Discrete Mathematics and Theoretical Computer Science DMTCS vol. **21**:3, 2019, #16

Equitable Coloring and Equitable Choosability of Planar Graphs without chordal 4- and 6-Cycles

Aijun Dong[1*] Jianliang Wu[2]

[1] *School of Science, Shandong Jiaotong University, Jinan, 250357, P. R. China*
[2] *School of Mathematics, Shandong University, Jinan, 250100, P. R. China*

received 5th June 2018, revised 6th Mar. 2019, accepted 19th Apr. 2019.

A graph G is equitably k-choosable if, for any given k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. A graph is equitably k-colorable if the vertex set $V(G)$ can be partitioned into k independent subsets V_1, V_2, $\cdots$, V_k such that $||V_i| - |V_j|| \leq 1$ for $1 \leq i, j \leq k$. In this paper, we prove that if G is a planar graph without chordal 4- and 6-cycles, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 7\}$.

Keywords: equitable choosability, planar graph, discharging

1 Introduction

The terminology and notation used but undefined in this paper can be found in Bondy and Murty (1976). Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and, minimum degree of G, respectively. Particularly, we use $F(G)$ to denote the face set of G when G is a plane graph. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (resp. face) x in G. A vertex (resp. face) x is called a k-$vertex$ (resp. k-$face$), k^+-$vertex$ (resp. k^+-$face$), k^--$vertex$ or k^{--}-$vertex$, if $d(x) = k$, $d(x) \geq k$, $2 \leq d(x) \leq k$, or $1 \leq d(x) \leq k$. We use $(d_1, d_2, \cdots, d_n)$ to denote a face f if $d_1, d_2, \cdots, d_n$ are the degrees of vertices incident with the face f where $3 \leq n \leq 5$. Let $\delta(f)$ denote the minimal degree of vertices incident with f. In the following, let $f_i(v)$ denote the number of i-faces incident with v for each $v \in V(G)$. Let $n_i(f)$ denote the number of i-vertices which are incident with f. A graph G is k-degenerate if every subgraph of G has a vertex of degree at most k. A cycle C of length k is called a k-$cycle$. Moreover, if there exists an edge $xy \in E(G) - E(C)$ and x, $y \in V(C)$, then the cycle C is called a chordal k-$cycle$.

A proper k-coloring of a graph G is a mapping π from the vertex set $V(G)$ to the set of colors $\{1, 2, \cdots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. A graph G is equitably k-colorable

*This work was supported by the National Natural Science Foundation of China (Grant No. 61773015). It was also supported by China Postdoctoral Science Foundation Funded Project (Grant No.2014M561909); the Nature Science Foundation of Shandong Province of China (Grant No. ZR2014AM028, ZR2017LEM014) .

ISSN 1365–8050

if G has a proper k-coloring such that the sizes of the color classes differ by at most 1. The equitable chromatic number of G, denoted by $\chi_e(G)$, is the smallest integer k such that G is equitably k-colorable. The equitable chromatic threshold of G, denoted by $\chi_e^*(G)$, is the smallest integer k such that G is equitably l-colorable for every $l \geq k$. It is obvious that $\chi_e(G) \leq \chi_e^*(G)$ for any graph G. However these two parameters may not be the same. For example, if $K_{2n+1,2n+1}$ (n is a positive integer) is a complete bipartite graph, then $\chi_e(K_{2n+1,2n+1}) = 2$, $\chi_e^*(K_{2n+1,2n+1}) = 2n+2$.

In many applications of graph coloring, it is desirable that the color classes are not too large. For example, when using a coloring model to find an optimal final exam schedule, one would like to have approximately equal number of final exams in each time slot because the whole exam period should be as short as possible and the number of classrooms available is limited. Recently, Pemmaraju (2001), Janson and Ruciński (2002) used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence. In all of these applications, the fewer colors we use, the better the deviation bound is. Equitable coloring has a well-known property that restricts the size of each color class by its definition.

In 1970, Hajnal and Szemerédi (1970) proved that $\chi_e^*(G) \leq \Delta(G) + 1$ for any graph G. This bound is sharp as the example of $K_{2n+1,2n+1}$ shows. In 1973, Meyer (1973) introduced the notion of equitable coloring and made the following conjecture.

Conjecture 1 *If G is a connected graph which is neither a complete graph nor odd cycle, then $\chi_e(G) \leq \Delta(G)$.*

In 1994, Chen et al. (1994) put forth the following conjecture.

