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Extremal digraphs on Meyniel-type condition for hamiltonian cycles in balanced bipartite digraphs

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Let D be a strong balanced digraph on $2a$ vertices. Adamus et al. have proved that D is hamiltonian if $d(u)+d(v)\geq 3a$ whenever $uv\notin A(D)$ and $vu\notin A(D)$. The lower bound $3a$ is tight. In this paper, we shall show that the extremal digraph on this condition is two classes of digraphs that can be clearly characterized. Moreover, we also show that if $d(u)+d(v)\geq 3a-1$ whenever $uv\notin A(D)$ and $vu\notin A(D)$, then D is traceable. The lower bound $3a-1$ is tight.

Keywords: bipartite digraph; degree sum condition; hamiltonian cycle

1 Terminology and introduction

In this paper, we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [5] for terminology not defined here. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x,y\in V(D)$, we will write $x\rightarrow y$ if $xy\in A(D)$, also write $x\leftrightarrow y$ if $x\rightarrow y$ and $y\rightarrow x$. For disjoint subsets X and Y of $V(D)$, $X\rightarrow Y$ means that every vertex of X dominates every vertex of Y, $X\Rightarrow Y$ means that there is no arc from Y to X and $X\mapsto Y$ means that both of $X\rightarrow Y$ and $X\Rightarrow Y$ hold. For a vertex set $S\subset V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ dominated by the vertices of S; i.e. $N^+(S)=\{u\in V(D): vu\in A(D)$ for some $v\in S\}$. Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ dominating vertices of S; i.e. $N^-(S)=\{u\in V(D): uv\in A(D)$ for some $v\in S\}$. If $S=\{v\}$ is a single vertex, the cardinality of $N^+(v)$ (resp. $N^-(v)$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the out-degree (resp. in-degree) of v in D. The degree of v is $d(v)=d^+(v)+d^-(v)$. For a pair of vertex sets X,Y of D, define $(X,Y)=\{xy\in A(D): x\in X, y\in Y\}$. Let $\overleftrightarrow{a}(X,Y)=|(X,Y)|+|(Y,X)|$.

Let $P=y_0y_1\dots y_k$ be a (y_0,y_k)-path of D. For $i\neq j$, $y_i,y_j\in V(P)$ we denote by y_iPy_j the subpath of P from y_i to y_j. If $0<i\leq k$, then the predecessor of y_i on P is the vertex y_{i-1} and is also denoted by y_i^-. If $0\leq i<k$, then the successor of y_i on P is the vertex y_{i+1} and is also denoted by y_i^+. A k-cycle is a cycle of order k. A cycle factor in D is a collection of vertex-disjoint cycles $C_1,C_2,\dots,C_t$ such that $V(C_1)\cup V(C_2)\cup\dots\cup V(C_t)=V(D)$.

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A digraph D is said to be strongly connected or just strong, if for every pair of vertices x, y of D, there is a path with endvertices x and y. A digraph D is called hamiltonian if it contains a hamiltonian cycle, i.e., a cycle that includes every vertex of D. A digraph D is traceable if D possesses a hamiltonian path. A digraph D is semicomplete, if for every pair of vertices x, y of D, xy or yx (or both) is in D.

A digraph D is bipartite when $V(D)$ is a disjoint union of independent sets V_1 and V_2. It is called balanced if $|V_1| = |V_2|$. A matching from V_1 to V_2 is an independent set of arcs with origin in V_1 and terminus in V_2 (u_1u_2 and v_1v_2 are independent arcs when $u_1 \neq v_1$ and $u_2 \neq v_2$). If D is balanced, one says that such a matching is perfect if it consists of precisely $|V_1|$ arcs. If D is bipartite and for every pair of vertices x, y from distinct partite sets, xy and yx are in D, then D is called complete bipartite.

The cycle problems for digraphs are one of the central problems in graph theory and its applications [5]. There are many degree or degree sum conditions for hamiltonicity in digraphs. The following result of Meyniel on the existence of hamiltonian cycles in digraphs is basic and famous.

Theorem 1.1 *[11] Let D be a strong digraph on n vertices where $n \geq 3$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices x, y in D, then D is hamiltonian.*

Recently, there is a renewed interest in various degree conditions for hamiltonicity in bipartite digraphs (see, e.g., [1], [2], [3], [4], [7], [10], [12], [13], [14]). In particular, In [4], Adamus et al. gave a Meyniel-type sufficient condition for hamiltonicity of balanced bipartite digraphs.

Definition 1.2 *Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. For an integer k, we will say that D satisfies the condition M_k when $d(u) + d(v) \geq 3a + k$, for all pairs of non-adjacent vertices u, v.*

Theorem 1.3 *[4] Let D be a balanced bipartite digraph on $2a$ vertices, where $a \geq 2$. Then D is hamiltonian provided one of the following holds:*

(a) *D satisfies the condition M_1, or*

(b) *D is strong and satisfies the condition M_0.*

In Section 3, we reduce the bound in Theorem 1.3(b) by 1 and prove that D is either hamiltonian or isomorphic to a digraph in $\mathcal{H}_1$ or the digraph H_2, see Examples 1.4 and 1.5 below. From this, we determine the extremal digraph of Theorem 1.3(b). We also prove that a strong balanced bipartite digraph of order $2a$ satisfying the condition M_{-1} is traceable. Our proofs are based on the arguments of [4].

Example 1.4 *For an odd integer $a \geq 3$, let $\mathcal{H}_1$ be a set of bipartite digraphs. For any digraph H_1 in $\mathcal{H}_1$, let V_1 and V_2 be partite sets of H_1 such that V_1 (resp. V_2) is a disjoint union of S, R (resp. U, W) with $|S| = |W| = \frac{a+1}{2}$, $|U| = |R| = \frac{a-1}{2}$ and $A(H_1)$ consists of the following arcs:*

(a) *rw and wr, for all $r \in R$ and $w \in W$;*

(b) *us and su, for all $u \in U$ and $s \in S$;*

(c) *ws, for all $w \in W$ and $s \in S$;*

(d) *there exist $r \in R$ and $u \in U$ such that $ur \in A(H_1)$. For every $r \in R$, $d_{H_1[U]}(r) \geq \frac{a-3}{2}$ and for every $u \in U$, $d_{H_1[R]}(u) \geq \frac{a-3}{2}$.*

Note that H_1 is strong and satisfies the condition M_{-1}, but since $|N^+(S)| = |U| < |S|$, there exists no perfect matching from V_1 to V_2. Thus, H_1 is non-hamiltonian.

