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Graphs with many Vertex-Disjoint Cycles

Dieter Rautenbach† and Friedrich Regen‡

Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany

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We study graphs G in which the maximum number of vertex-disjoint cycles $\nu(G)$ is close to the cyclomatic number $\mu(G)$, which is a natural upper bound for $\nu(G)$.

Our main result is the existence of a finite set $\mathcal{P}(k)$ of graphs for all $k \in \mathbb{N}_0$ such that every 2-connected graph G with $\mu(G) - \nu(G) = k$ arises by applying a simple extension rule to a graph in $\mathcal{P}(k)$. As an algorithmic consequence we describe algorithms calculating $\min\{\mu(G) - \nu(G), k + 1\}$ in linear time for fixed k.

Keywords: cyclomatic number; cycle; cycle packing; vertex-disjoint cycles

1 Introduction

We consider finite and undirected *graphs* G with *vertex set* $V(G)$ and *edge set* $E(G)$ that may contain *multiple edges* but no *loops*. We use standard terminology [15] and only recall a few notions. If an edge $e \in E(G)$ is incident with the two vertices u and v in $V(G)$, then we write $e = uv$. The *neighbourhood* $N_G(u)$ of a vertex $u \in V(G)$ is the set of vertices $v \in V(G)$ with $e = uv$ for some $e \in E(G)$. The *degree* $d_G(u)$ of a vertex $u \in V(G)$ is the number of edges incident with u. A *cycle* of G is a connected 2-regular subgraph of G. A *block* of G is a maximal 2-connected subgraph of G. A block is an *endblock* if it contains at most one *cutvertex* of G. A *cactus* is a connected graph all cycles of which are edge-disjoint, i.e. each of its blocks is a *bridge* or a cycle. An *ear* of G is a *path* in G whose *internal vertices* are all of degree 2. An ear is *maximal*, if it is not properly contained in another ear of G. If P is an ear of G and I is the set of internal vertices of P, then we say that G arises from $G' = (V(G) \setminus I, E(G) \setminus E(P))$ *by adding the ear* P and that G' arises from G *by removing the ear* P.

The *cyclomatic number* $\mu(G)$ of G is

$$\mu(G) = |E(G)| - |V(G)| + c(G)$$

where $c(G)$ is the number of components of G. A set $\mathcal{C}$ of vertex-disjoint cycles of G is a *cycle packing*. The set of edges of the cycles in $\mathcal{C}$ is denoted by $E(\mathcal{C})$. The maximum cardinality of a cycle packing of G is denoted by

$$\nu(G)$$

†Email: `dieter.rautenbach@uni-ulm.de`.
‡Email: `friedrich.regen@uni-ulm.de`.

and a cycle packing of cardinality $\nu(G)$ is called *optimal*.

Packing vertex-disjoint cycles in graphs is a very well-studied and classical graph-theoretical problem. There is a vast amount of literature concerning conditions in terms of for instance order, size, vertex degrees, degree sums, independence number, chromatic number, feedback vertex sets that are sufficient for the existence of some number of vertex-disjoint cycles, which may additionally contain specified elements or satisfy certain length conditions. We refer the reader to [2, 4–9, 11, 17–21], which is just a small selection. The algorithmic problems concerning cycle packings are typically hard [1, 10, 13, 14] and approximation algorithms were described [14]. Several authors mention practical applications in computational biology such as reconstruction of evolutionary trees or genomic analysis.

In the present paper we study graphs G in which the maximum number of vertex-disjoint cycles $\nu(G)$ is close to the cyclomatic number $\mu(G)$, which is a natural upper bound for $\nu(G)$. In fact $\mu(G)$ equals the minimum number of edges whose removal from G deletes all cycles of G, which easily implies $\mu(G) \geq \nu(G)$ with equality if and only if every component of G is a cactus and all cycles of G are vertex-disjoint.

As our main result we prove the existence of a finite set $\mathcal{P}(k)$ of graphs for all $k \in \mathbb{N}_0$ such that every 2-connected graph G with $\mu(G) - \nu(G) = k$ arises by applying a simple extension rule to a graph in $\mathcal{P}(k)$. As an algorithmic consequence we describe algorithms calculating $\min\{\mu(G) - \nu(G), k+1\}$ in linear time for fixed k, that is in time $O\left(f(k)(|V(G)| + |E(G)|)\right)$. While in [3, 12] we considered similar results concerning edge-disjoint cycles, the problem to find many vertex-disjoint cycles in a graph can not be reduced to its blocks unlike in the edge-disjoint case.

2 Results

In this section we will give a constructive characterization of the graphs in

$$\mathcal{G}(k) \quad = \quad \{G \mid \mu(G) - \nu(G) = k, V(G) \neq \emptyset, \text{ and } G \text{ is 2-connected}\}.$$

For $l \in \mathbb{N}_0$, a graph P is an *l-cycle-chain between u and v*, if

- P is a cactus with at most two endblocks,
- the set $\mathcal{C}(P)$ of cycles of P consists of l vertex-disjoint cycles,
- $u \neq v$, $d_P(u) = 1$, and $d_P(v) = 1$.

If G is a graph and $e = uv \in E(G)$, then the graph H is said *to arise from G by replacing the edge e with an l-cycle-chain P* (cf. Figure 1), if H arises from the disjoint union of G and an l-cycle-chain P between u' and v' by removing the edge e and identifying u with u' and v with v'. In this case H is said to *contain the l-cycle-chain P*. Note that subdividing an edge is the same as replacing it with a 0-cycle-chain. It is easy to see that if H arises from G by replacing the edge e with an l-cycle-chain P, then $\mu(H) = \mu(G) + l$ and $\nu(H) \in \{\nu(G) + l - 1, \nu(G) + l\}$.

We say that a graph H *extends* a graph G, if H arises from G by replacing every edge $e \in E(G)$ with an l_e-cycle-chain P_e such that $\mu(H) - \nu(H) = \mu(G) - \nu(G)$.

A graph H is called *reduced*, if H does not extend a graph G different from H. Let

$$\mathcal{P}(k) \quad = \quad \{G \mid G \in \mathcal{G}(k) \text{ and } G \text{ is reduced}\}.$$

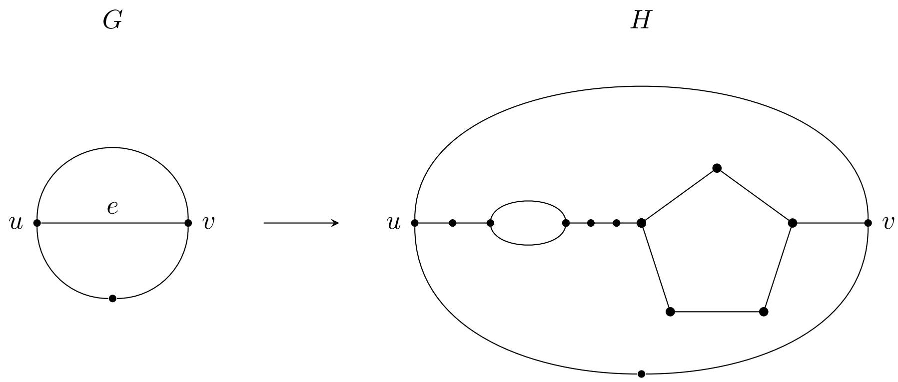

Fig. 1: Replacing the edge $e = uv \in E(G)$ with a 2-cycle-chain

The next lemma summarizes some important properties of the above extension notion.

Lemma 1 *Let H arise from G by replacing every edge $e \in E(G)$ with an l_e-cycle-chain P_e. Let*

$$l = \sum_{e \in E(G)} l_e \quad \textit{and} \quad \mathcal{C} = \bigcup_{e \in E(G)} \mathcal{C}(P_e).$$

(i) If H extends G, then $\mu(H) - \mu(G) = \nu(H) - \nu(G) = l$ and every optimal cycle packing of H contains all l cycles in $\mathcal{C}$.

(ii) H extends G if and only if G has an optimal cycle packing $\mathcal{C}_G$ such that $l_e = 0$ for all $e \in E(\mathcal{C}_G)$.

Proof: Let $\mathcal{C}_H$ be an optimal cycle packing of H. Let E be a set of l edges intersecting every cycle in $\mathcal{C}$. Removing the edges in E can delete at most l different cycles in $\mathcal{C}_H$, which implies

$$\nu(H) - \nu(G) \quad \leq \quad l. \tag{1}$$

Clearly, $\mu(H) - \mu(G) = l$.

(i) Since H extends G, we have $\mu(H) - \nu(H) = \mu(G) - \nu(G)$, which implies $\nu(H) - \nu(G) = l$. Furthermore, since (1) holds with equality for every choice of E, we obtain $E(\mathcal{C}) \subseteq E(\mathcal{C}_H)$. By the definition of a cycle-chain, this implies $\mathcal{C} \subseteq \mathcal{C}_H$.

(ii) If H extends G, then, by (i), the cycles in $\mathcal{C}_H \setminus \mathcal{C}$ are subdivisions of the cycles in an optimal cycle packing $\mathcal{C}_G$ of G. Clearly, $l_e = 0$ for all $e \in E(\mathcal{C}_G)$.

Conversely, if $\mathcal{C}_G$ is an optimal cycle packing of G such that $l_e = 0$ for all $e \in E(\mathcal{C}_G)$, then the cycles in H that are subdivisions of the cycles in $\mathcal{C}_G$ together with the cycles in $\mathcal{C}$ form a cycle packing of H, which implies $\nu(H) - \nu(G) \geq l$. Together with (1) it follows that $\nu(H) - \nu(G) = l$ and H extends G. □

By definition, extending a graph in $\mathcal{G}(k)$ results in a larger graph in $\mathcal{G}(k)$. Another important feature of the extension notion is that iterated extensions are not more powerful than a single extension as proved in the next lemma.

Lemma 2 *(i) If G_2 extends G_1 and G_1 extends G_0, then G_2 extends G_0.*

(ii) For $k \in \mathbb{N}_0$ every graph in $\mathcal{G}(k)$ extends a graph in $\mathcal{P}(k)$.

Proof:

(i) For $i = 1, 2$ let G_i extend G_{i-1} by replacing every edge $e \in E(G_{i-1})$ with an $l_e^{(i)}$-cycle-chain $P_e^{(i)}$. If $e \in E(G_0)$, $f \in E\left(P_e^{(1)}\right)$ and $l_f^{(2)} \geq 1$, then, by Lemma 1(i), f is a bridge of $P_e^{(1)}$. Therefore, if

$$l_e := l_e^{(1)} + \sum_{f \in E\left(P_e^{(1)}\right)} l_f^{(2)}$$

for every $e \in E(G)$, then G_2 extends G_0 by replacing every edge $e \in E(G_0)$ with an l_e-cycle-chain.

(ii) Let $H \in \mathcal{G}(k)$. By definition, there is a finite sequence $G_0, G_1, \ldots, G_s \in \mathcal{G}(k)$ such that G_i extends G_{i-1} for $1 \leq i \leq s$, $G_0 \in \mathcal{P}(k)$ and $H = G_s$. Repeated application of (i) implies that H extends G_0 and the proof is complete. □

In view of the observation about graphs G with $\mu(G) = \nu(G)$ made in the introduction it is easy to determine $\mathcal{G}(0)$ and $\mathcal{P}(0)$. Let $\mathtt{P}_n$ and $\mathtt{C}_n$ denote the *chordless path* and *chordless cycle* of order $n \in \mathbb{N}$.

Lemma 3 *(i) No reduced graph H contains a vertex $u \in V(H)$ with $d_H(u) = |N_H(u)| = 2$ or a 2-cycle-chain.*

(ii) $\mathcal{G}(0) = \{\mathtt{P}_1, \mathtt{P}_2\} \cup \{\mathtt{C}_n \mid n \geq 2\}$ and $\mathcal{P}(0) = \{\mathtt{P}_1, \mathtt{P}_2, \mathtt{C}_2\}$.

