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Generalized Petersen Graphs and Kronecker Covers[*]

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The family of generalised Petersen graphs $G(n,k)$, introduced by Coxeter et al. [4] and named by Watkins (1969), is a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. The Kronecker cover $\mathrm{KC}(G)$ of a simple undirected graph G is a special type of bipartite covering graph of G, isomorphic to the direct (tensor) product of G and K_2. We characterize all generalised Petersen graphs that are Kronecker covers, and describe the structure of their respective quotients. We observe that some of such quotients are again generalised Petersen graphs, and describe all such pairs. The results of this paper have been presented at EUROCOMB 2019 and an extended abstract has been published elsewhere.

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1 Introduction

The *generalised Petersen graphs*, introduced by Coxeter et al. [4] and named by Watkins [18], form a very interesting family of trivalent graphs that can be described by only two integer parameters. They include Hamiltonian and non-Hamiltonian graphs, bipartite and non-bipartite graphs, vertex-transitive and non-vertex-transitive graphs, Cayley and non-Cayley graphs, arc-transitive graphs and non-arc-transitive graphs, graphs of girth $3, 4, 5, 6, 7$ or 8. Their generalization to I-graphs does not introduce any new vertex-transitive graphs but it contains also non-connected graphs and has in special cases unexpected symmetries [2]. For further properties of I-graphs also see [8, 13].

Following the notation of Watkins [18], for given integers n and $k < \frac{n}{2}$, we can define a generalised Petersen graph $G(n,k)$ as a graph on vertex-set $\{u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}\}$. The edge-set may be naturally partitioned into three equal parts (note that all subscripts are assumed modulo n): the edges $E_O(n,k) = \{u_i u_{i+1}\}_{i=0}^{n-1}$ form the *outer rim*, inducing a cycle of length n; the edges $E_I(n,k) =$

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$\{v_i v_{i+k}\}_{i=0}^{n-1}$ form the *inner rims*, inducing $\gcd(n,k)$ cycles of length $\frac{n}{\gcd(n,k)}$; and the edges $E_S(n,k) = \{u_i v_i\}_{i=0}^{n-1}$, also called *spokes*, which induce a perfect matching in $G(n,k)$. Hence the edge-set may be defined as $E(G(n,k)) = E_O(n,k) \cup E_I(n,k) \cup E_S(n,k)$.

Various structural aspects of the mentioned family have been pointed out. Examples include identifying generalised Petersen graphs that are Hamiltonian [1] or Cayley [12, 15], or isomorphic [8, 13, 16], or finding their automorphism group [5]. Also, a related generalization to I-graphs has been introduced in the Foster census [3], and further studied by Boben et al. [2].

The theory of covering graphs became one of the most important and successful tools of algebraic graph theory. It is a discrete analog of the well known theory of covering spaces in algebraic topology. In general, covers depend on the values called voltages assigned to the edges of the graphs. Only in some cases the covering is determined by the graph itself. One of such cases is the recently studied *clone cover* [11]. The other, more widely known case is the Kronecker cover.

The *Kronecker cover* $\mathrm{KC}(G)$ (also called bipartite or canonical double cover) of a simple undirected graph G is a bipartite covering graph with twice as many vertices as G. Formally, $\mathrm{KC}(G)$ is defined as a tensor product $G \times K_2$, i.e. a graph on a vertex-set $V(\mathrm{KC}(G)) = \{v', v''\}_{v \in V(G)}$, and the edge-set $E(\mathrm{KC}(G)) = \{u'v'', u''v'\}_{uv \in E(G)}$. For $H = \mathrm{KC}(G)$, we also say that G is *a quotient* of H. Some recent work on Kronecker covers includes Gévay and Pisanski [6] and Imrich and Pisanski [9].

In this paper, we study the family of generalised Petersen graphs in conjunction with the Kronecker cover operation. Namely, in the next section we state our main theorem characterizing all generalised Petersen graphs that are Kronecker covers, and describing the structure of their corresponding quotient graphs. In Section 3 we focus on the necessary and sufficient conditions for a generalised Petersen graph to be a Kronecker cover while in Section 4 we complement the existence results with the description of the structure of the corresponding quotient graphs. We conclude the paper with some remarks and directions for possible future research.

2 Main result

In order to state the main result we need to introduce the graph $\mathcal{H}$ and two 2-parametric families of cubic, connected graphs.

Let $\mathcal{H}$ be the graph defined by the following procedure: Take the Cartesian product $K_3 \square P_3$, remove the edges of the triangle connecting the three vertices of degree 4, add a new vertex and connect it to the same three vertices. Note that the graph $\mathcal{H}$ is mentioned in [9] and is depicted in Figure 1.

As shown in [9], the Desargues graph $G(10,3)$ is the Kronecker cover of both $G(5,2)$ and $\mathcal{H}$. Note that in Figure 1 the edge-colored subgraphs of $\mathcal{H}$ lift to the corresponding edge-colored subgraphs of $G(10,3)$.

To describe the quotients of generalised Petersen graphs, we use the LCF notation, named by developers Lederberg, Coxeter and Frucht, for the representation of cubic hamiltonian graphs (for extended description see [14]).

> In a Hamiltonian cubic graph, the vertices can be arranged in a cycle, which accounts for two edges per vertex. The third edge from each vertex can then be described by how many positions clockwise (positive) or counter-clockwise (negative) it leads. The basic form of the LCF notation is just the sequence $[a_0, a_1, \ldots, a_{n-1}]$ of numbers of positions, starting from an arbitrarily chosen vertex and written in square brackets.