Conjecture 2 *For any connected graph G, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then $\chi_e^*(G) \leq \Delta(G)$.*

Chen et al. (1994); Chen and Lih (1994) proved Conjecture 2 for graphs with $\Delta(G) \leq 3$ or $\Delta(G) \geq \frac{|V(G)|}{2}$. Recently, Chen and Yen (2012) improved the former result and confirmed the Conjecture 2 for graphs with $\Delta(G) \geq \frac{|V(G)|}{3} + 1$. Yap and Zhang (1997, 1998) showed that Conjecture 2 holds for planar graphs with $\Delta(G) \geq 13$. Recently, Nakprasit (2012a) confirmed the Conjecture 2 for planar graphs with $\Delta(G) \geq 9$. Lih and Wu (1996) verified $\chi_e^*(G) \leq \Delta(G)$ for bipartite graphs other than complete bipartite graphs. Wang and Zhang (2000) proved Conjecture 2 for line graphs, and Kostochka and Nakprasit (2003, 2005) proved it for graphs with low average degree, and d-degenerate graphs with $\Delta(G) \geq 14d + 1$. Yan and Wang (2014) showed that Conjecture 2 holds for Kronecker products of complete multipartite graphs and complete graphs. Wu and Wang (2008), Luo et al. (2010) confirmed Conjecture 2 for some planar graphs with large girth, respectively. Recently, Li and Bu (2009), Zhu and Bu (2008), Dong et al. (2012a,b, 2013), Nakprasit (2012b) confirmed Conjecture 2 for some planar graphs with some forbidden cycles, respectively. Zhang and Wu (2011), Zhu et al. (2013) verified the Conjecture 2 for some series-parallel graphs and outerplanar graphs, respectively.

For a graph G and a list assignment L assigned to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors, an L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. A list assignment L for G is k-$uniform$ if $|L(v)| = k$ for all $v \in V(G)$. A graph G is equitably k-choosble if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices.

In 2003, Kostochka, Pelsmajer and West investigated the equitable list coloring of graphs. They proposed the following conjectures in Kostochka et al. (2003).

Conjecture 3 *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 4 *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{k,k}$ for some odd k.*

It has been proved that Conjecture 3 holds for graphs with $\Delta(G) \leq 3$ in Pelsmajer (2004); Wang and Lih (2004) and graphs with $\Delta(G) \leq 7$ in Kierstead and Kostochka (2012). Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in $K_{k,k}$) and $k \geq \max\{\Delta, \frac{|V(G)|}{2}\}$, or G is a connected interval graph and $k \geq \Delta(G)$ or G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$ in Kostochka et al. (2003). Pelsmajer proved that every graph is equitably k-choosable for any $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2} + 2$ in Pelsmajer (2004). Bu and his collaborators have established a series results for Conjecture 4 in class of planar graph as follows Li and Bu (2009); Zhu and Bu (2008); Zhu et al. (2013); Zhu and Bu (2010). Zhang and Wu proved Conjecture 4 for series-parallel graphs in Zhang and Wu (2011). Some improved results on planar graphs were obtained in Dong et al. (2012a), Dong et al. (2012b) and Dong et al. (2013).

In this paper, we improve the result in Li and Bu (2009) and confirm the Conjecture 2, Conjecture 4 for some planar graphs in which 4- and 6-cycles are allowed to exist, which shows that if G is a planar graph without chordal 4- and 6-cycles, then G is equitably k-colorable and equitably k-choosable where $k \geq \max\{\Delta(G), 7\}$.

2 Planar graphs without chordal 4- and 6-cycles

First let us introduce some lemmas.

Lemma 2.1 *Let G be a planar graph without chordal 4- or 6-cycles. Then in G, there is no 3-cycle adjacent to a 3-cycle, nor a 4-cycle adjacent to two 3-cycles. Furthermore, if $\delta(G) \geq 3$, then there is no 3-cycle adjacent to a 5-cycle, nor a 4-cycle adjacent to a 4-cycle.*

By Lemma 2.1, we have the following lemma.

Lemma 2.2 *Let G be a planar graph with $\delta(G) \geq 3$ and f be a 3-face which is incident with a 3-vertex in G. Then f is adjacent to at least one 6^+-face.*

Lemma 2.3 *Let G be a planar graph without chordal 4- and 6-cycles. If $\delta(G) \geq 4$, then G contains the configuration H depicted in Figure 1.*

Proof: Suppose to the contrary that G does not contain the configuration H depicted in Figure 1, i.e. none of the $(4,4,4)$-faces is adjacent to a $(4,4,4,4)$-face.