Example 1.5 *Let H_2 be a bipartite digraph with partite sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. The arc set $A(H_2)$ consist of the following arcs x_1y_2, y_2x_3, x_3y_3, y_3x_1 and the following 2-cycles $x_2 \leftrightarrow y_2$, $x_2 \leftrightarrow y_3$, $y_1 \leftrightarrow x_1$ and $y_1 \leftrightarrow x_3$. Note that H_2 is strong and the degree of every vertex in H_2 is 4. Thus H_2 satisfies the condition M_{-1} as $a = 3$. Observe that H_2 is non-hamiltonian (see Figure 1).*

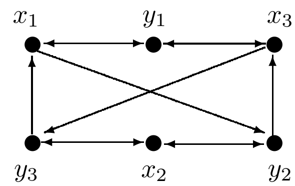

Figure 1. the digraph H_2.

2 Lemmas

The proof of the main result will be based on the following several lemmas.

Lemma 2.1 *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 2$. If D satisfies the condition M_{-1}, then either D contains a cycle factor or D is isomorphic to a digraph in $\mathcal{H}_1$.*

Proof: Let V_1 and V_2 denote two partite sets of D. Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a prefect matching from V_2 to V_1. In order to prove that D contains a perfect matching from V_1 to V_2 and a prefect matching from V_2 to V_1, by the Hall theorem, it suffices to show that $|N^+(S)| \geq |S|$ for every $S \subset V_1$ and $|N^+(T)| \geq |T|$ for every $T \subset V_2$.

If there exists a non-empty set $S \subset V_1$ such that $|N^+(S)| < |S|$, then we will show that D is isomorphic to a digraph in $\mathcal{H}_1$. Note that $V_2 \setminus N^+(S) \neq \emptyset$. If $|S| = 1$, write $S = \{x\}$, then $|N^+(S)| < |S|$ implies that $d^+(x) = 0$. It is impossible in a strong digraph. Thus $|S| \geq 2$. If $|S| = a$, then every vertex from $V_2 \setminus N^+(S)$ has in-degree zero, which again contradicts strong connectedness of D. Therefore, $2 \leq |S| \leq a - 1$.

For any $x_1, x_2 \in S$ and $w_1, w_2 \in V_2 \setminus N^+(S)$, by the hypothesis of the lemma,

$$3a - 1 \leq d(x_1) + d(x_2) \leq 2|N^+(S)| + 2a \tag{1}$$

and

$$3a - 1 \leq d(w_1) + d(w_2) \leq 2a + 2(a - |S|). \tag{2}$$

From these, we have $|N^+(S)| \geq \frac{a-1}{2}$ and $|S| \leq \frac{a+1}{2}$. If a is even, then $|N^+(S)| \geq \frac{a}{2}$ and $|S| \leq \frac{a}{2}$, which is a contradiction to $|N^+(S)| < |S|$. Thus a is odd and $\frac{a-1}{2} \leq |N^+(S)| \leq |S| - 1 \leq \frac{a+1}{2} - 1 = \frac{a-1}{2}$, which means $|N^+(S)| = \frac{a-1}{2}$ and $|S| = \frac{a+1}{2}$. Moveover, all equalities hold in (1) and (2), which means that $d^+(x_1) = d^+(x_2) = |N^+(S)|$, $d^-(x_1) = d^-(x_2) = a$, $d^-(w_1) = d^-(w_2) = a - |S|$ and $d^+(w_1) = d^+(w_2) = a$. By the strong connectedness of D and the hypothesis of this lemma, D is isomorphic to a digraph in $\mathcal{H}_1$. □

From the proof of Theorem 1.2 in [4], we have the following lemma. We provide its proof for completeness.

Lemma 2.2 *Let D be a bipartite digraph with partite sets V_1 and V_2. Suppose that C_i and C_j are two vertex-disjoint cycles in D. If C_i and C_j cannot be mergered into a cycle with vertex set $V(C_i) \cup V(C_j)$, then $\overleftrightarrow{a}(V(C_i), V(C_j)) \leq \frac{|V(C_i)|\cdot|V(C_j)|}{2}$. Moreover, if $\overleftrightarrow{a}(V(C_i), V(C_j)) = \frac{|V(C_i)|\cdot|V(C_j)|}{2}$, then for any $x_i \in V(C_i) \cap V_q$ and $x_j \in V(C_j) \cap V_q$, $|\{x_i x_j^+, x_j x_i^+\} \cap A(D)| = 1$, with $q \in \{1, 2\}$.*

Proof: Let $q \in \{1, 2\}, x_i \in V(C_i) \cap V_q$ and $x_j \in V(C_j) \cap V_q$ be arbitrary. Let x_i^+ be the successor of x_i in C_i and let x_j^+ be the successor of x_j in C_j. Let $\mathcal{Z}_q(x_i, x_j)$ be defined as $\{x_i x_j^+, x_j x_i^+\} \cap A(D)$. If $|\mathcal{Z}_q(x_i, x_j)| = 2$ for some x_i, x_j, then the cycles C_i and C_j can be merged into one cycle by deleting the arcs $x_i x_i^+$ and $x_j x_j^+$ and adding the arcs $x_i x_j^+$ and $x_j x_i^+$, a contradiction. So we may assume that

$$|\mathcal{Z}_q(x_i, x_j)| \leq 1, \text{ for all } x_i \in V(C_i) \cap V_q \text{ and } x_j \in V(C_j) \cap V_q. \tag{1}$$

Now, consider an arc $uv \in (V(C_i), V(C_j))$ and assume $u \in V_q$. Let v^- denote the predecessor of v in C_j. Then $uv \in \mathcal{Z}_q(u, v^-)$. Similarly, if $uv \in (V(C_j), V(C_i))$, $u \in V_q$, and v^- is the predecessor of v in C_i, then $uv \in \mathcal{Z}_q(v^-, u)$. Therefore

$$\overleftrightarrow{a}(V(C_i), V(C_j)) \leq \sum_{q=1}^{2} \sum_{x_i \in V(C_i) \cap V_q} \sum_{x_j \in V(C_j) \cap V_q} |\mathcal{Z}_q(x_i, x_j)|,$$

and hence, by (1),

$$\overleftrightarrow{a}(V(C_i), V(C_j)) \leq 2 \cdot \frac{|V(C_i)|}{2} \cdot \frac{|V(C_j)|}{2}.$$

Moreover, if $\overleftrightarrow{a}(V(C_i), V(C_j)) = \frac{|V(C_i)|\cdot|V(C_j)|}{2}$, then the equality holds in (1), that is to say, $|\{x_i x_j^+, x_j x_i^+\} \cap A(D)| = 1$, which completes the proof of the lemma. □

The next lemma shows two simple results.

Lemma 2.3 *Let $a_1, a_2, \ldots, a_t$ be non-negative integers with $a_1 \leq a_2 \leq \cdots \leq a_t$ and let A be a positive integer. If $a_1 + a_2 + \cdots + a_t \leq A$, then the following hold.*

(a) For any $l \in \{1, 2, \ldots, t\}$, $a_1 + a_2 + \cdots + a_l \leq \frac{lA}{t}$;

(b) If $a_1 + a_2 = \frac{2A}{t}$, then for all $i \neq j \in \{1, 2, \ldots, t\}$, we have $a_i = \frac{A}{t}$, $a_i + a_j = \frac{2A}{t}$ and $a_1 + a_2 + \cdots + a_t = A$.

Proof: (a) For a proof by contradiction, suppose that $a_1 + a_2 + \cdots + a_l > \frac{lA}{t}$. Then $a_l > \frac{A}{t}$, as otherwise $a_1 \leq a_2 \leq \cdots \leq a_l \leq \frac{A}{t}$ implies $a_1 + a_2 + \cdots + a_l \leq \frac{lA}{t}$, a contradiction. Then $\frac{A}{t} < a_l \leq a_{l+1} \leq \cdots \leq a_t$ implies $\frac{(t-l)A}{t} < a_{l+1} + \cdots + a_t \leq A - (a_1 + a_2 + \cdots + a_l) < A - \frac{lA}{t} = \frac{(t-l)A}{t}$, a contradiction. Hence $a_1 + a_2 + \cdots + a_l \leq \frac{lA}{t}$.

(b) If $t = 2$, then there is nothing to prove. Now assume $t \geq 3$. First $a_1 = a_2 = \frac{A}{t}$, as otherwise $a_2 > \frac{A}{t}$ implies $a_i > \frac{A}{t}$, for all $i \geq 3$. Then $\frac{(t-2)A}{t} < a_3 + \cdots + a_t \leq A - (a_1 + a_2) = \frac{(t-2)A}{t}$, a contradiction. So $a_i \geq \frac{A}{t}$, for all $i \geq 3$. Then $\frac{(t-2)A}{t} \leq a_3 + \cdots + a_t \leq A - (a_1 + a_2) = \frac{(t-2)A}{t}$. It follows that all equalities hold. Then $a_i = \frac{A}{t}$ for all $i \in \{1, 2, \ldots, t\}$. So $a_i + a_j = \frac{2A}{t}$ and $a_1 + a_2 + \cdots + a_t = A$. □

Theorem 2.4 *[8, 9] Let D be a strong semicomplete bipartite digraph. If D contains a cycle factor, then D is hamiltonian.*

3 Proof of the main result

Theorem 3.1 *Let D be a strong balanced bipartite digraph of order $2a$ with partite sets V_1 and V_2, where $a \geq 3$. If D satisfies the condition M_{-1}, then D is either hamiltonian or isomorphic to a digraph in $\mathcal{H}_1$ or the digraph H_2.*

Proof: Suppose that D is not isomorphic to a digraph in $\mathcal{H}_1$. By Lemma 2.1, D contains a cycle factor $C_1, C_2, \ldots, C_s$. Assume that s is minimum possible and D is not hamiltonian. So $s \geq 2$. Without loss of generality, assume that $|V(C_1)| \leq |V(C_2)| \leq \cdots \leq |V(C_s)|$. Clearly, $|V(C_1)| \leq a$. Denote $\overline{C}_1 = D - V(C_1)$. By Lemma 2.2, the following holds:

$$\begin{aligned}
&\overleftrightarrow{a}(V(C_1)\cap V_1, V(\overline{C}_1)) + \overleftrightarrow{a}(V(C_1)\cap V_2, V(\overline{C}_1)) \\
&= \overleftrightarrow{a}(V(C_1), V(\overline{C}_1)) = \sum_{i=2}^{s} \overleftrightarrow{a}(V(C_1), V(C_i)) \qquad (1)\\
&\leq \frac{|V(C_1)|(2a - |V(C_1)|)}{2}.
\end{aligned}$$

Without loss of generality, we may assume that

$$\overleftrightarrow{a}(V(C_1)\cap V_1, V(\overline{C}_1)) \leq \frac{|V(C_1)|(2a - |V(C_1)|)}{4}, \qquad (2)$$

as otherwise

$$\overleftrightarrow{a}(V(C_1)\cap V_2, V(\overline{C}_1)) \leq \frac{|V(C_1)|(2a - |V(C_1)|)}{4}. \qquad (3)$$

To complete the proof, we first give the following two claims.

Claim 1. For any two non-adjacent vertices x and y, if $d(x) \leq b$, then $d(y) \geq 3a - 1 - b$.

Proof: By the hypothesis of this theorem, $d(x) + d(y) \geq 3a - 1$. This together with $d(x) \leq b$ implies $d(y) \geq 3a - 1 - b$. □

Claim 2. If $s = 2$ and $D[V(C_1)]$ is either a complete bipartite digraph, or a complete bipartite digraph minus one arc with $|V(C_1)| \geq 6$, then there exists a vertex $z \in V(C_2)$ such that $d_{C_1}(z) = 0$.

Proof: Suppose, on the contrary, that for every $z \in V(C_2)$, $d_{C_1}(z) > 0$, where $d_{C_1}(z) = |N^+(z) \cap V(C_1)| + |N^-(z) \cap V(C_1)|$. Since D is strong, there exist arcs from C_2 to C_1. Without loss of generality, assume that $v \to x$, where $v \in V(C_2) \cap V_2$ and $x \in V(C_1) \cap V_1$. Let y be an arbitrary vertex in $V(C_1) \cap V_2$.

First, we observe that there exists a hamiltonian path Q from x to y in $D[V(C_1)]$. If $D[V(C_1)]$ is a complete bipartite digraph, it is obvious. Assume that $D[V(C_1)]$ is a complete bipartite digraph minus one arcs, say e, with $|V(C_1)| \geq 6$. Denote $m = \frac{|V(C_1)|}{2}$. Let x_iy_i, $i = 1, 2, \ldots, m$, be a perfect matching from V_1 to V_2 in $D[V(C_1)]$. Without loss of generality, assume $x = x_1$ and $y = y_m$ (This is possible as there are at least $m - 1$ arc-disjoint perfect matchings from V_1 to V_2 in $D[V(C_1)]$ and

$m \geq 3$). Let $P = x_1y_1x_2y_2 \ldots x_my_m$. If $e \notin \{y_ix_{i+1} : i = 1, 2, \ldots, m-1\}$, then P is the desired path. If $e \in \{y_ix_{i+1} : i = 1, 2, \ldots, m-1\}$, say $e = y_rx_{r+1}$, then $P \cup \{x_1y_r, y_rx_2, x_ry_1, y_1x_{r+1}\} \setminus \{x_1y_1, y_1x_2, x_ry_r, y_rx_{r+1}\}$ is the desired path.

According to the above observation, we can deduce that $y \nrightarrow v^+$, where v^+ is the successor of v in C_2, otherwise $vxQyv^+C_2v$ is a hamiltonian cycle of D, a contradiction. By the arbitrariness of y, this means that $d^-_{C_1}(v^+) = 0$. By $d_{C_1}(v^+) > 0$, we have $d^+_{C_1}(v^+) > 0$. Similarly, we can obtain that $d^-_{C_1}(w) = 0$, for every $w \in V(C_2)$, a contradiction to the fact that D is strong. The proof of the claim is complete. □

We now consider the following two cases.

Case 1. $|V(C_1)| = 2$.

Let $V(C_1) \cap V_1 = \{x_1\}$ and $V(C_1) \cap V_2 = \{y_1\}$. By (1),

$$d_{\overline{C}_1}(x_1) + d_{\overline{C}_1}(y_1) \leq 2a - 2 \tag{4}$$

and by (2),

$$d_{\overline{C}_1}(x_1) \leq a - 1. \tag{5}$$

So $d(x_1) = d_{C_1}(x_1) + d_{\overline{C}_1}(x_1) \leq a + 1$.

Assume $d(x_1) \leq a$. By Claim 1, $d(z) \geq 3a - 1 - a = 2a - 1$, for any $z \in V(D)$ such that z and x_1 are non-adjacent. It is easy to see that D is a semicomplete bipartite digraph. By Theorem 2.4, D is hamiltonian, a contradiction.

Now assume $d(x_1) = a + 1$. By Claim 1, for any $x' \in V_1 \setminus \{x_1\}$, $d(x') \geq 3a - 1 - (a + 1) = 2a - 2$. By (4) and $d_{\overline{C}_1}(x_1) = a - 1$, $d_{\overline{C}_1}(y_1) \leq a - 1$ and so $d(y_1) \leq a + 1$. Similarly, we can also obtain $d(y_1) = a + 1$. Hence, for any $y' \in V_2 \setminus \{y_1\}$, $d(y') \geq (3a - 1) - (a + 1) = 2a - 2$. In fact, we have shown that for any $w \in V(D) \setminus \{x_1, y_1\}$, $d(w) \geq 2a - 2$.

Assume that $|V(C_2)| = 2$. Write $C_2 = x_2y_2x_2$, where $x_2 \in V_1$ and $y_2 \in V_2$. Analogously, we can also obtain that $d(x_2) = d(y_2) = a+1$. Note that $d(x_2) \geq 2a-2$. Thus, $2a-2 \leq d(x_2) = a+1$. From this, we have $a \leq 3$ and so $a = 3$ and $s = 3$. Write $C_3 = x_3y_3x_3$, where $x_3 \in V_1$ and $y_3 \in V_2$. Analogously, we can also obtain that $d(x_3) = d(y_3) = a + 1$. By Theorem 2.4, D is not a semicomplete bipartite digraph. Hence, there exist two vertices from different partite sets such that they are not adjacent. Without loss of generality, assume that x_1 and y_2 are not adjacent. By $d(x_1) = a + 1 = 4$ and $d(y_2) = a + 1 = 4$, we have that $x_1 \leftrightarrow y_3$ and $x_3 \leftrightarrow y_2$. By Lemma 2.2, $\overleftrightarrow{a}(V(C_2), V(C_3)) \leq 2$. So x_2 and y_3 are not adjacent. Then $d(x_2) = a + 1 = 4$ implies that $x_2 \leftrightarrow y_1$. Note that D is hamiltonian, a contradiction (see Figure 2).

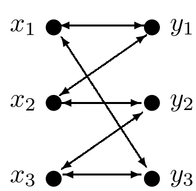

Figure 2. The case when $|V(C_2)| = 2$.

Next assume that $|V(C_2)| \geq 4$. From this, $|V(C_i)| \geq 4$, for $i = 3, \ldots, s$. Let $D' = D - \{x_1, y_1\}$ and $a' = a - 1$. First we claim that $s = 2$. It suffices to show that D' is hamiltonian. For $a' = 2$ and

$a' = 3$, it is obvious. Recall that for any $u \in V(D')$, $d_D(u) \geq 2a-2$. Thus, $d_{D'}(u) \geq 2a-4 = 2a'-2$. Thus, for any two non-adjacent vertices u and v in D', $d_{D'}(u) + d_{D'}(v) \geq 2(2a'-2)$. If $a' \geq 5$, then $2(2a'-2) \geq 3a'+1$. By Theorem 1.3(a), D' is hamiltonian. If $a' = 4$, then $2(2a'-2) \geq 3a'$. If D' is strong, then by Theorem 1.3(b), D' is hamiltonian. Next assume that D' is not strong. In this case, $s = 3$ and C_2, C_3 are both 4-cycles. Write $C_2 = x_2y_2x_3y_3x_2$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 2, 3$. Since D' is not strong, without loss of generality, assume that $C_2 \Rightarrow C_3$. So $d_{D'}(x_2) \leq 2+4 = 2a'-2$ and $d_{D'}(y_2) \leq 2a'-2$. Combining this with $d_{D'}(x_2) \geq 2a'-2$ and $d_{D'}(y_2) \geq 2a'-2$, we have that $d_{D'}(x_2) = d_{D'}(y_2) = 2a'-2 = 2a-4$. Recall that $d_D(x_2) \geq 2a-2$ and $d_D(y_2) \geq 2a-2$. So $x_2 \leftrightarrow y_1$ and $x_1 \leftrightarrow y_2$. This means that C_1 can be merged with C_2 by replacing the arc x_2y_2 on C_2 with the path $x_2y_1x_1y_2$, a contradiction. Hence $s = 2$. Write $C_2 = x_2y_2 \ldots x_ay_ax_2$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 2, \ldots, a$. By Claim 2, there exists a vertex $z \in V(C_2)$ such that $d_{C_1}(z) = 0$, say x_2. Thus, x_2 and y_1 are not adjacent and $d(x_2) \leq 2a-2$. From this with $d(x_2) \geq 2a-2$, we have that $d(x_2) = 2a-2$, which implies that $x_2 \leftrightarrow y_i$, for $i = 2, \ldots, a$. Recalling that $d_{C_2}(x_1) = d_{C_2}(y_1) = a-1$, that is to say, $\overleftrightarrow{a}(V(C_1), V(C_2)) = 2(a-1) = \frac{|V(C_1)|\cdot|V(C_2)|}{2}$. By Lemma 2.2, for any $x_i, y_i \in V(C_2)$,

$$|\{x_iy_1, x_1y_i\} \cap A(D)| = 1 \text{ and } |\{y_{i-1}x_1, y_1x_i\} \cap A(D)| = 1. \tag{6}$$

Since y_1 and x_2 are not adjacent, by (6), we have $y_a \to x_1$ and $x_1 \to y_2$.

First consider the case when $a = 3$. By $d_{C_2}(x_1) = d_{C_2}(y_1) = a-1$, we have $x_1 \mapsto y_2$ and $y_3 \mapsto x_1$ and $y_1 \leftrightarrow x_3$. If $x_3 \to y_2$, then $x_3y_2x_2y_3x_1y_1x_3$ is a hamiltonian cycle, a contradiction. Hence, $y_2 \mapsto x_3$. If $y_3 \to x_3$, then $y_3x_3y_1x_1y_2x_2y_3$ is a hamiltonian cycle, a contradiction. Hence $x_3 \mapsto y_3$. Then D is isomorphic to the digraph H_2 (see Figure 1.)

Next consider the case when $a \geq 4$. Assume that $x_a \to y_1$. By (6), $x_1 \nrightarrow y_a$. Furthermore, $y_a \nrightarrow x_3$, otherwise $x_1y_2x_2y_ax_3C_2x_ay_1x_1$ is a hamiltonian cycle, a contradiction. Hence $d(y_a) \leq 2a-2$. Combining this with $d(y_a) \geq 2a-2$, we have $d(y_a) = 2a-2$, which implies $x_3 \mapsto y_a$ and $y_a \leftrightarrow x_i$, for $i = 4, \ldots, a$. Moreover, $y_1 \nrightarrow x_3$, otherwise $y_1x_3y_ax_1y_2x_2y_3C_2x_ay_1$ is a hamiltonian cycle, a contradiction. From this, we see that $d(x_3) \leq 2a-2$. Combining this with $d(x_3) \geq 2a-2$, we have $d(x_3) = 2a-2$, which implies $y_3 \leftrightarrow x_3$. However, $x_3y_ax_4C_2x_ay_1x_1y_2x_2y_3x_3$ is a hamiltonian cycle, a contradiction. Now we assume $x_a \nrightarrow y_1$. Since $d_{C_2}(y_1) = a-1$ and y_1 and x_2 are not adjacent, there exists a vertex $x_i \in \{x_3, \ldots, x_{a-1}\}$ such that $x_i \to y_1$. Take $r = \max\{i : i \in \{3, \ldots, a-1\}$ and $x_i \to y_1\}$. By the choose of r, for every $j \in \{r+1, \ldots, a\}$, $x_j \nrightarrow y_1$. Then by (6), $x_1 \to y_j$. If $x_j \to y_2$, then $x_ry_1x_1y_jC_2x_2y_rC_2x_jy_2C_2x_r$ is a hamiltonian cycle, a contradiction. Hence $x_j \nrightarrow y_2$. Combining this with $x_j \nrightarrow y_1$ and $d(x_j) \geq 2a-2$, we have $d(x_j) = 2a-2$. Hence $x_j \to \{y_j, y_{j-1}\} \to x_j$. But $x_ry_1x_1y_ax_ay_{a-1} \ldots y_rx_2C_2x_r$ is a hamiltonian cycle, a contradiction.

Case 2. $|V(C_1)| \geq 4$.

In this case, $a \geq 4$. Let $x_1, x_2 \in V(C_1) \cap V_1$ be distinct and chosen so that $\overleftrightarrow{a}(\{x_1, x_2\}, V(\overline{C}_1))$ is minimum. By Lemma 2.3(a) and (2), $\overleftrightarrow{a}(\{x_1, x_2\}, V(\overline{C}_1)) \leq 2a - |V(C_1)|$, that is to say, $d_{\overline{C}_1}(x_1) + d_{\overline{C}_1}(x_2) \leq 2a - |V(C_1)|$. Since any vertex in C_1 has at most $|V(C_1)|$ arcs to other vertices in C_1 (as there are $\frac{|V(C_1)|}{2}$ vertices from V_2 in C_1) and $|V(C_1)| \leq a$, we get $3a-1 \leq d(x_1)+d(x_2) \leq 2a+|V(C_1)| \leq 3a$. From this $|V(C_1)| = a-1$ or $|V(C_1)| = a$. Before we consider these two cases, we claim the following. Clearly, $s = 2$.

Claim 3. For any $u \in V(C_2)$, $d_{C_1}(u) > 0$.

Proof: Suppose, on the contrary, that there exists $u_0 \in V(C_2)$ such that $d_{C_1}(u_0) = 0$. Then $3a - 1 \leq d(u_0) + d(x_i) \leq |V(C_2)| + |V(C_1)| + d_{C_2}(x_i)$, for $i = 1, 2$. From this, $d_{C_2}(x_i) \geq a - 1$. Thus $2(a-1) \leq d_{C_2}(x_1) + d_{C_2}(x_2) \leq 2a - |V(C_1)|$, which means $|V(C_1)| \leq 2$, a contradiction. □

From Claims 2 and 3, we know that $D[V(C_1)]$ is not a complete bipartite digraph. Let $y_1, y_2 \in V(C_1) \cap V_2$ be distinct and chosen such that $\overleftrightarrow{a}(\{y_1, y_2\}, V(C_2))$ is the minimum.

Claim 4. If $d_{C_2}(x_1) + d_{C_2}(x_2) = 2a - |V(C_1)|$, then $d_{C_2}(y_1) + d_{C_2}(y_2) \leq 2a - |V(C_1)|$.

Proof: If $d_{C_2}(x_1) + d_{C_2}(x_2) = 2a - |V(C_1)|$, then by Lemma 2.3(b) and (2), $\overleftrightarrow{a}(V(C_1) \cap V_1, V(C_2)) = \frac{|V(C_1)| \cdot (2a - |V(C_1)|)}{4}$. Then by (1), $\overleftrightarrow{a}(V(C_1) \cap V_2, V(C_2)) \leq \frac{|V(C_1)| \cdot (2a - |V(C_1)|)}{4}$. By Lemma 2.3(a), $\overleftrightarrow{a}(\{y_1, y_2\}, V(C_2)) \leq 2a - |V(C_1)|$, that is, $d_{C_2}(y_1) + d_{C_2}(y_2) \leq 2a - |V(C_1)|$. □

Now we return to the proof of the theorem and consider the following subcases.

Subcase 2.1. $|V(C_1)| = a - 1$.

In this case, $|V(C_1)| = a - 1 \geq 4$, that is $a \geq 5$, and $|V(C_2)| = a + 1$.

Claim 5. For any two non-adjacent vertices $u, v \in V(C_1)$, if $d_{C_2}(u) + d_{C_2}(v) \leq 2a - |V(C_1)|$, then $d_{C_2}(u) + d_{C_2}(v) = 2a - |V(C_1)|$ and $d_{C_1}(u) = d_{C_1}(v) = |V(C_1)|$.

Proof: By hypothesis, $3a - 1 \leq d(u) + d(v) = d_{C_2}(u) + d_{C_2}(v) + d_{C_1}(u) + d_{C_1}(v) \leq 2a - |V(C_1)| + 2|V(C_1)| = 3a - 1$. If follows that $d_{C_2}(u) + d_{C_2}(v) = 2a - |V(C_1)|$ and $d_{C_1}(u) = d_{C_1}(v) = |V(C_1)|$. □

By Claim 5, $d_{C_2}(x_1) + d_{C_2}(x_2) = 2a - |V(C_1)|$ and $d_{C_1}(x_1) = d_{C_1}(x_2) = |V(C_1)|$. By (2) and Lemma 2.3(b), for any $x', x'' \in V(C_1) \cap V_1$, $d_{C_2}(x') + d_{C_2}(x'') = 2a - |V(C_1)|$. By Claim 5, $d_{C_1}(x') = d_{C_1}(x'') = |V(C_1)|$. Then $D[V(C_1)]$ is a complete bipartite digraph, a contradiction.

Subcase 2.2. $|V(C_1)| = a$.

In this case, $|V(C_2)| = a$. By

$$\begin{aligned} 3a - 1 &\leq d(x_1) + d(x_2) \\ &= d_{C_1}(x_1) + d_{C_1}(x_2) + d_{C_2}(x_1) + d_{C_2}(x_2) \\ &\leq 2a + d_{C_2}(x_1) + d_{C_2}(x_2), \end{aligned} \tag{7}$$

we have $d_{C_2}(x_1) + d_{C_2}(x_2) \geq a - 1$. Combining this with $d_{C_2}(x_1) + d_{C_2}(x_2) \leq 2a - |V(C_1)| = a$, we have $d_{C_2}(x_1) + d_{C_2}(x_2) = a - 1$ or $d_{C_2}(x_1) + d_{C_2}(x_2) = a$.

First suppose $d_{C_2}(x_1) + d_{C_2}(x_2) = 2a - |V(C_1)| = a$. By Lemma 2.3(b) and (2), $d_{C_2}(x_i) + d_{C_2}(x_j) = a$, for any $x_i, x_j \in V(C_1) \cap V_1$. Since $D[V(C_1)]$ is not a complete bipartite digraph, there exists a vertex $x' \in V(C_1) \cap V_1$ such that $d_{C_1}(x') \leq a - 1$. For any $x_k \in (V(C_1) \cap V_1) \setminus \{x'\}$, $3a - 1 \leq d(x') + d(x_k) = (d_{C_1}(x') + d_{C_1}(x_k)) + (d_{C_2}(x') + d_{C_2}(x_k)) \leq (a - 1 + a) + a = 3a - 1$. So $d_{C_1}(x') = a - 1$ and $d_{C_1}(x_k) = a$, which implies that $D[V(C_1)]$ is a complete bipartite digraph minus one arc. According to Claims 2 and 3, $|V(C_1)| = 4$. Write $C_1 = x_1 y_1 x_2 y_2 x_1$ and $C_2 = x_3 y_3 x_4 y_4 x_3$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 1, 2, 3, 4$. Without loss of generality, assume that $d_{C_1}(x_1) = 3$ and $y_2 \mapsto x_1$. According to Claim 4, $d_{C_2}(y_1) + d_{C_2}(y_2) \leq a$. Then $3a - 1 \leq d(y_1) + d(y_2) \leq 2a - 1 + a = 3a - 1$ implies that $d_{C_2}(y_1) + d_{C_2}(y_2) = a$, which means $d_{C_1}(x_3) + d_{C_1}(x_4) = a$. By symmetry, we can deduce that $D[V(C_2)]$ is a complete bipartite digraph minus one arc. Without loss of generality, assume that $d_{C_2}(x_3) = a-1$. Then $3a-1 \leq d(x_1)+d(x_3) \leq 2(a-1)+d_{C_2}(x_1)+d_{C_1}(x_3)$, that is, $d_{C_2}(x_1)+d_{C_1}(x_3) \geq a+1$. Without loss of generality, assume that $d_{C_2}(x_1) \geq \frac{a}{2}+1$. Combining this with $d_{C_2}(x_1)+d_{C_2}(x_2) = a$, we have $d_{C_2}(x_2) \leq \frac{a}{2}-1$. Then $3a-1 \leq d(x_2)+d(x_3) \leq (a+\frac{a}{2}-1)+(a-1+d_{C_1}(x_3))$ implies that $d_{C_1}(x_3) \geq \frac{a}{2}+1$. From this with $d_{C_1}(x_3)+d_{C_1}(x_4) = a$, we have $d_{C_1}(x_4) \leq \frac{a}{2}-1$. But $d(x_2)+d(x_4) \leq 2a+2(\frac{a}{2}-1) = 3a-2$, a contradiction.

Now suppose $d_{C_2}(x_1)+d_{C_2}(x_2) = a-1$. From (7), $d_{C_1}(x_1) = d_{C_1}(x_2) = a$. If $a = 4$, then $D[V(C_1)]$ is a complete bipartite digraph, a contradiction. Next assume that $a \geq 6$. By Claims 2 and 3, $D[V(C_1)]$ is not a complete bipartite digraph minus one arc. Denote $V(C_1) \cap V_1 = \{x_1, x_2, \ldots, x_{\frac{a}{2}}\}$ and without loss of generality, assume that $d_{C_2}(x_1) \leq d_{C_2}(x_2) \leq \cdots \leq d_{C_2}(x_{\frac{a}{2}})$. By the choice of x_1 and x_2 and $d_{C_2}(x_1)+d_{C_2}(x_2) = a-1$, we know that $d_{C_2}(x_1) \leq \frac{a}{2}-1$. Denote $d_{C_2}(x_1) = \frac{a}{2}-k$, with $k \geq 1$. So $d_{C_2}(x_2) = \frac{a}{2}+k-1$ and $d_{C_2}(x_i) \geq \frac{a}{2}+k-1$, for $i = 3, \ldots, \frac{a}{2}$. By (2),

$$d_{C_2}(x_1)+d_{C_2}(x_2)+\cdots+d_{C_2}(x_{\frac{a}{2}}) \leq \frac{a^2}{4}. \tag{8}$$

Since $D[V(C_1)]$ is neither a complete bipartite digraph nor a complete bipartite digraph minus one arc, either there exists a vertex $x_i \in V(C_1) \cap V_1$ such that $d_{C_1}(x_i) \leq a-2$ or there exist at least two vertices x_i and x_j such that $d_{C_1}(x_i) = a-1$ and $d_{C_1}(x_j) = a-1$. If $a = 6$, then $d_{C_1}(x_3) \leq a-2$. By $3a-1 \leq d(x_1)+d(x_3) \leq (a+\frac{a}{2}-k)+(a-2+d_{C_2}(x_3))$, we have $d_{C_2}(x_3) \geq \frac{a}{2}+k+1$. So $d_{C_2}(x_1)+d_{C_2}(x_2)+d_{C_2}(x_3) \geq a-1+\frac{a}{2}+k+1 = \frac{3a}{2}+k$. According to (8), $\frac{3a}{2}+k \leq \frac{a^2}{4}$. It is impossible as $k \geq 1$ and $a = 6$, a contradiction. Hence $a \geq 8$. By $\frac{a}{2}-k+(\frac{a}{2}-1)(\frac{a}{2}+k-1) \leq \sum_{i=1}^{\frac{a}{2}} d_{C_2}(x_i) \leq \frac{a^2}{4}$, we have $k \leq 1$ and so $k = 1$. So $d_{C_2}(x_1) = \frac{a}{2}-1$ and $d_{C_2}(x_i) \geq \frac{a}{2}$, for $i \geq 2$. Suppose that there exists a vertex $x' \in V(C_1) \cap V_1$ such that $d_{C_1}(x') \leq a-2$. By Claim 1 and $d(x_1) = d_{C_1}(x_1)+d_{C_2}(x_1) = a+\frac{a}{2}-1 = \frac{3a}{2}-1$, we have $d(x') \geq \frac{3a}{2}$ and so $d_{C_2}(x') = d(x')-d_{C_1}(x') \geq \frac{a}{2}+2$. Then $\frac{a^2}{4}+1 = (\frac{a}{2}-1)+(\frac{a}{2}-2)\frac{a}{2}+\frac{a}{2}+2 \leq \sum_{i=1}^{\frac{a}{2}} d_{C_2}(x_i) \leq \frac{a^2}{4}$, a contradiction. Thus there exist two vertices $x_i, x_j \in V(C_1) \cap V_1$ such that $d_{C_1}(x_i) = a-1$ and $d_{C_1}(x_j) = a-1$. Then by Claim 1, $d_{C_2}(x_i) \geq (3a-1)-d(x_1)-d_{C_1}(x_i) = (3a-1)-(\frac{3a}{2}-1)-(a-1) = \frac{a}{2}+1$ and $d_{C_2}(x_j) \geq \frac{a}{2}+1$. Then $\frac{a^2}{4}+1 = (\frac{a}{2}-1)+(\frac{a}{2}-3)\frac{a}{2}+2(\frac{a}{2}+1) \leq \sum\limits_{i=1}^{\frac{a}{2}} d_{C_2}(x_i) \leq \frac{a^2}{4}$, a contradiction. We have considered all cases and completed the proof of the theorem. $\square$

From Theorem 3.1, we can obtain the following.

Theorem 3.2 *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If D satisfies the condition M_{-1}, then D is traceable.*

Proof: By Theorem 3.1, D is either hamiltonian or isomorphic to a digraph in $\mathcal{H}_1$ or the digraph H_2. If D is hamiltonian, there is nothing to prove. If D is isomorphic to the digraph H_2 (see Figure 1), then $x_1y_1x_3y_3x_2y_2$ is a hamiltonian path. Suppose that D is isomorphic to a digraph in $\mathcal{H}_1$ (see Example 1.4). Note that both $D[S \cap U]$ and $D[R \cap W]$ are complete bipartite digraphs and $|S| = |W| = \frac{a+1}{2} \geq 2$. Clearly, for any $x_1, x_2 \in S$, there is a hamiltonian path Q_1 from x_1 to x_2 in $D[S \cap U]$ and for any $w_1, w_2 \in W$, there is a hamiltonian path Q_2 from w_1 to w_2 in $D[R \cap W]$. Then $w_1Q_2w_2x_1Q_1x_2$ is a hamiltonian path in D. $\square$

The bound in Theorem 3.2 is sharp, as can be seen in the following example.

Example 3.3 *Let $a \geq 4$ be an even integer and let H_3 be a balanced bipartite digraph with partite sets V_1 and V_2 such that V_1 (resp. V_2) is a disjoint union of S, R (resp. U, W) with $|S| = |W| = \frac{a+2}{2}$, $|U| = |R| = \frac{a-2}{2}$, and $A(H_3)$ consists of the following arcs:*

(a) *ry and yr, for all $r \in R$ and $y \in V_2$,*

(b) *ux and xu, for all $u \in U$ and $x \in V_1$, and*

(c) *ws, for all $w \in W$ and $s \in S$.*

Then $d(r) = d(u) = 2a$ for all $r \in R$ and $u \in U$, and $d(s) = d(w) = \frac{3a-2}{2}$ for all $s \in S$ and $w \in W$ and so H_3 satisfies the condition M_{-2}. Notice that H_3 is strong, but contains no hamiltonian path, as the size of a maximum matching from V_1 to V_2 is $a - 2$.

4 Related problems

Let $\mathcal{D}_{a,k}$ denote all strong balanced bipartite digraphs on $2a$ vertices such that $d(u) + d(v) \geq 3a - k$ for all non-adjacent vertices u, v. If $D \in \mathcal{D}_{a,0}$, then by Theorem 1.3, D is hamiltonian. A hamiltonian digraph must possess a cycle factor. In this present paper, we have shown that if $D \in \mathcal{D}_{a,1}$ and D contains a cycle factor, then D is hamiltonian unless D is the digraph H_2. A natural question would be if there are at most a finite number (depending on k) of digraphs in $D \in \mathcal{D}_{a,k}$ containing a cycle factor but not a hamiltonian cycle.

Theorem 2.4 implies that a strong bipartite tournament containing a cycle factor, is hamiltonian. Let D be a balanced bipartite oriented graph of order $2a$. An another natural question would be if there exists an integer $k \geq 1$ such that D satisfying the condition $d(x) + d(y) \geq 2a - k$ for any pair of non-adjacent vertices x, y in D and containing a cycle factor, is hamiltonian.

To conclude the paper, we mention two related problems. In [6], Bang-Jensen et al. conjectured the following strengthening of a classical Meyniel theorem.

Conjecture 4.1 *[6] If D is a strong digraph on n vertices in which $d(u) + d(v) \geq 2n - 1$ for every pair of non-adjacent vertices u, v with a common out-neighbour or a common in-neighbour, then D is hamiltonian.*

In [1], Adamus proved a bipartite analogue of the conjecture.

Theorem 4.2 *[1] Let D be a strong balanced bipartite digraph of order $2a$ with $a \geq 3$. If $d(x) + d(y) \geq 3a$ for every pair of vertices x, y with a common out-neighbour or a common in-neighbour, then D is hamiltonian.*

A natural problem is to characterize the extremal digraph on the condition in Theorem 4.2.

A balanced bipartite digraph containing cycles of all even length is called bipancyclic. In [2], Adamus proved that the hypothesis of Theorem 4.2 implies bipancyclicity of D, except for a directed cycle of length $2a$ (Theorem 4.3 below).

Theorem 4.3 *[2] Let D be a strong balanced bipartite digraph of order $2a$ with $a \geq 3$. If $d(x) + d(y) \geq 3a$ for every pair of vertices x, y with a common out-neighbour or a common in-neighbour in D, then D either is bipancyclic or is a directed cycle of length $2a$.*

In the same paper, the author presented the following problem: if for every $1 \leq l < a$ there is an integer $k \geq 1$ such that every strong balanced bipartite digraph on $2a$ vertices contains cycles of all even lengths up to $2l$, provided $d(x) + d(y) \geq 3a - k$ for every pair of vertices x, y with a common out-neighbour or a common in-neighbour.

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