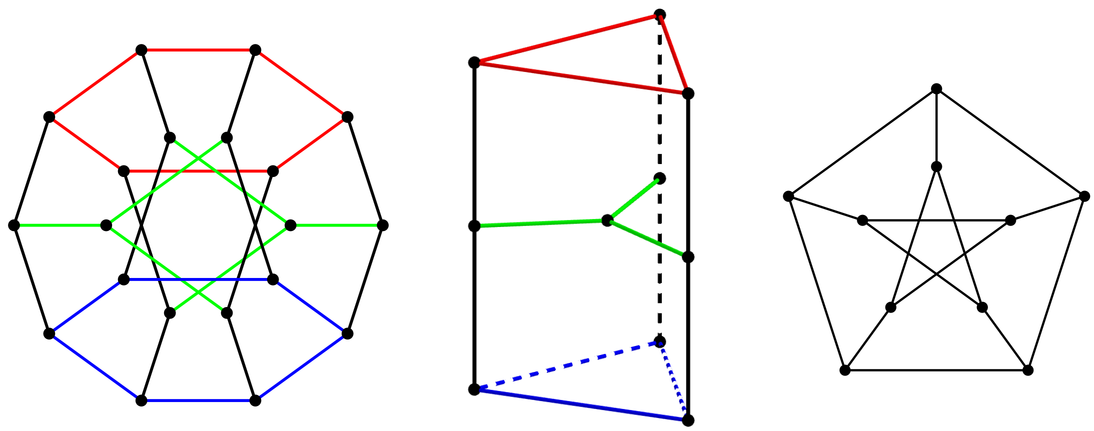

Fig. 1: The Desargues graph and both its quotients; $\mathcal{H}$ and the Petersen graph.

To state our results, we only use a special type of such LCF-representable graphs, namely $C^{+}(n,k)$ and $C^{-}(n,k)$, which we define below.

Definition 1. Let n be a positive even integer and let $k < n/2$ be a positive integer such that $k^2 \equiv 1 \pmod{n}$. Assuming all numbers are modulo n, define graphs

$$C^{+}(n,k) = \left[\frac{n}{2}, \frac{n}{2} + (k-1), \frac{n}{2} + 2(k-1), \ldots, \frac{n}{2} + (n-1)(k-1)\right],$$

and similarly

$$C^{-}(n,k) = \left[\frac{n}{2}, \frac{n}{2} - (k+1), \frac{n}{2} - 2(k+1), \ldots, \frac{n}{2} - (n-1)(k+1)\right].$$

It is easy to show that, under the mentioned restrictions on k from above, both graphs $C^{+}(n,k)$ and $C^{-}(n,k)$ are well defined. For the case of $C^{+}(n,k)$, pick an i-th chord, and (from its LCF-representation) look at the values on its endpoints a_i and a_{i+a_i} – namely $n/2+i(k-1)$ and $n/2+(i+n/2+i(k-1))(k-1)$. The claim follows from the fact that their sum is divisible by n.

In [9] it was proven that $G(10,3)$ is the Kronecker cover of two non-isomorphic graphs. Here we prove among other things that this is the only generalised Petersen graph that is a multiple Kronecker cover. Every other generalised Petersen graph is either a Kronecker cover of a single graph or it is not a Kronecker cover at all. More precisely;

Theorem 1. *Among the members of the family of generalised Petersen graphs, $G(10,3)$ is the only graph that is the Kronecker cover of two non-isomorphic graphs, the Petersen graph and the graph $\mathcal{H}$. For any other $G \simeq G(n,k)$, the following holds:*

$a)$ *If $n \equiv 2 \pmod{4}$ and k is odd, G is a Kronecker cover. In particular*

- $a_1)$ *if $4k < n$, the corresponding quotient graph is $G\left(\frac{n}{2}, k\right)$, and*
- $a_2)$ *if $n < 4k < 2n$ the quotient graph is $G\left(\frac{n}{2}, \frac{n}{2} - k\right)$.*

$b)$ *If* $n \equiv 0 \pmod 4$ *and* k *is odd,* G *is a Kronecker cover if and only if* $n \mid \frac{k^2-1}{2}$*. Moreover,*

- $b_1)$ *if* $k \equiv 1 \pmod 4$ *the corresponding quotient is* $C^+(n,k)$ *while*
- $b_2)$ *if* $k \equiv 3 \pmod 4$ *the quotient is* $C^-(n,k)$*.*

$c)$ *Any other generalised Petersen graph is not a Kronecker cover.*

For $k = 1$ and even n each $G(n,1)$ is a Kronecker cover. If $n = 4t$ case $b_1)$ applies and the quotient graph is the Möbius ladder M_n (see [7]). For $G(4,1)$ the quotient is $K_4 = M_4$. Similarly, the 8-sided prism $G(8,1)$ is the Kronecker cover of M_8. In case $n = 4t+2$ the case $a_1)$ applies and the quotient is $G(n/2,1)$. For instance, the 6-sided prism is the Kronecker cover of the 3-sided prism. For $k > 1$ the smallest cases stated in Theorem 1 are presented in Table 1.

It is well-known that any automorphism of a connected bipartite graph either preserves the two sets of bipartition or interchanges the two sets of bipartition. In the former case we call the automorphism *colour preserving* and in the latter case *colour-reversing*. Clearly, the product of two color-reversing automorphisms is a color preserving automorphism and the collection of all color preserving automorphisms determines a subgroup of the full automorphism group that is of index at most 2.

3 Identifying the Kronecker involutions

Before we state an important condition that classifies Kronecker covers we give the following definition.

Definition 2. A color-reversing involution ω from the automorphism group of a bipartite graph is called a *Kronecker involution*, if the vertices v and $\omega(v)$ are non-adjacent for every vertex v.

We proceed by a well-known proposition from [9], regarding the existence of Kronecker covers.

Theorem 2. *For a bipartite graph* G*, there exists* G' *such that* $\mathrm{KC}(G') \simeq G$*, if and only if* $Aut(G)$ *admits a Kronecker involution. Furthermore, the corresponding quotient graph may be obtained by contracting all pairs of vertices, naturally coupled by a given Kronecker involution.*

The following result is well-known. One can find it, for instance in [8].