By Euler's formula, we have

$$\sum_{v\in V(G)}(2d(v)-6) + \sum_{f\in F(G)}(d(f)-6) = -6(|V|-|E|+|F|) = -12. \tag{1}$$

Define an initial charge function w on $V(G) \cup F(G)$ by setting $w(v) = 2d(v) - 6$ if $v \in V(G)$ and $w(f) = d(f) - 6$ if $f \in F(G)$, then $\sum_{x\in V(G)\cup F(G)} w(x) = -12$ by Equation (1). Now redistribute the charges according to the following discharging rules.

$D1$. If f is a 3-face incident with a vertex v, then v gives 1 to f if $d(v) = 4$ and f is a $(4,4,4)$-face, v gives $\frac{3}{4}$ if $d(v) = 4$ and f is a 3-face of another type, and v gives $\frac{3}{2}$ if $d(v) \geq 5$.

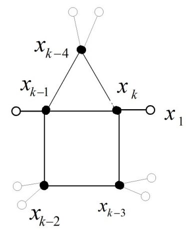


Fig. 1:

$D2$. If f is a 4-face incident with a vertex v, then v gives $\frac{1}{2}$ to f if $d(v) = 4$ and f is a $(4,4,4,4)$-face, v gives $\frac{2}{5}$ if $d(v) = 4$ and f is a 4-face of another type, and v gives $\frac{4}{5}$ if $d(v) \geq 5$.
$D3$. Transfer $\frac{1}{5}$ from each vertex v to the 5-face which is incident with v.

Let the new charge of each element $x \in V(G) \cup F(G)$ be $w'(x)$. In the following, we will show that $\sum_{x \in V(G) \cup F(G)} w'(x) \geq 0$, a contradiction to Equation (1). This will complete the proof.

Consider any vertex $v \in V(G)$, **suppose** $d(v) = 4$**.** Then $w(v) = 2$, $f_3(v) \leq 2$ by Lemma 2.1.
First, we assume that $f_3(v) = 2$. Then $f_4(v) = 0$ and $f_5(v) = 0$ by Lemma 2.1. Thus $w'(v) \geq 2 - 1 \times 2 = 0$ by $D1$.
Now we assume that $f_3(v) = 1$. Then $f_4(v) \leq 2$ by Lemma 2.1. If $f_4(v) = 2$, then $f_5(v) \leq 1$. Since G does not contain the configuration H depicted in Figure 1, thus $w'(v) \geq 2 - 1 - \frac{2}{5} \times 2 - \frac{1}{5} = 0$ or $w'(v) \geq 2 - \frac{3}{4} - \frac{1}{2} \times 2 - \frac{1}{5} = \frac{1}{20} > 0$ by $D1$, $D2$ and $D3$. If $f_4(v) \leq 1$, then $f_5(v) \leq 1$ by Lemma 2.1. Thus $w'(v) \geq 2 - 1 - \frac{1}{2} - \frac{1}{5} = \frac{3}{10} > 0$ by $D1$, $D2$ and $D3$.
Now we assume that $f_3(v) = 0$. Then $f_4(v) \leq 2$, $f_5(v) \leq 4$ by Lemma 2.1. Thus $w'(v) > 2 - \frac{1}{2} \times 2 - \frac{1}{5} \times 4 = \frac{1}{5} > 0$ by $D2$ and $D3$.

Suppose $d(v) = 5$**.** Then $w(v) = 4$, $f_3(v) \leq 2$ by Lemma 2.1. If $f_3(v) = 2$, then $f_4(v) \leq 1$ and $f_5(v) = 0$ by Lemma 2.1. Thus $w'(v) \geq 4 - \frac{3}{2} \times 2 - \frac{4}{5} = \frac{1}{5} > 0$ by $D1$ and $D2$. If $f_3(v) = 1$, then $f_4(v) \leq 2$ and $f_5(v) \leq 2$ by Lemma 2.1. Thus $w'(v) \geq 4 - \frac{3}{2} - \frac{4}{5} \times 2 - \frac{1}{5} \times 2 = \frac{1}{2} > 0$ by $D1$, $D2$ and $D3$. If $f_3(v) = 0$, then $f_4(v) \leq 2$ and $f_5(v) \leq 5$ by Lemma 2.1. Thus $w'(v) > 4 - \frac{4}{5} \times 2 - \frac{1}{5} \times 5 = \frac{7}{5} > 0$ by $D2$ and $D3$.

Suppose $d(v) \geq 6$**.** Then $w(v) = 2d(v) - 6$, $f_4(v) \leq d(v) - 2f_3(v)$, $f_5(v) \leq d(v) - 2f_3(v)$ by Lemma 2.1. So $w'(v) \geq 2d(v) - 6 - \frac{3}{2}f_3(v) - \frac{4}{5}f_4(v) - \frac{1}{5}f_5(v) \geq d(v) - 6 + \frac{1}{2}f_3(v) \geq d(v) - 6 \geq 0$ by $D1$, $D2$ and $D3$.

Consider any face $f \in F(G)$, **suppose** $d(f) = 3$**.** Then $w(f) = -3$. If f is a $(4,4,4)$-face, then $w'(f) = -3 + 1 \times 3 = 0$ by $D1$. Otherwise, $w'(f) \geq -3 + \frac{3}{4} + \frac{3}{4} + \frac{3}{2} = 0$ by $D1$.

Suppose $d(f) = 4$**.** Then $w(f) = -2$. If f is a $(4,4,4,4)$-face, we have that $w'(f) \geq -2 + \frac{1}{2} \times 4 = 0$ by $D2$. Otherwise, $w'(f) \geq -2 + \frac{2}{5} \times 3 + \frac{4}{5} = 0$ by $D2$.

Suppose $d(f) = 5$**.** Then $w(f) = -1$. We have $w'(f) \geq -1 + \frac{1}{5} \times 5 = 0$ by $D3$.

Suppose $d(f) \geq 6$**.** Then $w'(f) = w(f) \geq 0$. □

Lemma 2.4 *(Zhu and Bu (2008)) Let $S = \{x_1, x_2, \cdots, x_k\}$ be a set of k different vertices in G such that $G - S$ has an equitable k-coloring. If $|N_G(x_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable k-coloring.*

Lemma 2.5 *(Kostochka et al. (2003)) Let G be a graph with a k-uniform list assignment L. Let $S = \{x_1, x_2, \cdots, x_k\}$, where $x_1, x_2, \cdots, x_k$ are distinct vertices in G. If $G - S$ has an equitable L-coloring and $|N_G(x_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable L-coloring.*

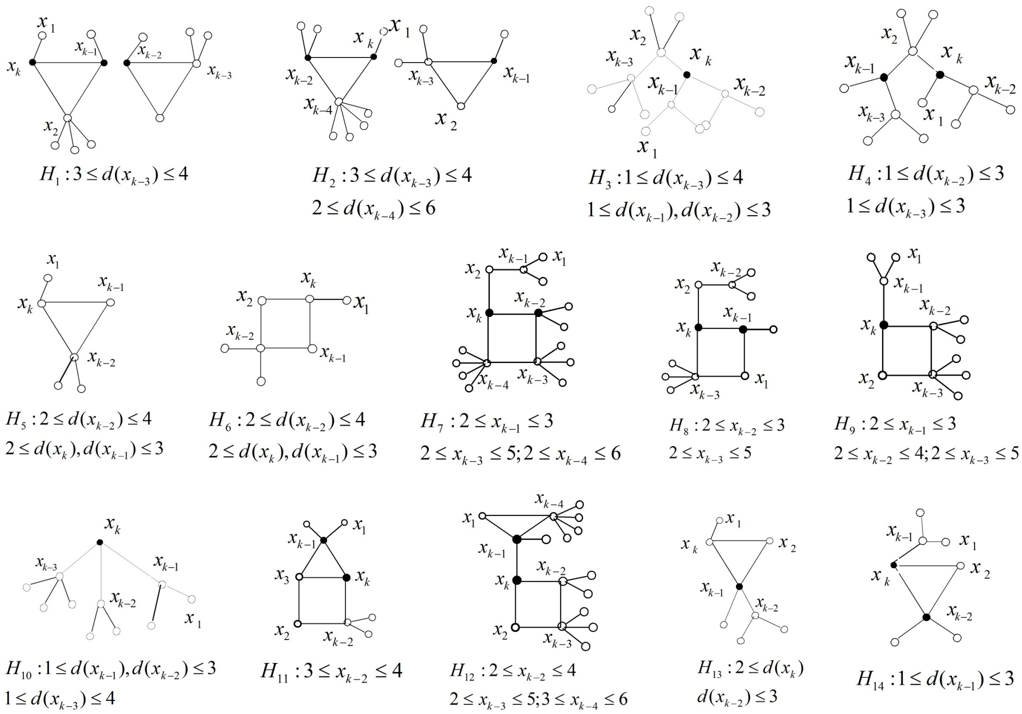
Lemma 2.6 *(Borodin (1996)) Every planar graph without adjacent triangles is 4-degenerate.*

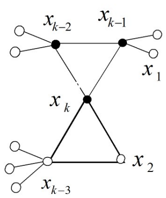
By Lemma 2.6, we have the following corollary.

Corollary 2.7 *Let G be a planar graph without chordal 4-cycles. Then G is 4-degenerate.*

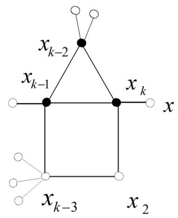
Lemma 2.8 *Let G be a connected planar graph with order at least 5 and without chordal 4- and 6-cycles. If $\delta(G) \leq 3$, then G has at least one of the configurations depicted in Figure 2.*

Proof: Suppose to the contrary that G does not contain the configurations $H_1 \ldots H_{41}$ depicted in Figure 2.

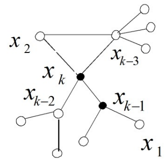




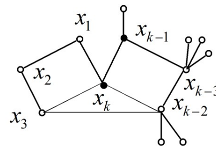
$H_{15}: 2 \le d(x_{k-3}) \le 5$



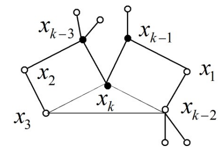
$H_{16}: 2 \le d(x_{k-3}) \le 5$



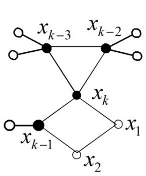
$H_{17}: 1 \le d(x_{k-2}) \le 3$
$2 \le d(x_{k-3}) \le 5$



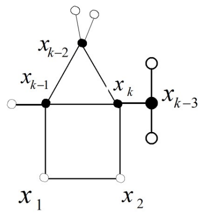
$H_{18}: 4 \le x_{k-2} \le 5$
$4 \le x_{k-3} \le 5$



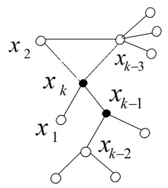
$H_{19}: 4 \le x_{k-2} \le 5$



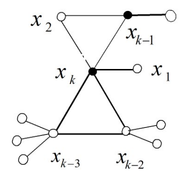
H_{20}



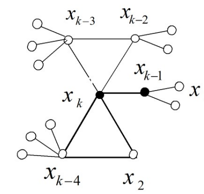
H_{21}



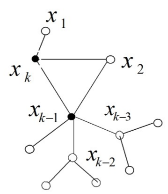
$H_{22}: 1 \le d(x_{k-2}) \le 3$
$2 \le d(x_{k-3}) \le 5$



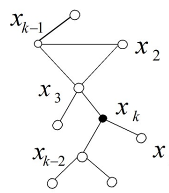
$H_{23}: 2 \le d(x_{k-2}) \le 4$
$2 \le d(x_{k-3}) \le 5$



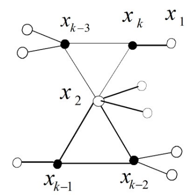
$H_{24}: 2 \le d(x_{k-2}) \le 4$
$2 \le d(x_{k-3}), d(x_{k-4}) \le 5$



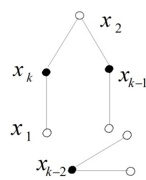
$H_{25}: 2 \le d(x_{k-2}), d(x_{k-3}) \le 3$



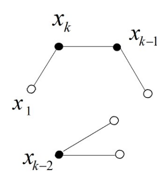
$H_{26}: 2 \le d(x_{k-1}) \le 3$
$1 \le d(x_{k-2}) \le 3$



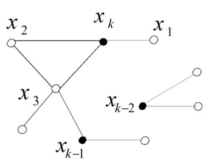
H_{27}



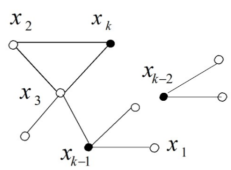
H_{28}



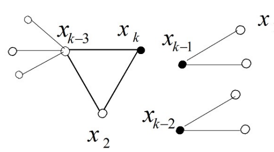
H_{29}



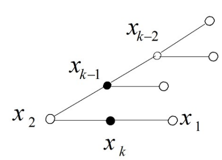
H_{30}



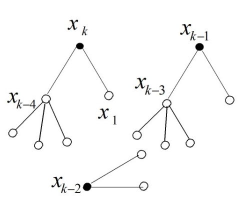
H_{31}



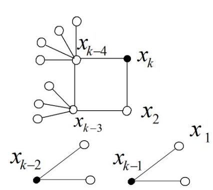
$H_{32}: 2 \le d(x_{k-3}) \le 5$



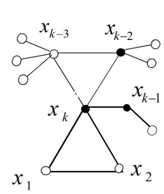
$H_{33}: 1 \le d(x_{k-2}) \le 3$



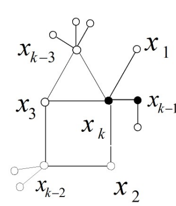
$H_{34}: 3 \le d(x_{k-3}), d(x_{k-4}) \le 4$



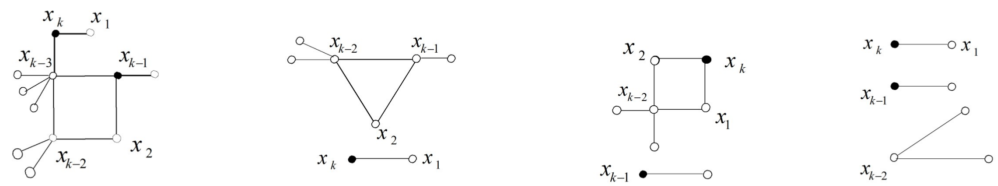
$H_{35}: 3 \le d(x_{k-3}) \le 5,$
$3 \le d(x_{k-4}) \le 6$



$H_{36}: 2 \le d(x_{k-3}) \le 5$



H_{37}



$H_{38}: 2 \le d(x_{k-2}) \le 4$
$5 \le d(x_{k-3}) \le 6$

$H_{39}: 2 \le d(x_{k-1}) \le 3$
$2 \le d(x_{k-2}) \le 4$

$H_{40}: 2 \le d(x_{k-2}) \le 4$

$H_{41}: 1 \le d(x_{k-2}) \le 2$

Fig. 2:

Each configuration depicted in Figure 2 is such that: (1) the vertices labelled x_k, x_{k-1}, x_{k-2} are distinct and the other vertices may coincide if they have the same degree and multiple edges cannot be resulted in; (2) solid vertices have no incident edges other than the ones shown; and (3) except for being specially pointed, the degree of a hollow vertex may be any integer from $[d, \Delta(G)]$, where d is the number of edges incident with the hollow vertex shown in the configuration; (4) the order of the vertices on the boundary of a 4-face can be rearranged except for the vertex which is also adjacent to other labelled vertex that is not on the boundary of the 4-face.

A face is said to be a $special\ face$ if it is a $(3,3,5^+)$-face, $(3,4,4)$-face, $(3,4,5)$-face or a $(3,4,6)$-faces. In the following, we call a 3-vertex a $special$ 3-$vertex$ if it is incident with a special face, otherwise, it is called a $simple$ 3-$vertex$.

Since G contains neither H_1 nor H_2, we obtain the following property.

Claim 1 *There is at most one special face in G.*

By Claim 1, G has at most two special 3-vertices. For convenience, let $n_3(v)$ denote the number of simple 3-vertices adjacent to v for each $v \in V(G)$. Since G contains neither H_3 nor H_4, we can conclude the following properties.

Claim 2 *For each $v \in V(G)$ with $d(v) \ge 4$, if v is adjacent to a simple 3-vertex which is adjacent to two 3^{--}-vertices, then it is not adjacent to another 4^{--}-vertex.*

Claim 3 *For any $v \in V(G)$ with $d(v) \ge 4$, v is adjacent to at most one simple 3-vertex which is adjacent to another 3^{--}-vertex.*

By Euler's formula $|V| - |E| + |F| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E|$, thus

$$\sum_{v \in V(G)} (3d(v) - 10) + \sum_{f \in F(G)} (2d(f) - 10) = -10(|V| - |E| + |F|) = -20. \quad (2)$$

Define an initial charge function w on $V(G) \cup F(G)$ by setting $w(v) = 3d(v) - 10$ if $v \in V(G)$ and $w(f) = 2d(f) - 10$ if $f \in F(G)$.

In the following, we divide the proof into four cases.

Case 1. $\delta(G) = 3$.

Since G does not contain the configuration H_5, G has the following property.

Fact 1 *Any* 3*-face in* G *is a* $(3,3,5^+)$*-,* $(3,4^+,4^+)$*- or* $(4^+,4^+,4^+)$*-face, i.e. there is no* $(3,3,4^-)$*-face.*

Since G does not contain the configuration H_6, G has the following property.

Fact 2 *Any* 4*-face in* G *is a* $(3,3,5^+,5^+)$*-,* $(3,4^+,4^+,4^+)$*- or* $(4^+,4^+,4^+,4^+)$*-face, i.e. there is no* $(3,3,3,3^+)$*- or* $(3,3,4,4^+)$*-face.*

For convenience, if a face is a $(3,3,5,5^+)$- or $(3,4,5^-,6^-)$-face, then we call it a $bad\ face$. The 3-vertex which is incident with a bad face is said to be a bad 3-$vertex$. If a vertex v is adjacent to a bad 3-vertex w and v is not incident with the bad face f which is incident with the vertex w, then we say that v is $weakly\ incident$ with the bad face f.

Now redistribute the charge according to the following discharging rules.

- $R1$. **Transfer** 1 **from each** 5^+**-vertex to every adjacent simple** 3**-vertex which is adjacent to exactly two** 3^{--}**-vertices**.

- $R2$. **Transfer** $\frac{1}{2}$ **from each** 4^+**-vertex to every adjacent simple** 3**-vertex which is adjacent to exactly one** 3^{--}**-vertex**.

- $R3$. **Transfer** $\frac{1}{3}$ **from each** 4^+**-vertex to every adjacent simple** 3**-vertex which is not adjacent to any** 3^{--}**-vertex**.

- $R4$. **Transfer** $\frac{1}{3}$ **from each** 6^+**-face** f **to every adjacent** 3**-face and** 4**-face via each common edge**.

- $R5$. **If** f **is a** 4**-face incident with a vertex** v**, then** v **gives** $\frac{1}{2}$ **to** f **if** $d(v) = 4$ **and** f **is a** $(3,4,5^-,5^-)$**- or** $(4,4,4,4^+)$**-face,** $\frac{1}{3}$ **if** $d(v) = 4$ **and** f **is either a** $(3,4,4^+,6^+)$**- or a** $(4,4^+,5^+,5^+)$**-face;**

  $\frac{1}{2}$ **if** $d(v) = 5$ **and** f **is a** $(5,5^+,5^+,5^+)$**-face;** $\frac{2}{3}$ **if** $d(v) = 5$ **and** f **is a** 4**-face of another type**;

  1 **if** $d(v) = 6$ **and** f **is a** $(3,3,6,6^+)$**- or** $(3,4,6,4^+)$**-face,** $\frac{2}{3}$ **if** $d(v) = 6$ **and** f **is a** $(3,6,5^+,5^+)$**- or** $(4^+,6,4^+,4^+)$**-face**;

  $\frac{4}{3}$ **if** $d(v) \geq 7$.

- $R6$. **If** f **is a** 3**-face incident with a vertex** v **with** $d(v) = 4$**, then** v **gives** $\frac{2}{3}$ **to** f if f is a $(3,4,4^+)$-face, $\frac{4}{3}$ if f is a $(4,4,4)$-face, 1 if f is a $(4,4,5^+)$- or $(4,5,5^+)$-face, 0 if f is a $(4,6^+,6^+)$-face;

  If f **is a** 3**-face incident with a vertex** v **with** $d(v) = 5$**, then** v **gives** $\frac{11}{6}$ **to** f **if** f **is a** $(3,5,3^+)$**-face,** 2 **if** f **is a** $(4,4,5)$**- or** $(4,5,5)$**-face,** $\frac{4}{3}$ **if** f **is a** $(5,5,5)$**-face,** 1 **if** f **is a** $(4,5,6^+)$**-,** $(5,5,6^+)$**- or** $(5,6^+,6^+)$**-face**;

  If f **is a** 3**-face incident with a vertex** v **with** $d(v) = 6$**, then** v **gives** 2 **to** f;

  If f **is a** 3**-face incident with a vertex** v **with** $d(v) \geq 7$**, then** v **gives** 3 **to** f **if** f **is a** $(3,3,7^+)$**- or** $(3,4,7^+)$**-face,** 2 **if** f **is a** $(3,5^+,7^+)$**- or** $(4^+,4^+,7^+)$**-face**.

- $R7$. **If** f **is a bad face and** v **is weakly incident with** f**, then** v **gives charge** $\frac{1}{2}$ **to** f.

In the following, let us check the new charge of each element x for $x \in V(G) \cup F(G)$.

For convenience, we use $f_k^\alpha(v)$ (respectively, $n_3^\alpha(v)$) to denote the number of k-faces (respectively, 3-vertices) which are incident with v and receive charge at least α from v according to the discharging rules.

By Claim 2, Claim 3, $R1$, $R2$ and $R3$, we have the following fact.

Fact 3 *For each $v \in V(G)$, obviously, $n_3^{\frac{1}{2}}(v) \le 1$, and if $n_3^1(v) \ne 0$, then $n_3(v) = 1$ and the degrees of other neighbors of v are at least 5.*

Since G contains no configurations H_7 and H_8, thus the following fact holds.

Fact 4 *For each $v \in V(G)$, v is weakly incident with at most one bad face. Furthermore, if v is weakly incident with a bad face, then $n_3(v) = 1$.*

Let $v \in V(G)$. **Suppose** $d(v) = 3$. Then $w(v) = -1$. Since G contains no configuration H_9, v is not weakly incident with any bad face. Since G contains no configuration H_{10}, v is adjacent to at least one 5^+-vertex or is adjacent to at least two 4^+-vertices. If v is a simple 3-vertex, then $w'(v) = -1 + 1 = 0$ by $R1$, $w'(v) = -1 + \frac{1}{2} \times 2 = 0$ by $R2$ or $w'(v) = -1 + \frac{1}{3} \times 3 = 0$ by $R3$. Otherwise, i.e. if v is a special 3-vertex, then $w'(v) = w(v) = -1$.

Suppose $d(v) = 4$. Then $w(v) = 2$.

First, we assume that v is weakly incident with a bad face. Since G contains no configuration H_{11}, we have $f_3(v) \le 1$. Additionally, if $f_3^\alpha = 1$, we have $\alpha = 0$ because G contains no configuration H_{12} and by $R6$. By Lemma 2.1, we have $f_4(v) \le 1$. Clearly, $w'(v) \ge 2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \frac{1}{2} > 0$ by Fact 4, $R2$, $R5$ and $R7$.

Now we assume that v is not weakly incident with a bad face. Clearly, we have $f_3(v) \le 2$. For convenience, we divide the proof into the following cases.

Case 1.1 $f_3(v) = 2$. Then $n_3(v) \le 1$, $f_4(v) = 0$ for the reason that G contains no configuration H_{13}, by Fact 1 and Lemma 2.1. If $f_3(v) = 2$, $n_3(v) = 1$, then we have that $f_3^{\frac{4}{3}}(v) = 0$ and $n_3^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configurations H_{15}, H_{14} and by $R6$, $R2$, $R1$. Clearly, $w'(v) \ge 2 - \frac{2}{3} - 1 - \frac{1}{3} = 0$ by $R6$ and $R3$. If $f_3(v) = 2$, $n_3(v) = 0$ and $f_3^{\frac{4}{3}}(v) \ne 0$, then we have that $w'(v) \ge 2 - \frac{4}{3} = \frac{2}{3} > 0$ for the reason that G contains no configuration H_{15} and by $R6$. If $f_3(v) = 2$, $n_3(v) = 0$ and $f_3^{\frac{4}{3}}(v) = 0$, then we have that $w'(v) \ge 2 - 1 \times 2 = 0$ by $R6$.

Case 1.2 $f_3(v) = 1$. Then $f_4(v) \le 2$ by Lemma 2.1.

Case 1.2.1 $f_4(v) = 2$.

If $f_4(v) = 2$ and the 3-face incident with v is a $(3, 4, 4^+)$-face, then $n_3(v) = 1$ and $n_3^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configurations H_{13}, H_{14} and by $R2$, $R1$. Thus $w'(v) \ge 2 - \frac{2}{3} - \frac{1}{2} \times 2 - \frac{1}{3} = 0$ by $R6$, $R5$, $R3$.

If $f_4(v) = 2$ and the 3-face incident with v is a $(4, 4, 4)$-face, then $n_3(v) = 0$, $f_4^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configuration H_{16} and $R5$. