \ell = n$ such that σ is the permutation $\pi_1 + \ell, \pi_2 + \ell, \ldots, \pi_k + \ell, \pi'_1, \pi'_2, \ldots, \pi'_\ell$. A permutation is *indecomposable* if it is not decomposable, *co-indecomposable* if it is not co-decomposable, and *fully indecomposable* if it is both indecomposable and co-indecomposable.

A *fixed point* of a permutation σ is an entry i such that $\sigma(i) = i$. It is a *free fixed point* if the corresponding point is not a bottom-left or upper-right record. Let $f(\sigma)$ denote the number of free fixed points of σ.

A *colored permutation* is a permutation with a (possibly empty) subset of its free fixed points that are colored. In particular a colored square permutation is a colored permutation whose underlying permutation is square, and we identify standard permutations with colored permutations without colored fixed point.

Theorem 5 *(Bernini* et al *[2]) There is a bijection ϕ between*

- *convex permutominoes of size n, and*

- *colored square permutations of size n that are co-indecomposable.*

To obtain the colored permutation associated to a convex permutomino P, one should first color the bottom turnpoint on the leftmost edge of P in black, and then alternatively color all other turnpoints of P in white or black, following the boundary of P in clockwise order, so that adjacent turnpoints have different colors. Then $\phi(P)$ is the permutation represented by black points, with fixed points that belong to the upper walk of P as colored fixed points (resp. belonging to the lower boundary of P as non-colored fixed points). See an example in Figure 6.

As announced in the introduction, this theorem leads to a relation between the generating function of convex permutominoes with respect to the size and the generating function of co-indecomposable square permutations with respect to the size and number of free fixed points. However, this relation does not explain the similarity between Formulas (1) and (2).

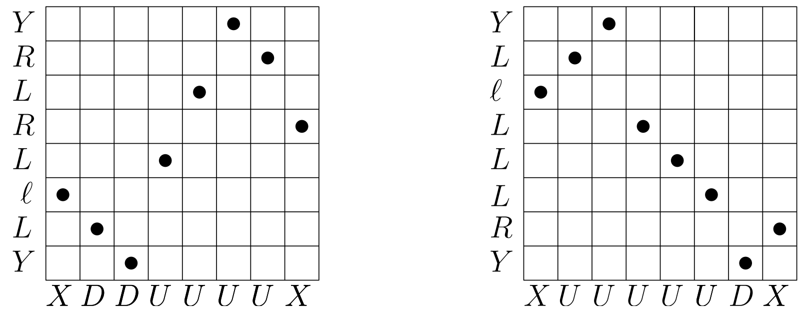


Fig. 7: Two square permutations of size 8 whose marked biwords are respectively $(X,Y)(D,L)(D,L)(U,L)(U,R)(U,L)(U,R)(X,Y)$ with $m = 3$ and $(X,Y)(U,R)(U,L)(U,L)(U,L)(U,L)(D,L)(X,Y)$ with $m = 6$.

3 Main results

3.1 The horizontal-vertical encoding

We now define the horizontal-vertical encoding of a colored square permutation. We will later study this encoding for three subsets of colored square permutations: square permutations (that is, colored square permutations without colored points), indecomposable square permutations, and colored co-indecomposable square permutations (which are in bijection with convex permutominoes as previously discussed).

Let us say that a point (x,y) in a set S is an *upper point* if it is a upper-left or a upper-right record of S, and it is a *left point* if it is a upper-left or bottom-left record of S. Given a colored permutation σ, we denote by $H(\sigma) = Xu_2 \dots u_{n-1}X$ the *horizontal profile* of σ, where

$$u_i = \begin{cases} U, & \text{if } \sigma(i) \text{ is an upper point that is not colored;} \\ D, & \text{otherwise} \end{cases}$$

with $i \in \{2, \dots, n-1\}$, and the letter X codes for the horizontal projection of extremal points of σ, and we denote by $V(\sigma) = Yv_2 \dots v_{n-1}Y$ the *vertical profile* of σ, where

$$v_j = \begin{cases} L, & \text{if } (\sigma^{-1}(j), j) \text{ is a left point that is not colored;} \\ R, & \text{otherwise;} \end{cases}$$

with $j \in \{2, \dots, n-1\}$, and the letter Y codes for the vertical projection of extremal points of σ.

Finally let the *code* of a permutation σ be the triple $(H(\sigma), V(\sigma), \sigma(1))$: we will see later that the encoding $\sigma \mapsto (H(\sigma), V(\sigma), \sigma(1))$ is indeed injective for the families of permutations we are interested in.

In order to state our results let us consider the following set of words: Let $\mathcal{W} = \mathcal{A}^*$ denote the set of (bi)words on the alphabet $\mathcal{A} = \{U, D\} \times \{L, R\}$, and let $\mathcal{M}$ denote the set of marked words (w, m) consisting of

- a word $w = (u_1, v_1) \cdots (u_n, v_n) \in \{(X, Y)\} \cdot \mathcal{W} \cdot \{(X, Y)\}$
- and a mark m with $1 \leq m \leq n$ and $v_m \in \{L, Y\}$.

Observe then that the triple $(H(\sigma), V(\sigma), \sigma(1))$ can alternatively be viewed as a marked biword (w, m) of $\mathcal{W}$ by taking $w = (X, Y)(u_2, v_2) \dots (u_{n-1}, v_{n-1})(X, Y)$ and $m = \sigma(1)$ where, by construction, the mark m is on a letter L or X of the vertical profile (see Figure 7 for an example, where the marked letter L is represented by ℓ).

The main contribution of this paper is to prove that the above horizontal/vertical encoding of square permutations and of convex permutominoes by words of $\mathcal{M}$, is injective and to describe the complementary sets of *non-coding words*, that is, the elements of $\mathcal{M}$ that are not codewords in both cases:

Theorem 6 *The horizontal/vertical encoding defines bijections*

$$\mathcal{S}q \equiv \mathcal{M} \; - \; \mathcal{T}^{\swarrow} \cdot \{(D,L),(D,R)\} \cdot \mathcal{W} \cdot \{(X,Y)\} \; - \; \mathcal{T}^{\nwarrow} \cdot \{(U,R),(D,L)\} \cdot \mathcal{W} \cdot \{(X,Y)\}$$

$$\mathcal{C}p \equiv \mathcal{M} \; - \; \mathcal{D}^{\swarrow} \cdot \{(D,L)\} \cdot \mathcal{W} \cdot \{(X,Y)\} \; - \; \mathcal{D}^{\nwarrow +} \cdot \mathcal{W} \cdot \{(X,Y)\}$$

where $\mathcal{T}^{\swarrow}$ and $\mathcal{T}^{\nwarrow}$ are languages encoding variants of triangular permutations *(namely permutations obtained from triangular permutations by a 180^o or 90^o rotation respectively), while $\mathcal{D}^{\swarrow}$ and $\mathcal{D}^{\nwarrow +}$ are languages encoding variants of* directed convex permutominoes *(namely permutations obtained respectively from colored triangular permutations by a 180^o rotation or from indecomposable triangular permutations by a 90^o rotation respectively).*

3.2 Enumerative consequences and random sampling

Let now $M(t; x, y)$ and $W(t; x, y)$ be respectively the generating functions of $\mathcal{M}$ and $\mathcal{W}$ with respect to the length (var. t), number of U and X (var. x) and number of L and Y (var. y). Then

$$W(t; x, y) = \frac{1}{(1 - (1 + x)(1 + y)t)}$$

and, upon dealing separately with the case $m \in \{1, n\}$ where the mark is on a letter Y from the case $1 < m < n$ where the mark is on a letter L, we have that

$$M(t; x, y) = 2 \cdot txy \cdot W(t; x, y) \cdot txy + (txy) \cdot W(t; x, y) \cdot t(1 + x)y \cdot W(t; x, y) \cdot txy.$$

In particular $M(t, 1, 1)$ gives an *a priori unrelated* combinatorial interpretation of the dominant rational term in Formula (1) and (2) since

$$W(t; 1, 1) = \frac{1}{1 - 4t} \quad \text{and} \quad M(t; 1, 1) = \frac{t^2}{1 - 4t}\left(2 + \frac{2t}{1 - 4t}\right).$$

From Theorem 6 it already appears that the two sets of words that do not encode objects belonging to $\mathcal{S}q$ and to $\mathcal{C}p$ according to their *horizontal/vertical encoding*, that we call *non-coding words* for $\mathcal{S}q$ and for $\mathcal{C}p$, have a similar structure. In fact the analogy between the two results goes further since the languages $\mathcal{T}^{\swarrow}$ and $\mathcal{T}^{\nwarrow}$, $\mathcal{D}^{\swarrow}$, $\mathcal{D}^{\nwarrow +}$ have essentially the same univariate generating functions:

$$T^{\swarrow}(t) = T^{\nwarrow}(t) = (2D^{\swarrow}(t) + 1)t = 2D^{\nwarrow +}(t) - t = \frac{t}{\sqrt{1 - 4t}} \tag{3}$$

and these results together with Theorem 6 immediately imply Formulas (1) and (2).

The evaluations (3) can be obtained from algebraic decompositions of the respective classes of objects, but we choose also to obtain them by re-using the same horizontal/vertical encoding to relate all these four generating functions to the generating series $N(t;x,y)$ of Narayana numbers, defined by the equation

$$N(t;x,y) = t \cdot (1 + xN(t;x,y))(1 + yN(t;x,y)).$$

From this analysis we obtain the refinements of Formulas (1) and (2):

Corollary 7 *Let $Sq(t,x,y)$ be the generating function of square permutations with respect to the size (t), the number of upper points (x) and the number of left points (y), and let $Cp(t;x,y)$ be the generating function of convex permutominoes with respect to the size (t), the number of upper sides (x) and the number of left sides (y). Then*

$$\begin{aligned} Sq(t;x,y) = M(t;x,y) - {} & \widetilde{T}^{\swarrow}(t,x,y) \cdot t \cdot (y+1) \cdot W(t,x,y) \cdot txy \\ - {} & T^{\nwarrow}(t,x,y) \cdot t \cdot (x+y) \cdot W(t,x,y) \cdot txy \\ Cp(t;x,y) = M(t;x,y) - {} & D^{\swarrow}(t,x,y) \cdot t \cdot y \cdot W(t,x,y) \cdot txy \\ - {} & D^{\nwarrow +}(t,x,y) \cdot W(t,x,y) \cdot txy \end{aligned}$$

where the auxiliary series are explicit rational functions of x, y and the Narayana function evaluations $N(t;x,y)$ and $N(t;xy,1)$. In particular

$$\begin{aligned} T^{\nwarrow}(t;x,y) &= \frac{xyN(t;x,y)}{(1 - xyN(t;x,y))(1 + (x + y - xy)N(t;x,y))} \\ \widetilde{T}^{\swarrow}(t;x,y) &= \frac{xyN(t;xy,1)}{(1 - yN(t;xy,1))(1 + N(t;xy,1))}. \end{aligned}$$

Remark 8 *Observe that, when specializing $x = y = 1$, we obtain $\widetilde{T}^{\swarrow}(t;1,1) = T^{\swarrow}(t;1,1)$.*

In the proof of Theorem 6 we provide a linear time decoding procedure that, given a word in $\mathcal{M}$, outputs the corresponding square permutation or convex permutomino. Consequently, since Formulas (1)–(2) immediately imply that Sq_n and Cp_n behave asymptotically as the number of words of size n in $\mathcal{M}$, we also have the following corollary:

Corollary 9 *There is a random sampling algorithm to generate uniform random square permutations with n points or uniform random convex permutominoes of size n in expected time linear in n.*

This result implies in turn that there is an algorithm to produce in linear time generic random horizontally and vertically convex configurations of points and self-avoiding polygons on a large grid.

We would like to point out that the properties of large random square permutations are studied by Borga and Slivken in [4], where they proved that their shape is typically rectangular.

4 A decoding algorithm for square permutations

We first give a detailed description of the decoding algorithm for square permutations (without colors), in the other cases the decoding algorithm will be similar.

We already pointed out that the set of triples coding square permutations of size n forms a subset of $\mathcal{M}$. However, this subset is not easy to describe directly. We provide an algorithmic description via the following decoding algorithm.

Let us take an element of $\mathcal{M}$ of length n, viewed as a triple (u, v, ℓ). We are going to construct a partial permutation $\sigma = \sigma(1) \dots \sigma(k)$, with $k \leq n$. The idea is to insert each point on the grid from left to right as long as a compatible decoding is possible. The rules to construct σ depend on the letters of u and v: the letter L (resp. R, U, D) means that the point of the permutation we are constructing belongs to a left path (resp. right, lower, upper path) of the corresponding permutation, while the letters X or Y means that this point is extremal. Then (see Figure 8) :

- Write u (resp. v) along the upper edge (resp. left edge) of the grid $n \times n$ starting from the left (resp. starting from the bottom).
- Start reading the word u: u_1 is necessarily an X and it should correspond to the leftmost point of the permutation. Accordingly insert a point $\sigma(1)$ in the leftmost column and in row ℓ (the row of the marked letter of v).
- Continue reading each letter u_i of u, for $i \geq 2$:

1. If $i = n$ then insert the point $\sigma(n)$ in the unique remaining empty row and end the algorithm.
2. If $u_i = U$ and we have not yet inserted the highest point of the permutation, then take the first row j above $\sigma(i-1)$ that is labeled with L or Y and insert the point $\sigma(i) = j$.
3. If $u_i = U$ and we have already inserted the highest point of the permutation, there are two cases:
   - if $\sigma(1) \dots \sigma(i-1)$ is contained in the grid $(i-1) \times (i-1)$ attached to the upper-left corner of the grid $n \times n$ then the insertion is valid only if the row $n - i$ is labeled with L or Y, and then $\sigma(i) = n - i$; otherwise the insertion is invalid and the algorithm stops (see the first case in Figure 8, where the invalid insertion is represented with a white bullet);
   - otherwise take the first row j under $\sigma(i-1)$ that is labeled with R or Y and insert the point $\sigma(i) = j$.
4. If $u_i = D$ and we have not yet inserted the lowest point of the permutation, then take the first row j that is labeled with L or Y under $\sigma(i-1)$ and insert the point $\sigma(i) = j$: if $\sigma(1) \dots \sigma(i-1)$ is contained in the grid $(i-1) \times (i-1)$ attached to the upper-left corner of the grid $n \times n$ and $j = i$ then the insertion is invalid and the algorithm stops (see the second case in Figure 8).
5. If $u_i = D$ and we have already inserted the lowest point of the permutation, then take the first row j that is labeled with R or Y above $\sigma(i-1)$ and insert the point $\sigma(i) = j$: if $\sigma(1) \dots \sigma(i-1)$ is contained in the grid $(i-1) \times (i-1)$ attached to the bottom-left corner of the grid $n \times n$ then the insertion is invalid and the algorithm stops (see the third case in Figure 8).

Let us suppose that the algorithm ends after reading i letters of u and v, $i \in \{1, \dots, n\}$. Then we can state the following proposition :

Proposition 10 *Given an element (w, ℓ) of $\mathcal{M}$ of size n viewed as triple (u, v, ℓ) the algorithm stops and yields:*

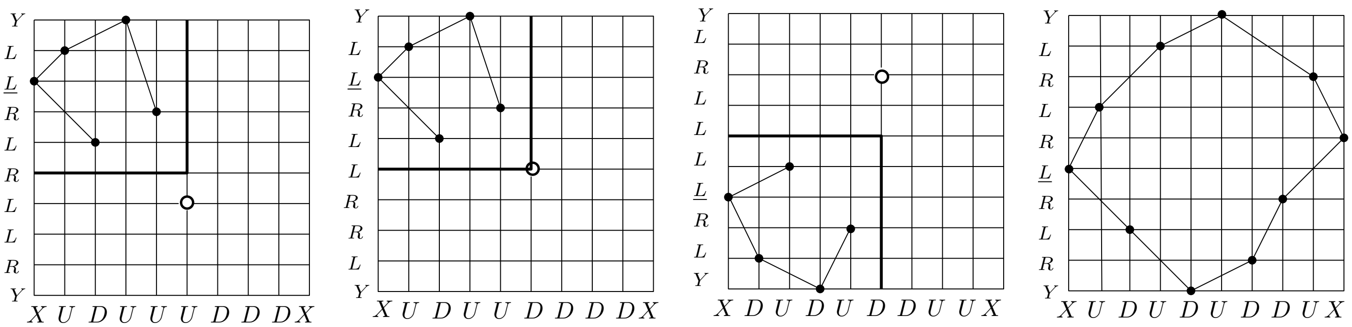


Fig. 8: Four examples of triples (u, v, ℓ), only the fourth represents an encoding of a square permutation.

- *(in case of success) a permutation* $\sigma(1)\ldots\sigma(n) \in \mathcal{S}q_n$ *whose code is* (w, ℓ)*, or*

- *(in Cases 3 and 4 above) a triple* $(\rho, (u_i, v_{n-i}), (u'', v''))$*, where* $\rho = \mathrm{Standard}(\sigma(1)\ldots\sigma(i-1)) \in T^{\nwarrow}_{i-1}$*, with* $2 \le i \le n$*, is the standardization of* σ *on* $\{1, \ldots, i-1\}$*, i.e.,* $\rho_l = \sigma(l) - n + i - 1$ *with* $l \in \{1, \ldots, i-1\}$*,* $(u_i, v_{n-i}) \in \{(U, R), (D, L)\}$*, and* (u'', v'') *is a pair of words of length* $n - i - 1$ *where* $u'' = u_{i+1}\ldots u_{n-1}X$ *and* $v'' = Y v_2 \ldots v_{n-i-1}$*, or*

- *(in Case 5 above) a triple* $(\rho, (u_i, v_i), (u'', v''))$*, where* $\rho = \sigma(1)\ldots\sigma(i-1) \in T^{\swarrow}_{i-1}$ *with* $2 \le i \le n$*,* $(u_i, v_i) \in \{(D, L), (D, R)\}$*, and* (u'', v'') *is a pair of words of length* $n - i - 1$*, where* $u'' = u_{i+1}\ldots u_{n-1}X$ *and* $v'' = v_{i+1}\ldots v_{n-1}Y$*.*

Proof Proof of Proposition 10: By construction the permutation obtained by applying the algorithm is uniquely determined and it has no interior points. Indeed, the first point σ_1 is determined by the marked letter and Case 2 (resp. 3,4,5) of the previous algorithm uniquely describes the insertion of a point in the left-upper path (resp. right-upper path, left-lower path, right-lower path) of σ. We just need to prove that row j in Case 2 (resp. 3, 4, 5) does exist. Suppose that it does not, then for Case 2 (resp. 4) it means that the point at the row labeled by Y, i.e., the higher (resp. lower) point of the permutation it has already been inserted, thus contradicting the hypothesis that it has not. For Case 3 (resp. 5) we have two possibilities :

- the lower (resp. higher) point of the permutation has already been inserted, therefore the leftmost left-upper and left-lower paths have been constructed, i.e., there is no L row free, thus contradicting the hypothesis that there is no R or Y row free;

- the lower (resp. higher) point of the permutation has not been inserted, therefore the lower (resp. higher) row, labeled by Y, is still free, thus contradicting the hypothesis.

Now we have to distinguish the two ways in which the algorithm can stop :

1. $i = n$. In this case we obtain by construction a permutation in $\mathcal{S}q_n$ and (u, v, ℓ) is the code of σ.

2. $i < n$. In this case we obtain from the previous algorithm a partial permutation σ of size $i - 1$ and a pair of words (u'', v'') of length $n - i - 2$. If the algorithm stops in Case 3 or 4 then the partial permutation σ of the grid $n \times n$ is contained in the grid $(i-1) \times (i-1)$ on the upper-left corner.

Since σ is contained in the $i-1$ upper row, the lowest row has not been touched and the construction of a right-lower path has not started: $\rho = \mathrm{Standard}(\sigma)$ is a rotated triangular permutation $\in T_{i-1}^{\nwarrow}$. If the algorithm stops in Case 5, $\rho = \sigma_1 \ldots \sigma_{i-1}$ is directly a permutation of $\{1, \ldots, i-1\}$ and similarly as above $\rho \in T_{i-1}^{\searrow}$ since the construction of a right-upper path has not started.

□ This yields a bijection corresponding to the first statement in Theorem 6:

$$\mathcal{M} \equiv \mathcal{S}q \;+\; \mathcal{T}^{\swarrow} \cdot \{(D,L),(D,R)\} \cdot \mathcal{W} \cdot \{(X,Y)\} \;+\; \mathcal{T}^{\nwarrow} \cdot \{(U,R),(D,L)\} \cdot \mathcal{W} \cdot \{(X,Y)\}$$

In turn this bijection implies an equation for refined generating series:

$$\begin{aligned} M(t;x,y) = Sq(t;x,y) \;&+\; \widetilde{T}^{\swarrow}(t;x,y) \cdot t \cdot (1+y) \cdot W(t;x,y) \cdot txy \\ &+\; T^{\nwarrow}(t;x,y) \cdot t \cdot (x+y) \cdot W(t;x,y) \cdot txy \end{aligned}$$

where $\widetilde{T}^{\swarrow}(t;x,y) = (txy)^{-1} T_1^{\swarrow}(t;x,y) - 1$ is a modified version of the bivariate generating series of south west triangular permutations in which the rightmost point contributes to the parameter x only if it is maximal, and $T_1^{\swarrow}(t;x,y)$ denotes the generating series of south west triangular permutations ending with a maximal point. More generally we use the subscript 1 to indicate that a subset of triangular or parallel permutations having a unique (extremal) point on the trivial face.

Corollary 11

$$\{(w,n) \in \mathcal{M}\} \;\equiv\; (\bullet \mathcal{T}^{\searrow}) + (\bullet \mathcal{P}^{\searrow}) \times \{(U,R),(D,L)\} \times \mathcal{W} \times \{(X,Y)\}, \tag{4}$$

Proof: Indeed, observe that the permutations of $T^{\searrow}$ of size $n-1$ are exactly the permutations of Sq that can be obtained from permutations of $\mathcal{S}q$ of size n that start with the maximal value n. Moreover, the permutations of $\mathcal{P}^{\searrow}$ of size $k-1$ are exactly the permutations that can be obtained from permutations of $T^{\nearrow}$ of size k that start with the maximal value. Then by applying Theorem 6 we obtain the result. □

Consequently,

$$txy \cdot W(t;x,y) \cdot txy = (T_1^{\searrow}(t;x,y) - txy) + P_1^{\searrow}(t;x,y) \cdot t(x+y) \cdot W(t;x,y) \cdot txy. \tag{5}$$

These relations are enough to compute the univariate generating functions (with $x = y = 1$): indeed by symmetry of the various classes of triangular permutations and by knowing that $\mathcal{P}^{\searrow}$ are counted by Catalan numbers we directly obtain a proof of Formula 1.

In order to obtain the refined generating series a bit more information is necessary because the four symmetry classes of triangular permutations have different refined generating series: the symmetry along the co-diagonal exchanges the parameter counted by x and y, so that $T_1^{\searrow}(t;x,y) = T_1^{\searrow}(t;y,x)$ and $T_1^{\nwarrow}(t;x,y) = T_1^{\nwarrow}(t;y,x)$ while $T_1^{\nearrow}(t;x,y) = T_1^{\swarrow}(t;y,x)$, but symmetries do not imply such direct relations between these three classes of series.

Forcing the initial point to be in row one yields a second equation that allows to compute the refined generating series $T_1^{\nearrow}(t;x,y)$ (and $T_1^{\swarrow}(t;x,y)$):

Corollary 12

$$\{(w,1) \in \mathcal{M}\} \;\equiv\; (\bullet \mathcal{T}^{\nearrow}) + (\bullet \mathcal{P}^{\nearrow}) \times \{(D,L),(D,R)\} \times \mathcal{W} \times \{(X,Y)\}, \tag{6}$$

Consequently,

$$txy \cdot W(t;x,y) \cdot txy = (T_1^{\nearrow}(t;x,y) - txy) + \widetilde{P}_1^{\nearrow}(t;x,y) \cdot t(1+y) \cdot W(t;x,y) \cdot txy, \tag{7}$$

where $\widetilde{P}_1^{\nearrow}(t;x,y) = \widetilde{P}_1^{\searrow}(t;xy,1)$.

However, the approach has to be modified in order to deal with the last series $T^{\nwarrow}(x,y)$: by symmetry with respect to the main diagonal this amounts to counting elements of $\mathcal{T}^{\searrow}$ with respect to the number of bottom points and right points; however note that it is not possible to simply change the encoding into D for lower points and R for right points while keeping the left to right decoding procedure because the later becomes ambiguous. Instead one can rely on the fact that $\mathcal{T}^{\nwarrow}$ can be described in terms of $\mathcal{P}^{\nwarrow}$ and the set $\mathcal{C}^{\nwarrow}$ of co-indecomposable elements of $\mathcal{T}^{\nwarrow}$, and similarly for $\mathcal{T}_1^{\searrow}$ and $\mathcal{C}^{\searrow}$: more precisely we have

$$\mathcal{T}^{\nwarrow} \equiv \mathcal{C}^{\nwarrow} \times (1 + \mathcal{P}^{\nwarrow}), \qquad \mathcal{T}_1^{\searrow} \equiv \bullet + \mathcal{P}_1^{\searrow} \times \mathcal{C}^{\searrow}. \tag{8}$$

These decompositions yield the refined identities:

$$T^{\nwarrow}(t;x,y) = C^{\nwarrow}(t;x,y)(1 + \widetilde{P}^{\nwarrow}(t;x,y)) = C^{\nwarrow}(t;x,y)P_1^{\searrow}(t;x,y) \tag{9}$$

$$T_1^{\searrow}(t;x,y) = txy + P_1^{\searrow}(t;x,y)\widetilde{C}^{\searrow}(t;x,y) \tag{10}$$

Then due to the fact that co-indecomposable permutations have no points that are simultaneously upper and left points, apart from their uppermost and leftmost points (which are distinct), we have $C^{\nwarrow}(t;x,y) = \widetilde{C}^{\searrow}(txy; \frac{1}{y}, \frac{1}{x})xy$.

As a conclusion,

$$T^{\nwarrow}(t;x,y) = P_1^{\searrow}(t;x,y)\frac{T_1^{\searrow}(txy, \frac{1}{y}, \frac{1}{x})xy - txy}{P_1^{\searrow}(txy, \frac{1}{y}, \frac{1}{x})}$$

In order to finish the computation we need to identify the various refinements of the Catalan generating series. Let $N(t;x,y)$ be the Narayana generating series, the unique formal power series solution of the equation

$$N(t;x,y) = t(1 + xN(t;x,y))(1 + yN(t;x,y)).$$

Then

$$P_1^{\nwarrow}(t;x,y) = \frac{xyN(t;x,y)}{1 + (x+y-xy)N(t;x,y)}.$$

Observe also that $N(txy, \frac{1}{y}, \frac{1}{x}) = xyN(t;x,y)$ so that $\widetilde{P}_1^{\nearrow}(t;x,y) = P_1^{\nwarrow}(t;xy,1) = txy(1+xyN(t;xy,1))$, so that all expressions are rational in terms of x, y, $N(t;x,y)$ and $N(t;xy,1)$.

In particular we obtain that

$$T^{\nwarrow}(t;x,y) = \frac{xyN(t;x,y)}{(1 - xyN(t;x,y))(1 + (x+y+xy)N(t;x,y))} \tag{11}$$

$$\widetilde{T}^{\swarrow}(t;x,y) = \frac{xyN(t;xy,1)}{(1 - yN(t;xy,1))(1 + N(t;xy,1))} \tag{12}$$

which together with

$$\begin{aligned} Sq(t;x,y) = M(t;x,y) \; &- \; \widetilde{T}^{\swarrow}(t;x,y) \cdot t \cdot (1+y) \cdot W(t;x,y) \cdot txy \\ &- \; T^{\nwarrow}(t;x,y) \cdot t \cdot (x+y) \cdot W(t;x,y) \cdot txy \end{aligned}$$

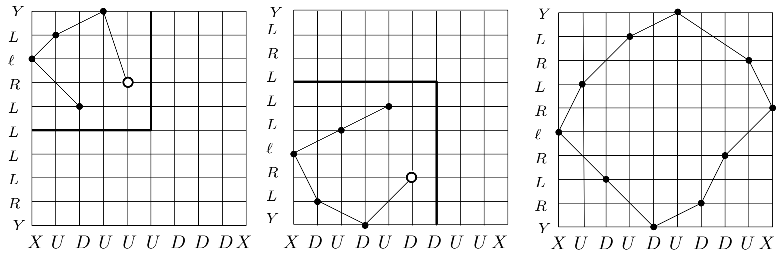


Fig. 9: Three examples of triples (u, v, ℓ). Only the third one gives a square permutation that is indecomposable and co-indecomposable.

gives the result.

Corollary 13 *Uniform random square permutations can be generated in expected linear time.*

Proof: To generate a permutation of size n, one generates triples (u, v, ℓ) from $\mathcal{M}_n$ and applies the decoding algorithm until a square permutation of size n is obtained. Each generation and decoding step takes linear time in n. The probability of rejection goes to zero as n goes to infinity since it corresponds to partially decoded permutations that are asymptotically negligible with respect to the total number of square permutations. □

5 A decoding algorithm for fully indecomposable square permutations

By using the previous encoding for square permutations we can specialize the bijection of Theorem 6 to the class of indecomposable square permutations, co-indecomposable square permutations or fully indecomposable square permutations (*i.e.* square permutations that are both indecomposable and co-indecomposable). The idea is to adapt the previous decoding algorithm in order to remove triples that do not correspond to objects of these class: each resulting algorithm is similar to the decoding algorithm for square permutations, with slightly different stopping conditions. We present here the case of fully indecomposable square permutations (see Figure 9 for an example):

- in Case 2 above one should stop as soon as $\sigma_1 \dots \sigma_{i-1}$ is contained in the grid $(i-1) \times (i-1)$ attached to the lower-left corner, regardless of the other conditions;
- in Cases 3 and 4 above, one should stop as soon as $\sigma_1 \dots \sigma_{i-1}$ is contained in the grid $(i-1)\times(i-1)$ attached to the upper-left corner, regardless of the other conditions.

Let $\mathcal{F}$ be the class of fully indecomposable square permutations, then from this decoding algorithm we obtain the following bijection:

$$\mathcal{M} \equiv \mathcal{F} \;+\; \mathcal{C}^{\swarrow} \cdot \mathcal{W} \cdot \{(X, Y)\} \;+\; \mathcal{C}^{\nwarrow} \cdot \mathcal{W} \cdot \{(X, Y)\}$$

and consequently the following equation for its refined generating function:

$$\begin{aligned} M(t; x, y) = F(t; x, y) \;&+\; \widetilde{C}^{\swarrow}(t; x, y) \cdot W(t; x, y) \cdot txy \\ &+\; C^{\nwarrow}(t; x, y) \cdot W(t; x, y) \cdot txy \end{aligned}$$

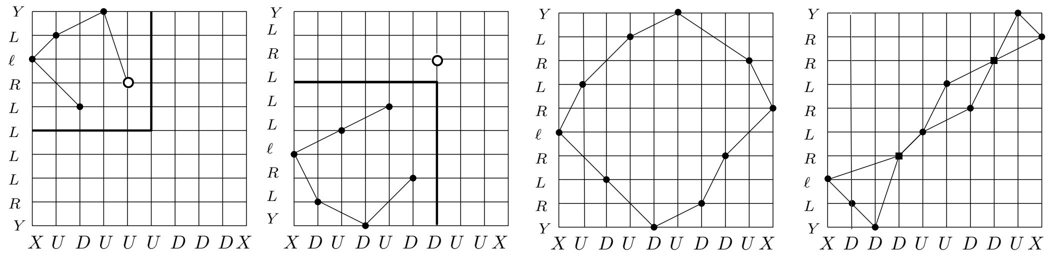


Fig. 10: Four examples of triples (u, v, ℓ). The third one and the fourth one belong to the class of colored co-indecomposable square permutations.

where $\widetilde{C}^{\swarrow}(t; x, y)$ and $C^{\nwarrow}(t; x, y)$ can obtained by Formula (12)–(11) via Equation (9). In particular for $x = y = 1$ we see that the (known) can be written in a similar form as $Sq(t)$ and $Cp(t)$:

$$F(t; 1, 1) = \frac{t^2}{1-4t}\left(1 + \frac{2t}{1-4t}\right) - \frac{t^2}{(1-4t)^{3/2}}$$

6 The encoding for convex permutominoes and a decoding algorithm

The decoding algorithm for convex permutominoes, viewed as colored co-indecomposable square permutations, is similar to the decoding algorithm for square permutations, with slightly different stopping conditions (see Figure 10 for an example):

- in Cases 3 and 4 above, one should stop as soon as $\sigma_1 \ldots \sigma_{i-1}$ is contained in the grid $(i-1)\times(i-1)$ attached to the upper-left corner, regardless of the other conditions, to eliminate co-decomposable permutations;
- in Case 5 above, one should declare the insertion invalid only if $j \neq i$, to allow for colored fixed points.

As a result, we obtain the bijection

$$\mathcal{M} \;\equiv\; \mathcal{C}p + \mathcal{D}^{\nwarrow +} \cdot W \cdot \{(X, Y)\} + \mathcal{D}^{\swarrow} \cdot \{(D, L)\} \cdot W \cdot \{(X, Y)\}$$

where $\mathcal{D}^{\swarrow}$ denotes the subclass of colored permutations that can be obtained from colored square permutations ending with a maximal value by removing this maximal value, and $\mathcal{D}^{\nwarrow +}$ denotes the subclass of co-indecomposable colored permutations that can be obtained from colored co-indecomposable square permutations ending with a minimal value by removing this last value and standardizing the permutation.

In order to deal with $\mathcal{D}^{\nwarrow +}$ and $\mathcal{D}^{\swarrow}$ we use again that the symmetric permutation classes $\mathcal{D}^{\searrow +}$ and $\mathcal{D}^{\nearrow}$ can be obtained as adapted special cases of the above construction of $\mathcal{C}p$:

- $\mathcal{D}^{\searrow +}$, co-indecomposable colored square permutations starting with the maximal value ($\sigma_1 = n$ for a permutation of size n):

$$\{(w, n) \in \mathcal{M}\} \equiv\; (\bullet\mathcal{D}^{\searrow +}) + (\bullet\mathcal{Q}^{\searrow +}) \cdot W \cdot \{(X, Y)\} + \bullet \cdot \{(D, L), (D, R)\} \cdot W \cdot \{(X, Y)\}$$

where $\mathcal{Q}^{\searrow+}$ denotes the class of permutations obtained from co-indecomposable permutations of $\mathcal{C}p$ that start with the maximal value and end with the minimal value by removing these extremal values.

- $\mathcal{D}^{\nearrow}$, colored square permutations starting with the minimal value ($\sigma \in \mathcal{S}q$ s.t. $\sigma_1 = 1$), are obtained when the decoding is applied to pairs $(w, 1)$:

$$\{(w,1) \in \mathcal{M}\} \equiv \ (\bullet\mathcal{D}^{\nearrow}) + (\bullet(1+\mathcal{Q}^{\nearrow})) \cdot \{(D,L)\} \cdot W \cdot \{(X,Y)\}$$

where $\mathcal{Q}^{\nearrow}$ denotes the class of permutations that are obtained from colored permutations of $\mathcal{C}p$ that start with the minimal value and end with the maximal value by removing these extremal values.

Finally the classes $\mathcal{Q}^{\searrow}$ and $\mathcal{Q}^{\nearrow}$ are again simple variants of the class of permutations avoiding the pattern 123, that for which various algebraic decompositions or bijective constructions are available.

Generating functions Taking into account the U, X, L and Ys, the above bijection translate into generating function relations. For convex permutominoes:

$$\begin{aligned} M(t;x,y) = Cp(t;x,y) \ &+ \ \widetilde{D}^{\swarrow}(t;x,y) \cdot ty \cdot W(t;x,y) \cdot txy \\ &+ \ D^{\nwarrow+}(t;x,y) \cdot W(t;x,y) \cdot txy. \end{aligned}$$

For the set $D^{\searrow}$, and $\mathcal{D}^{\nearrow+}$, the bijection yields

$$txy \cdot W(t;x,y) \cdot txy = D_1^{\nearrow}(t;x,y) + Q^{\nearrow}(t;x,y) \cdot ty \cdot W(t;x,y) \cdot txy$$
$$txy \cdot W(t;x,y) \cdot txy = D_1^{\searrow+}(t;x,y) + txy \cdot t(1+y) \cdot W(t;x,y) \cdot txy + Q^{\searrow+}(t;x,y) \cdot W(t;x,y) \cdot txy$$

The rest of the computation is analogous to the case of square permutations using symmetries, and leads again to a parametrization in terms of the same refined Narayana series $N(t;x,y)$ and $N(t;xy,1)$.

Acknowledgements

We warmly thank the referees for their constructive suggestions as well as Gilles Schaeffer for his help during the preparation of this paper.

References

[1] M. H. Albert, S. Linton, N. Ruškuc, V. Vatter, and S. Waton, On convex permutations, *Discrete Math.* **311** (2011), 715–722.

[2] A. Bernini, F. Disanto, R. Pinzani, and S. Rinaldi, Permutations defining Convex Permutominoes, *Journal of Integer Sequences*, **10** (2007), Article 07.9.7

[3] P. Boldi, V. Lonati, R. Radicioni, M. Santini, The number of convex permutominoes, Proc. of *LATA 2007, International Conference on Language and Automata Theory and Applications*, Tarragona, Spain, (2007).

[4] J. Borga, E. Slivken, Square permutations are typycally rectangular, *https://arxiv.org/abs/1904.03080* (2019).

[5] M. Bousquet-Mélou, R. Brak, Exactly Solved Models, in A. J. Guttmann (Ed.), *Polygons, Polyominoes and Polycubes*, Lecture Notes in Physics (2009) Springer–Verlag 43–78.

[6] M. Bousquet-Mélou, A. J. Guttmann, Enumeration of three-dimensional convex polygons, *Ann. Comb.* **1** (1997) 27–53.

[7] F. Disanto, A. Frosini, R. Pinzani, S. Rinaldi, A closed formula for the number of convex permutominoes, *El. J. Combinatorics*, **14** (2007) #R57.

[8] E. Duchi, D. Poulalhon, On square permutations, in *Fifth Colloquium on Mathematics and Computer Science: Proc. MathInfo 2008* Discrete Math. Theor. Comput. Sci., Vol. 97, 2008, pp. 207–222.

[9] T. Mansour, S. Severini, Grid polygons from permutations and their enumeration by the kernel method, in *Proc. of the 19-th International Conference on Formal Power Series and Algebraic Combinatorics*, 2007.

[10] S. Waton, On permutation classes generated by token passing networks, gridding Matrices and pictures: three flavours of involvement, *PhD Thesis*, University of St. Andrews, Scotland, 2007.