Theorem 3. *A generalised Petersen graph* $G(n,k)$ *is bipartite if and only if* n *is even and* k *is odd.*

We also include the classification concerning symmetries of generalised Petersen graphs, which follows from the work of Frucht et al. [5], Nedela and Škoviera [12], and Lovrečič-Saražin [15].

Theorem 4 ([5, 12, 15])**.** *Let* $G(n,k)$ *be a generalised Petersen graph. Then*

a) *it is symmetric if and only if*

$$(n,k) \in \{(4,1),(5,2),(8,3),(10,2),(10,3),(12,5),(24,5)\},$$

b) *it is vertex-transitive if and only if* $k^2 \equiv \pm 1 \pmod n$ *or if* $n = 10$ *and* $k = 2$*,*

c) *it is a Cayley graph if and only if* $k^2 \equiv 1 \pmod n$*.*

In general the word *symmetric* means arc-transitive. For cubic graphs this is equivalent to saying vertex-transitive and edge-transitive. For a generalised Petersen graph symmetric is equivalent to edge-transitive.

In order to understand which generalised Petersen graphs are Kronecker covers we have to identify all Kronecker involutions for each $G(n,k)$. In what follows, for a given pair (n,k), our arguments rely on the structure of the automorphism group $A(n,k)$ of $G(n,k)$. We define three types of permutations on the vertex set of a generalised Petersen graph which are useful for describing elements of its automorphism group.

Definition 3. For $i \in [0, n-1]$, define the permutations α, β and γ on $V(G(n,k))$ by

$$\begin{aligned} \alpha(u_i) &= u_{i+1}, & \alpha(v_i) &= v_{i+1}, \\ \beta(u_i) &= u_{-i}, & \beta(v_i) &= v_{-i}, \\ \gamma(u_i) &= v_{ki}, & \gamma(v_i) &= u_{ki}. \end{aligned}$$

Let us paraphrase Theorem 5 of Lovrečič-Saražin [15] that follows from Frucht et al. [5]

Theorem 5. *If (n,k) is not one of $(4,1),(5,2),(8,3),(10,2),(10,3),(12,5)$, or $(24,5)$, then the following holds:*

- *if $k^2 \equiv 1$ mod n, then*

$$A(n,k) = \langle \alpha, \beta, \gamma | \alpha^n = \beta^2 = \gamma^2 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma\alpha^k, \beta\gamma = \gamma\beta \rangle$$

- *if $k^2 \equiv -1$ mod n, then*

$$A(n,k) = \langle \alpha, \beta, \gamma | \alpha^n = \beta^2 = \gamma^4 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma\alpha^k, \beta\gamma = \gamma\beta \rangle$$

  In this case $\beta = \gamma^2$.

- *In all other cases the graph $G(n,k)$ is not vertex-transitive and*

$$A(n,k) = \langle \alpha, \beta | \alpha^n = \beta^2 = 1, \alpha\beta = \beta\alpha^{-1} \rangle$$

Since all of α, β and γ preserve the set of spokes, it is clear that the automorphism group of a symmetric graph is not generated by α, β and γ. As there is only five symmetric bipartite generalised Petersen graphs, we simply checked their automorphism group by using SageMath software [17], identified the Kronecker involutions, and the corresponding quotient graphs. It turns out that the quotients of generalised Petersen graphs may be obtained by Kronecker involutions from $\langle \alpha, \beta, \gamma \rangle$, except in the case $n = 10$ and $k = 3$ by another involution which we call Δ, and corresponds to the quotient graph $\mathcal{H}$.

However, for the non-symmetric members of generalised Petersen graphs, Theorem 5 implies that any element of the automorphism group (including any Kronecker involution) may be expressed in terms of α, β and γ. In fact, in the next lemma we show that any such element may be expressed in a canonical way.

Lemma 6. *Let $n \geq 3$ and $1 \leq k < n/2$ be integers such that (n,k) is not one of the pairs $(4,1),(5,2),(8,3),(10,2),(10,3),(12,5)$, or $(24,5)$. Then any automorphism ω from $A(n,k)$ may associated unique triple $(a,b,c) \in \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\omega = \alpha^a \beta^b \gamma^c$.*

n	k	case	involution	quotient
4	1	b_1	$\alpha^2\gamma$	$C^+(4,1)$
6	1	a_1	α^3	$G(3,1)$
8	1	b_1	$\alpha^4\gamma$	$C^+(8,1)$
10	1	a_1	α^5	$G(5,1)$
10	3	$\cdot$	α^5, Δ	$G(5,2), \mathcal{H}$
12	1	b_1	$\alpha^6\gamma$	$C^+(12,1)$
12	5	b_1	$\alpha^6\gamma$	$C^+(12,5)$
14	1	a_1	α^7	$G(7,1)$
14	3	a_1	α^7	$G(7,3)$
16	1	b_1	$\alpha^8\gamma$	$C^+(16,1)$
18	1	a_1	α^9	$G(9,1)$
18	3	a_1	α^9	$G(9,3)$
18	5	a_2	α^9	$G(9,4)$
20	1	b_1	$\alpha^{10}\gamma$	$C^+(20,1)$
20	9	b_1	$\alpha^{10}\gamma$	$C^+(20,9)$
22	1	a_1	α^{11}	$G(11,1)$
22	3	a_1	α^{11}	$G(11,3)$
22	5	a_1	α^{11}	$G(11,5)$
24	1	b_1	$\alpha^{12}\gamma$	$C^+(24,1)$
24	7	b_2	$\alpha^{12}\beta\gamma$	$C^-(24,7)$
26	1	a_1	α^{13}	$G(13,1)$
26	3	a_1	α^{13}	$G(13,3)$
26	5	a_1	α^{13}	$G(13,5)$
26	7	a_2	α^{13}	$G(13,6)$
28	1	b_1	$\alpha^{14}\gamma$	$C^+(28,1)$
28	13	b_1	$\alpha^{14}\gamma$	$C^+(28,13)$

Tab. 1: The smallest generalised Petersen graphs that are Kronecker covers, together with their corresponding Kronecker involutions ω and the quotient graphs.

Proof: Let $G(n,k)$ be a generalised Petersen graph and let a,b,c be arbitrary integers. Then, by definition of the three generators α,β,γ (or the three permutations) it clearly holds

1. $\beta\alpha^a = \alpha^{-a}\beta$,
2. $\gamma\alpha^a = \alpha^{ak}\gamma$. If $k^2 \equiv 1$ mod n.
3. $\gamma\alpha^a = \alpha^{-ak}\gamma$. If $k^2 \equiv -1$ mod n.
4. $\gamma\beta = \beta\gamma$,

We omit the arguments for (1) and (4) as they are repeated from the definition. Property (2) follows from the facts $\alpha\gamma = \gamma\alpha^k$ and $k^2 \equiv 1$ mod n. Since $\alpha^a\gamma = \gamma\alpha^{ak}$ for any a, take $a = k$ and we get $\alpha^k\gamma = \gamma\alpha^{k^2} = \gamma\alpha$ and the result follows. In a similar way we prove (3).

By using the commuting rules (1–4) above we may transform any product of permutations α,β,γ to a form $\alpha^a\beta^b\gamma^c$ with $0 \le b,c \le 1$. In non-vertex-transitive case we have $c = 0$ while in vertex-transitive non-Cayley case, one could have γ,γ^2,γ^3. However, we may always use the fact that $\gamma^2 = \beta$ and the result follows readily.

A shorter proof was suggested by a referee: Namely, from Theorem 5 it clearly follows that both β and γ normalize α, and so the fact that the intersection of $\langle\alpha\rangle$ and $\langle\beta,\gamma\rangle$ (which is clearly isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathbb{Z}_4$) is trivial implies that $A(n,k)$ is a semidirect product of $\mathbb{Z}_n$ by $\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_4$. □

Note that in a bipartite $G(n,k)$ automorphisms α and γ are color-reversing, while β is color preserving.

Proposition 7. *For a bipartite generalised Petersen graph $G(n,k)$, the following statements hold:*

1. *α^a is a Kronecker involution if and only if $a = n/2$ and $n \equiv 2 \pmod 4$;*
2. *$\alpha^a\beta$ is not a Kronecker involution;*
3. *if $k^2 \equiv -1 \pmod n$, then neither $\alpha^a\gamma$ nor $\alpha^a\beta\gamma$ is a Kronecker involution, for any admissible a.*

Proof: We prove the claims separately.

(1) Let $\omega = \alpha^a$ be a Kronecker involution. It is clear that ω does not fix any edge, and since ω is an involution we trivially have $a = \frac{n}{2}$. But since ω must be color-reversing, a must at the same time be odd, hence the conclusion.

(2) Let $\omega = \alpha^a\beta$ be a Kronecker involution. Since ω is color-reversing, a must be odd. Letting $i = \frac{a-1}{2}$, it is enough to observe that an edge u_iu_{i+1} is fixed by ω.

(3) In both cases, the resulting squared permutation can be written in form $\alpha^{a'}\beta$, which contradicts the fact that the original permutation is an involution.

□

In every generalised Petersen graph $G(n,k)$ the permutations α and β are automorphisms. Moreover, they generate the dihedral group D_n of order $2n$ of automorphisms which is, in general, a subgroup of the full automorphism group $A(n,k)$. The two vertex orbits under D_n are exactly the outer rim and the union of the inner rims. The three edge orbits are outer-rim, inner-rim and the spokes. Clearly, Proposition 7 deals with Kronecker involutions from D_n and in particular implies the condition for $G(n,k)$ being

Kronecker cover described in $a)$ of Theorem 1. But additional Kronecker involutions may exist by the fact that the automorphism group of a generalised Petersen graph may be larger then D_n. In the next subsection we describe these additional Kronecker involutions, which may (see (3) of Proposition 7) only happen when $k^2 \equiv 1 \pmod{n}$.

3.1 Additional Kronecker involutions with $k^2 \equiv 1 \pmod{n}$

In what follows, we assume $k^2 \equiv 1 \pmod{n}$ and define Q, such that $k^2 - 1 = Qn$. The only two permissible types of involutions are $\alpha^a\gamma$ and $\alpha^a\beta\gamma$.

For an integer i let $b(i)$ be the maximal integer such that $2^{b(i)}$ divides i. In particular, we have

$$b(n) = b(k+1) + b(k-1) - b(Q). \tag{1}$$

In the following two subsections, we prove the condition for a generalised Petersen graph being a Kronecker cover, described in (b_1) and (b_2) of Theorem 1, respectively.

Involutions of type $\alpha^a\gamma$

We have $\omega_a = \alpha^a\gamma$ such that $\omega_a(v_i) = u_{ki+a}$ and $\omega_a(u_i) = v_{ki+a}$, so let us for easier notation define a function $\Omega_a : \mathbb{Z}_n \to \mathbb{Z}_n$ such that $\Omega_a(i) = ki + a$. By these definitions, Theorem 2 translates to the following properties:

- P1 Permutation ω_a is color-reversing if and only if $\Omega_a(i) \equiv i \pmod{2}$, in other words if a is even.
- P2 Permutation ω_a is an involution if and only if $\Omega_a(\Omega_a(i)) \equiv i \pmod{n}$, i.e. if $a(k+1) \equiv 0 \pmod{n}$.
- P3 Permutation ω_a may not fix any spoke. In particular, ω_a fixes some edge if and only if there exists an integer i, such that $\Omega_a(i) \equiv i \pmod{n}$. In other words, a is not allowed to be any multiple of $(k-1)$ in $\mathbb{Z}_n$.

We now describe an alternative reformulation of the property P3.

Lemma 8. *Assume that properties* P1 *and* P2 *hold. Then the property* P3 *is equivalent to* $b(\gcd(n,a)) < b(\gcd(n,k-1))$.

Proof: First observe that $\gcd(k-1,k+1) = 2$, while $n \mid (k+1)(k-1)$. But, by P2 also $n \mid (k+1)a$ holds, hence every odd divisor of $\gcd(n,k-1)$ must also divide $\gcd(n,a)$. To conclude the proof it is enough to observe that whenever a is some multiple of $(k-1)$ in $\mathbb{Z}_n$ (i.e. the negation of P3), it must also hold that $b(\gcd(n,a)) \geq b(\gcd(n,k-1))$. □

It is now easy to derive the existence version of our main theorem, for the involutitons of type $\alpha^a\gamma$.

Corollary 9. *Let* $G = G(n,k)$ *with* $k^2 \equiv 1 \pmod{n}$*, and let a be an integer. Then G admits a Kronecker involution of type* $\alpha^a\gamma$ *if and only if* $n \mid \frac{k^2-1}{2}$ *and* $k \equiv 1 \pmod{4}$.

Proof: By Theorem 2 and Lemma 8, a permutation $\alpha^a\gamma$ is a rim-switching Kronecker involution if and only if $b(\gcd(n,a)) < b(\gcd(n,k-1))$, in addition to the properties P1 and P2.

It is easy to see that in the case when these properties hold for $\alpha^a\gamma$, we also have $n \mid \frac{k^2-1}{2}$ and $k \equiv 1 \pmod{4}$. Indeed, as n and a are both even, it follows that $4 \mid \gcd(n,k-1)$, in particular $4 \mid n$, and $k \equiv 1$

$\pmod 4$, and hence trivially $b(k+1) = 1$. Now observe that by Lemma 8 the only possible values for $b(k-1)$ and $b(a)$ are $b(n)$ and $b(n)-1$, respectively. But then (1) implies $b(Q) = 1$, i.e. $2n \mid k^2 - 1$.

Conversely, assume that $b(Q) = 0$. Then (1) implies $b(k-1) \leq b(n) - 1$, while Lemma 8 in turn implies $b(a) \leq b(n) - 2$, for otherwise P3 is violated. But $b(a) \leq b(n) - 2$ then violates Property P2. $\square$

Involutions of type $\alpha^a\beta\gamma$

In this section we focus on Kronecker involutions that also include the reflection β. While this fact requires some adjustments by the fact that we are now considering involutions of type $\alpha^a\beta\gamma$, the subsection is mostly a compact transcript of the previous one.

Define $\omega'_a = \alpha^a\beta\gamma$ i.e. $\omega'_a(v_i) = u_{a-ki}$ and $\omega'_a(u_i) = v_{a-ki}$, and let $\Omega'_a : \mathbb{Z}_n \to \mathbb{Z}_n$ be a function defined as $\Omega'_a(i) = a - ki$. In this case, the requirements for the ω'_a being Kronecker involution are by Theorem 2 equivalent to:

- $\mathrm{P1}'$ Permutation ω'_a is color-reversing if and only if a is even.
- $\mathrm{P2}'$ Permutation ω'_a is an involution if and only if $a - ak \equiv 0 \pmod n$.
- $\mathrm{P3}'$ Permutation ω'_a may only fix an i-th spoke if and only if there exists an integer i, such that $\Omega'_a(i) \equiv i \pmod n$. As before, this is equivalent to saying that a is not allowed to be any multiple of $(k+1)$ in $\mathbb{Z}_n$.

We now describe an alternative reformulation of the property $\mathrm{P3}'$.

Lemma 10. *Assume that properties* $\mathrm{P1}'$ *and* $\mathrm{P2}'$ *hold. Then the property* $\mathrm{P3}'$ *is equivalent to*

$$b(\gcd(n, a)) < b(\gcd(n, k+1)).$$

Proof: First observe that $\gcd(k-1, k+1) = 2$, while $n \mid (k+1)(k-1)$. But, by $\mathrm{P2}'$ also $n \mid (k-1)a$ holds, hence every odd divisor of $\gcd(n, k+1)$ must also divide $\gcd(n, a)$. To conclude the proof it is enough to observe that whenever a is some multiple of $(k+1)$ in $\mathbb{Z}_n$ (i.e. the negation of P3), it must also hold that $b(\gcd(n, a)) \geq b(\gcd(n, k+1))$. $\square$ It is now easy to derive the existence version of our main theorem, for the involutitons of type $\alpha^a\beta\gamma$.

Corollary 11. *Let* $G = G(n, k)$ *with* $k^2 \equiv 1 \pmod n$, *and let a be an integer. Then G admits a Kronecker involution of type* $\alpha^a\beta\gamma$ *if and only if* $n \mid \frac{k^2-1}{2}$ *and* $k \equiv 3 \pmod 4$.

Proof: By Theorem 2 and Lemma 10, a permutation $\alpha^a\beta\gamma$ is a rim-switching Kronecker involution if and only if $b(\gcd(n, a)) < b(\gcd(n, k+1))$, in addition to the properties $\mathrm{P1}'$ and $\mathrm{P2}'$.

It is easy to see that in the case when these properties hold for $\alpha^a\beta\gamma$, we also have $n \mid \frac{k^2-1}{2}$ and $k \equiv 3 \pmod 4$. Indeed, as n and a are both even, it follows that $4 \mid \gcd(n, k+1)$, in particular $4 \mid n$, and $k \equiv 3 \pmod 4$, and hence trivially $b(k-1) = 1$. Now observe that by Lemma 10 the only possible values for $b(k+1)$ and $b(a)$ are $b(n)$ and $b(n) - 1$, respectively. But then (1) implies $b(Q) = 1$, i.e. $2n \mid k^2 - 1$.

Conversely, assume that $b(Q) = 0$. Then (1) implies $b(k+1) \leq b(n) - 1$, while Lemma 10 implies $b(a) \leq b(n) - 2$, for otherwise $\mathrm{P3}'$ is violated. But $b(a) \leq b(n) - 2$ then violates Property $\mathrm{P2}'$. $\square$

In the next section we prove that for any generalised Petersen graph except $G(10, 3)$, all quotients are isomorphic.

4 The quotients of generalised Petersen graphs

For a given generalised Petersen graph, so far we identified all its Kronecker involutions. In this section we determine the structure of the corresponding quotient graphs, for each of these involutions. Namely, the next two subsections deal with the structural part of the statements $a)$ and $b)$ of Theorem 1, respectively.

4.1 Involutions of D_n

We already know that the only Kronecker involution in the Dihedral group is the rotation $\alpha^{n/2}$, which is realized whenever $n \equiv 2 \pmod 4$ and k is odd. In order to prove $a)$ of Theorem 1, it is enough to show the following proposition, which describes the corresponding quotient graph explicitly.

Proposition 12. *For an odd n and an integer $k < \frac{n}{2}$, we have*

$$\mathrm{KC}\left(G\left(n,k\right)\right) \simeq \begin{cases} G\left(2n,k\right); & k \text{ is odd;} \\ G\left(2n,n-k\right); & k \text{ is even.} \end{cases}$$

Proof Proof of $a)$ from Theorem 1: Let $G \simeq G(n,k)$ and $G' \simeq \mathrm{KC}\left(G\right)$, for an odd integer n and $k < \frac{n}{2}$. The edges of $\mathrm{KC}\left(G\right)$ are naturally partitioned into the following three groups:

(E1) $u_i'v_i''$ and $u_i''v_i'$;

(E2) $u_i'u_{i+1}''$ and $u_i''u_{i+1}'$;

(E3) $v_i'v_{i+k}''$ and $v_i''v_{i+k}'$.

For easier notation, define k' to be equal to k or $n-k$, depending on whether k is odd or even, respectively. Furthermore, let $H \coloneqq G\left(2n,k'\right)$ and denote its vertex set with

$$V(H) = \left\{a_0, \ldots, a_{2n-1}, b_0, \ldots, b_{2n-1}\right\},$$

while its edge set consists of edges of the form a_ia_{i+1}, a_ib_i and $b_i, b_{i+k'}$. To show the left implication of Proposition 12, it is enough to show that $G' \simeq H$. Throughout the proof all subscripts for vertices of H (on the left-hand side) are assumed to be modulo $2n$, while all subscripts for vertices of G' (on the right-hand side) are assumed to be modulo n. To show an equivalence, we introduce a bijection $f : V(H) \to V(G')$, such that

$$a_i \quad \mapsto \quad \begin{cases} u_i' & \text{if } i \text{ is even,} \\ u_i'' & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad b_i \mapsto \begin{cases} v_i'' & \text{if } i \text{ is even,} \\ v_i' & \text{if } i \text{ is odd.} \end{cases}$$

for any $0 \leq i < 2n$. Since n is odd, f is clearly a bijection and it is enough to show that f is a homomorphism between H and G'. We now check that all edges of H map to edges in G'. First observe that in H, edges of types a_ia_{i+1} and a_ib_i map to these in (E2) and (E1), respectively. Indeed, by definition we have

$$f\left(a_ia_{i+1}\right) = \begin{cases} u_i'u_{i+1}'' & \text{if } i \text{ is even,} \\ u_i''u_{i+1}' & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad f\left(a_ib_i\right) = \begin{cases} u_i'v_i'' & \text{if } i \text{ is even,} \\ u_i''v_i' & \text{if } i \text{ is odd.} \end{cases}$$

Finally, for edges of type $b_i, b_{i+k'}$, we now observe that

$$f\left(b_i b_{i+k'}\right) = \begin{cases} v_i'' v_{i+k'}' & \text{if } i \text{ is even,} \\ v_i' v_{i+k'}'' & \text{if } i \text{ is odd.} \end{cases} \tag{2}$$

Indeed, if k is odd or even, we have

$$b_{i+k} \mapsto \begin{cases} v_{i+k}'' & \text{if } i+k \text{ is even,} \\ v_{i+k}' & \text{if } i+k \text{ is odd.} \end{cases} \quad \text{and} \quad b_{i-k+n} \mapsto \begin{cases} v_{i+n-k}'' & \text{if } i-k+n \text{ is even,} \\ v_{i+n-k}' & \text{if } i-k+n \text{ is odd,} \end{cases}$$

respectively. Keep in mind that all subscripts on the right hand side are modulo n. By (2) we conclude that edges of type $b_i b_{i+k'}$ correspond to the edges of type (E3) in G'. Since both G' and H are by definition cubic and of the same cardinality, the isomorphism follows. $\square$

It remains to describe the behavior of the rest of Kronecker involutions satisfying the conditions $n \equiv 0 \pmod 4$ and $n \mid \frac{k^2-1}{2}$, while $k < \frac{n}{2}$. In the next subsection we describe their equivalence (for fixed n, k), and also the corresponding quotient structure.

4.2 The rim-switching Kronecker involutions

Let us now turn to the Kronecker involutions containing permutation α, which are described by item b) in Theorem 1, so we assume that $k^2 \equiv 1 \pmod n$ and $Q = \frac{k^2-1}{n}$ is even. In addition, we assume that our involution is of type $\alpha^i\gamma$ (or $\alpha^i\beta\gamma$), whenever $k \equiv 1 \pmod n$ (or $k \equiv 3 \pmod n$), respectively. We will say that a positive integer a is *nice* whenever $\alpha^a\gamma$ or $\alpha^a\beta\gamma$, is a Kronecker involution.

In contrast with involutions from D_n, whenever $G(n,k)$ admits a rim-switching Kronecker involution, there may exist several different such involutions, so we will first enumerate all such involutions. These involutions could potentially give rise to several non-isomorphic quotients. In order to show that this is not the case, we will need the following extension of the LCF notion.

Definition 4. For an involution g without fixed points of type $[n] \to [n]$, we define $f(i) = g(i) - i$ and write, for short, $[f]$ instead of $[f(0), f(1), \ldots, f(n-1)]$.

It is easy to see that both graphs $C^+(n,k)$ and $C^-(n,k)$ from Definition 1 correspond to $[\frac{n}{2} + (k-1)x]$ and $[\frac{n}{2} - (k+1)x]$, respectively. In order to complete the proof of the main theorem, it remains to show that for all possible Kronecker involutions, the corresponding quotient is unique. We split the further analysis into two cases, depending on the value of $k \pmod 4$.

Case 1: $k \equiv 1 \pmod 4$

Define $a_{\min} = n/\gcd(n, k+1)$. The next lemma describes all nice values and hence enumerates the Kronecker involutions for this case.

Lemma 13. *The value $a_{\min}$ is the smallest nice value. Furthermore, a is nice whenever $a \equiv sa_{\min} \pmod n$ for some odd integer s.*

Proof: From P2 it follows that $a(k+1)$ is a multiple of n. In other words, there exists a positive integer C, such that $a = \frac{Cn}{k+1}$. It is clear that a is minimized whenever $Cn = \operatorname{lcm}(k+1, n)$, i.e.

$$a_{\min} = \frac{\operatorname{lcm}(k+1,n)}{k+1} = \frac{n}{\gcd(n,k+1)}.$$

Note that in general C may be some s-th multiple of $\mathrm{lcm}(k+1,n)/n$, however, by Lemma 8, such value of s corresponds to a nice value if and only if s is positive and odd. □

After showing that any positive odd s defines $a = sa'_{\min}$ and subseqently a Kronecker involution of the form $\alpha^a\gamma$, with $\Omega_a(i) = a + ki$, let us look at the corresponding quotient graph G'. By Definition 1 and Theorem 2 the graph G' is isomorphic to an outer-rim, augmented by a matching edges of type $i \sim \Omega_a^{-1}(i)$, which implies $G' \simeq [f_a]$, where

$$f_a(i) = \Omega_a^{-1}(i) - i = ik + a - i.$$

To show that for any odd s, all instances of corresponding Kronecker involutions are equivalent, we first prove the following lemma.

Lemma 14. *Let* $a' = a + \gcd(k-1, Q) \cdot a_{\min}$. *Then* $[f_a] \simeq [f_{a'}]$.

Proof: First notice $a_{\min} = \frac{Qn}{\gcd(Qn,Q(k+1))} = \frac{k-1}{\gcd(Q,k-1)}$. To prove the claim it is enough to observe that the LCF sequence of graph $[f_{a'}]$ is equivalent to the LCF sequence of $[f_a]$, cyclically shifted by one, i.e. $f_{a'}(i) = f_a(i+1)$. Indeed, this is clear as

$$\begin{aligned} f_{a'}(i) &= a + \gcd(k-1,Q)a_{\min} + i(k-1) \\ &= a + (i+1)(k-1) = f_a(i+1). \end{aligned}$$

□

We are now ready to show item $b_1)$ of Theorem 1.

Proposition 15. *Let* $k^2 \equiv 1 \pmod n$ *with* $n \mid \frac{k^2-1}{2}$ *and* $k \equiv 1 \pmod 4$. *Then* $G(n,k)$ *have unique quotient* $[f]$.

Proof Proof of $b_1)$ of Theorem 1: Let S be the set of all $\gcd(n,k+1)/2$ Kronecker involutions. Lemma 14 partitions S into equivalence classes with respect to the relation of having the corresponding quotients isomorphic. We show that all elements of S are members of the same equivalence class.

In other words, this is equivalent to being in an additive group of order $\frac{\gcd(n,k+1)}{2}$ and calculating the order of the element $\frac{\gcd(Q,k-1)}{2}$. Clearly, all classes of such partition of S have the same cardinality, while the number of these classes is equal to

$$\begin{aligned} \gcd\left(\frac{\gcd(n,k+1)}{2}, \frac{\gcd(Q,k-1)}{2}\right) &= \gcd\left(\frac{n}{2}, \frac{k+1}{2}, \frac{k-1}{2}, \frac{Q}{2}\right) \\ &\leq \gcd\left(\frac{k+1}{2}, \frac{k-1}{2}\right) = 1. \end{aligned}$$

But then any Kronecker involution corresponds to the unique quotient $C^+(n,k)$. For this case of $k \equiv 1 \pmod 4$, an easy example of such an involution is $\alpha^{n/2}\gamma$. □

Case 2: $k \equiv 3 \pmod 4$

Define $a'_{\min} = n/\gcd(n, k+1)$. We will say that a positive integer a is *nice* whenever $\omega_a = \alpha^a \beta\gamma$ is a Kronecker involution. We proceed by enumerating the Kronecker involutions for this case.

Lemma 16. *The value $a'_{\min}$ is the smallest nice value. Furthermore, a is nice whenever $a \equiv sa_{\min} \pmod n$ for some odd integer s.*

Proof: From P2$'$ it follows that $a(k-1)$ is a multiple of n. In other words, there exists a positive integer C, such that $a = \frac{Cn}{k-1}$. It is clear that a is minimized whenever $Cn = \operatorname{lcm}(k-1, n)$, i.e.

$$a'_{\min} = \frac{\operatorname{lcm}(k-1,n)}{k-1} = \frac{n}{\gcd(n, k-1)}.$$

Note that in general C may be some s-th multiple of $\operatorname{lcm}(k-1, n)/n$, however, by Lemma 10, such value of s corresponds to a nice value if and only if s is positive and odd. $\square$

After showing that any odd s defines $a = sa'_{\min}$ and subseqently a Kronecker involution of the form $\omega'_a = \alpha^a\beta\gamma$, with $\Omega'_a(i) = a - ki$. By Definition 1 and Theorem 2 the corresponding quotient graph G' is isomorphic to an outer-rim, augmented by matching edges of type $i \sim \Omega_a^{-1}(i)$, which implies $G' \simeq [f_a]$, where

$$f_a(i) = \Omega_a'^{-1}(i) - i = a - ik - i.$$

Before we can show the item b_2) of Theorem 1, we need tp prove the following lemma.

Lemma 17. *Let $a' = a + \gcd(k+1, Q) \cdot a'_{\min}$. Then $[f_a] \simeq [f_{a'}]$.*

Proof: Again notice $a'_{\min} = \frac{Qn}{\gcd(Qn, Q(k-1))} = \frac{k+1}{\gcd(Q,k+1)}$. We similarly prove the claim by observing $f_{a'}(i) = f_a(i-1)$. Indeed, we have

$$\begin{aligned} f_{a'}(i) &= a + \gcd(k+1, Q)a'_{\min} - i(k+1) \\ &= a - (i-1)(k+1) = f_a(i-1). \end{aligned}$$

$\square$

The following proposition is equivalent to the item b_2) of Theorem 1.

Proposition 18. *Let $k^2 \equiv 1 \pmod n$ with $n \mid \frac{k^2-1}{2}$ and $k \equiv 3 \pmod 4$. Then $G(n,k)$ have the unique quotient $C^-(n,k)$.*

Proof Proof of b_2) of Theorem 1: Let S be the set of all $\frac{\gcd(n,k-1)}{2}$ Kronecker involutions. Again, we show that Lemma 17 eventually covers the whole set S.

In this case one may consider an additive group of order $\frac{\gcd(n,k-1)}{2}$ and calculate the order of the element $\frac{\gcd(k+1,Q)}{2}$. The number of such orbits is equal to

$$\gcd\left(\frac{\gcd(n, n-k+1)}{2}, \frac{\gcd(k+1, Q)}{2}\right) = 1.$$

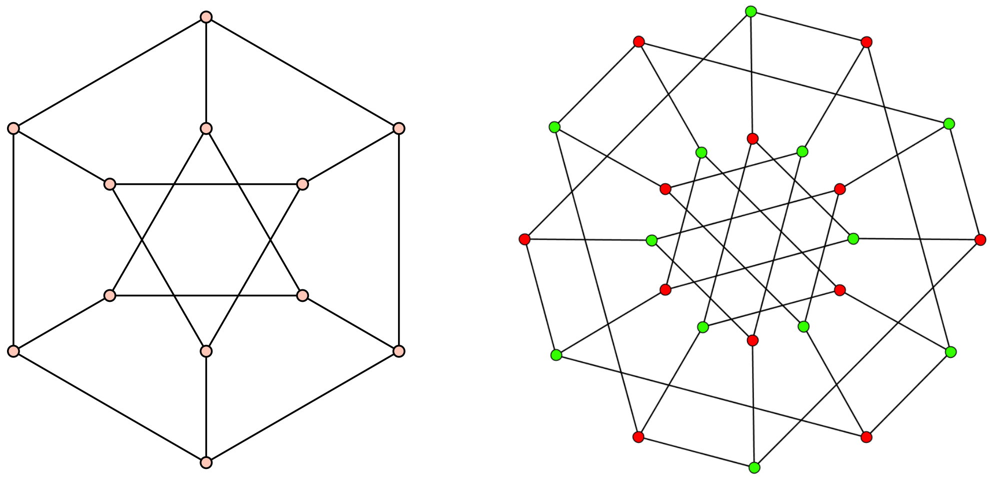

Fig. 2: The Dürer graph $G(6,2)$ and its Kronecker cover $KC(G(6,2))$ with proper vertex two-coloring.

We finally conclude that also in this case of $k \equiv 3 \pmod 4$, any Kronecker involution corresponds to the unique quotient $C^{-}(n,k)$. □ Since we described the quotients of all existing

Kronecker involutions, this concludes the proof of Theorem 1. We conclude with an observation about an easy example of rim-switching Kronecker involutions whenever they exist, by simply setting $a = n/2$.

Corollary 19. *Let* $G(n,k)$ *admit a rim-switching Kronecker involution. Then either* $\alpha^{n/2}\gamma$ *or* $\alpha^{n/2}\beta\gamma$ *is such a Kronecker involution.*

Proof: Assume $k \equiv 1 \pmod 4$ and observe that setting $a = n/2$ immediately satisfies P1 and P2. To conclude the proof it is hence enough to observe that $b(n/2) = b(n) - 1$, which is by Lemma 8 equivalent to P3. We omit the case of $k \equiv 3 \pmod 4$ as it is the same. □

5 Concluding remarks and future work

In this paper, we classified parameters (n,k) such that $G(n,k)$ is a Kronecker cover of some graph, and described the corresponding quotients. From our main result it easily follows:

Corollary 20. $\mathrm{KC}(G(n,k))$ *is itself a generalised Petersen graph if and only if* n *is odd.*

We analyzed the problem of Kronecker covers of the family of generalised Petersen graphs. It would be interesting to transfer this problem to the family of I-graphs [2, 3, 13, 8] or Rose-Window graphs [19], or some other families of cubic or quartic graphs.

Graphs $\mathrm{KC}(G(n,k))$ that are not generalised Petersen graphs, in other words if n is even, fall into two known classes, depending on the parity of k. If k is odd, we have $\mathrm{KC}(G(n,k)) = 2G(n,k)$. It would be interesting to investigate the family of graphs $\mathrm{KC}(G(n,k))$ with both n and k even. The smallest case is depicted in Figure 2. This is the Kronecker cover of the Dürer graph $G(6,2)$.

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