Proof:

(i) Let H be a reduced graph. If $u \in V(H)$ is such that $d_H(u) = |N_H(u)| = 2$, then contracting an edge incident with u results in a graph G such that H extends G, which is a contradiction. If H contains a 2-cycle-chain P, then every optimal cycle packing of H contains both cycles contained in P. Therefore, if G arises from H by contracting one cycle C in P together with one further edge incident with C (cf. Figure 2), then H extends G, which is a contradiction.

(ii) Let $G \in \mathcal{G}(0)$. As noted in the introduction, $\mu(G) = \nu(G)$ implies that every component of G is a cactus. Since G is 2-connected, it follows that G is either $\mathtt{P}_1$, or $\mathtt{P}_2$, or a chordless cycle $\mathtt{C}_n$ for $n \geq 2$. By (i), $\mathtt{P}_1$, $\mathtt{P}_2$, and $\mathtt{C}_2$ are the only reduced graphs in $\mathcal{G}(0)$, which implies (ii). □

After these preparations, we are ready to prove our main result.

Theorem 1 *$\mathcal{P}(k)$ is finite for every $k \in \mathbb{N}_0$.*

Proof: We prove the result by induction on k. For $k = 0$, the result follows from Lemma 3(ii).

If $k > 0$, we argue that the number of edges in any graph $H \in \mathcal{P}(k)$ is bounded in terms of the number of edges in some graph in $\mathcal{P}(k-1)$.

Whitney [22] proved that a graph of order at least 2 is 2-connected if and only if it has an *ear decomposition*, i.e. it arises from $\mathtt{P}_2$ by iteratively adding ears. Since removing an ear from H reduces $\mu(H)$ by exactly 1 and $\nu(H)$ by at most 1, iteratively removing the ears of an ear decomposition of H, we obtain a sequence of 2-connected graphs $G_0, G_1, \ldots, G_l = H$, such that

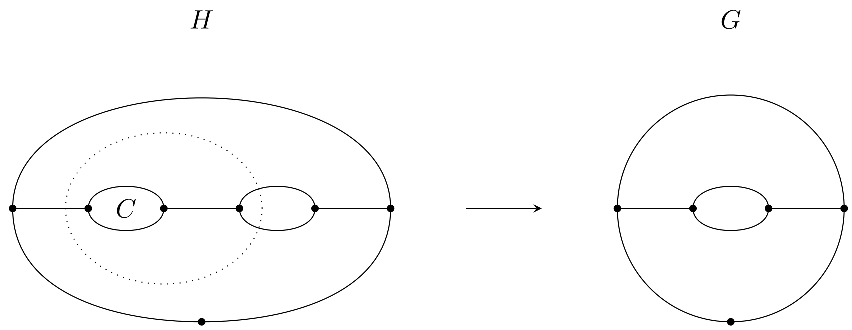

Fig. 2: Contraction in the proof of Lemma 3(i)

- for each $i \in \{1, \dots, l\}$, G_i arises by adding the ear P_i to G_{i-1},
- $\nu(G_{i-1}) = \begin{cases} \nu(G_i) & \text{, if } i = 1 \\ \nu(G_i) - 1 & \text{, if } i > 1. \end{cases}$

The second condition implies that $G_0 \in \mathcal{G}(k-1)$ and $G_i \in \mathcal{G}(k)$ for $i \in \{1, \dots, l\}$. By Lemma 2(ii), G_0 extends some graph $G \in \mathcal{P}(k-1)$.

Let $\mathcal{C}_l$ be an optimal cycle packing of G_l. If $l \geq 2$, then the ear P_l is contained in a unique cycle C_l of $\mathcal{C}_l$ and $\mathcal{C}_l \setminus \{C_l\}$ is an optimal cycle packing of G_{l-1}. Repeating this argument for indices from l down to 2, we obtain vertex-disjoint cycles $C_2, \dots, C_l \in \mathcal{C}_l$ such that P_i is contained in C_i for $2 \leq i \leq l$. Since H is reduced, Lemma 3(i) implies that $\mathcal{E} := \{P_2, \dots, P_l\}$ is a set of edges.

Claim. *The graph G_1 does not contain a 2-cycle-chain.*

Proof of the Claim: For contradiction, we assume that G_1 contains a 2-cycle-chain P. Since $G_l = H$ is reduced, Lemma 3(i) implies that $l \geq 2$. It suffices to show that G_2 contains a 2-cycle-chain. Repeating this argument, it follows that H contains a 2-cycle-chain, which is a contradiction.

Let C' and C'' denote the two cycles in P. Clearly, the optimal cycle packing $\mathcal{C}_1$ of G_1 contains both these cycles. Let P' denote the path in P between C' and C''. Recall that P_2 is contained in the cycle C_2, which is vertex-disjoint to all cycles in $\mathcal{C}_1$. Therefore, if P_2 has no endvertex in P', then G_2 contains a 2-cycle-chain contained in P, and, if P_2 has an endvertex in P', then P_2 has both its endvertices in P' and G_2 contains even a 3-cycle-chain, which completes the proof of the claim. □

Since G_1 arises from G_0 by adding the ear P_1, the claim implies that the graph G_0 does not contain a 6-cycle-chain. Since every l-cycle-chain for $l \leq 5$ contains at most $2 \cdot 5 + 6 = 16$ maximal ears, the number of maximal ears of G_0 is at most $16|E(G)|$. Hence the number of maximal ears of G_1 is at most $16|E(G)| + 3$.

Since H is reduced, all internal vertices of a maximal ear P of G_1 must be endvertices of edges in $\mathcal{E}$. At most two internal vertices can be contained in some $P_i \in \mathcal{E}$ such that C_i contains an endvertex of P. Each further internal vertex must be incident with an edge $P_i \in \mathcal{E}$ such that C_i consists of P_i and a subpath of P. Hence, since H is reduced, Lemma 3(i) implies that each maximal ear of G_1 contains at most four internal vertices. Therefore, each maximal ear contributes at most five edges to G_1, i.e. $|E(G_1)| \leq 5(16|E(G)|+3)$. Finally, since the edges in $\mathcal{E}$ are vertex-disjoint and $|V(G_1)| \leq |E(G_1)|$, we obtain $|\mathcal{E}| \leq \frac{5}{2}(16|E(G)|+3)$, which implies $|E(H)| \leq 8(16|E(G)|+3)$. □

Procedure `Difference(k)`

Input: A graph G
Output: $\min\{\mu(G)-\nu(G), k+1\}$

1 **begin**
2   **while** *G contains a bridge $e \in E(G)$* **do**
3   Delete e.
4   **while** *G contains a vertex u with $d_G(u) = |N_G(u)| = 2$* **do**
5   Contract one of the edges incident with u.
6   **while** *G contains a 2-cycle-chain P* **do**
7   Contract one cycle C in P together with one further edge incident with C.
8   **while** *G contains a component C with $C \in \{P_1, C_2\}$* **do**
9   Delete C.
10   **if** $V(G) = \emptyset$ **then return** 0.
11   Select an endblock B of G.
12   **if** $\mu(B) - \nu(B) \geq k+1$ **then return** $k+1$.
13   If B contains a cutvertex, then let $u \in V(B)$ be the cutvertex, otherwise let $u \in V(B)$ be any vertex. Let u be contained in s blocks of G.
14   $\Delta k \longleftarrow \mu(B) - \nu(B)$.
15   **if** *u is contained in every optimal cycle packing of B* **then**
16     $\Delta k \longleftarrow \Delta k + d_{G-E(B)}(u) - (s-1)$;
17     $G' \longleftarrow G - V(B)$;
18   **if** $\Delta k \geq k+1$ **then return** $k+1$;
19   **else**
20     $G' \longleftarrow G - (V(B) \setminus \{u\})$;
21   Let k' be the output of DIFFERENCE$(k - \Delta k)$ applied to G'.
22   **return** $\min\{\Delta k + k', k+1\}$.
23 **end**

We proceed to an algorithmical consequence of Theorem 1: For fixed k the finiteness of the sets $\mathcal{P}(k)$ allows to decide $\mu(G) - \nu(G) \leq k$ in linear time.

Theorem 2 *For every $k \in \mathbb{N}_0$ the algorithm* DIFFERENCE(k) *works correctly and has linear running time.*

Proof of correctness: By induction on the recursive depth, we may assume that the output of the recursive call performed in line 21 is correct.

Up to line 9, G is modified such that the difference $\mu(G) - \nu(G)$ does not change (cf. the argument in the proof of Lemma 3(i)). Note that after these preprocessing steps, G contains neither a bridge, nor a vertex u with $d_G(u) = |N_G(u)| = 2$, nor a 2-cycle-chain, nor a component that is an isolated vertex or a chordless cycle.

Clearly, it is correct to return 0 in line 10.

Since $\mu(G) - \nu(G) \geq \mu(B) - \nu(B)$, it is correct to return $k + 1$ in line 12.

If u is contained in every optimal cycle packing of B, then there is an optimal cycle packing of G that is the union of an optimal cycle packing of $G - V(B)$ and an optimal cycle packing of B. Since

$$\mu(G) = \mu(G - V(B)) + \mu(B) + d_{G-E(B)}(u) - (s - 1),$$

we obtain

$$\begin{aligned} \mu(G) - \nu(G) &= \mu(G - V(B)) - \nu(G - V(B)) \\ &\quad +\mu(B) - \nu(B) \\ &\quad +d_{G-E(B)}(u) - (s - 1) \end{aligned}$$

and the return value in line 18 or line 22 is correct.

If u is not contained in every optimal cycle packing of B, then there is an optimal cycle packing of G that is the union of an optimal cycle packing of $G - (V(B) \setminus \{u\})$ and an optimal cycle packing of $B - \{u\}$. Since

$$\begin{aligned} \mu(G) &= \mu(G - (V(B) \setminus \{u\})) + \mu(B) \text{ and} \\ \nu(B) &= \nu(B - \{u\}), \end{aligned}$$

we obtain

$$\begin{aligned} \mu(G) - \nu(G) &= \mu(G - (V(B) \setminus \{u\})) - \nu(G - (V(B) \setminus \{u\})) \\ &\quad +\mu(B) - \nu(B) \end{aligned}$$

and the return value in line 22 is correct.

This completes the proof of correctness. □

Proof of linear running time: If B is a component of G or u is not contained in every optimal cycle packing of B, then, by Lemma 3(ii) and the preprocessing, $\mu(B) - \nu(B) > 0$. If B is contained in $s \geq 2$ blocks of G, then, by the preprocessing, G has no bridge and hence $d_{G-E(B)}(u) - (s - 1) > 0$. This implies that $\Delta k > 0$ in line 21. Therefore, the recursive depth is at most k and it suffices to show that all steps until line 20 can be done in linear time.

Since the block-cutvertex tree of G can be determined in linear time [16], the deletion of bridges (line 3), the deletion of trivial components (line 9), the selection of B (line 11) and the selection of u (line 13) can be done in linear time. Furthermore, it is easy to see that the contractions in the preprocessing (lines 5 and 7) can be done in linear time.

By Lemma 2(ii), if $\mu(B) - \nu(B) \leq k$, then there exists a graph $B' \in \mathcal{P} := \bigcup_{i=0}^{k} \mathcal{P}(i)$ such that B extends B'. Since B contains at most one vertex v with $d_G(v) = |N_G(v)| = 2$ — the cutvertex u — and since G contains no 2-cycle-chain after the preprocessing, B contains no 4-cycle-chain. Therefore, in order to obtain B each edge of B' is replaced by a subgraph with at most 11 edges. Since, by Theorem 1,

$\mathcal{P}$ is finite, $\mu(B) - \nu(B) \leq k$ can only hold, if B belongs to a finite set of graphs depending on k and lines 12, 14, and 15 can be done in constant time. This completes the proof. □

It is easy to modify DIFFERENCE(k) so that it also returns an optimal cycle packing of the instance graph G in linear time provided that $\mu(G) - \nu(G) \leq k$. In fact, such a packing could consist of the cycles contracted in line 7, the cycles of length 2 deleted in line 9, an optimal cycle packing of B, which, if possible, avoids u and an optimal cycle packing of G' obtained recursively.

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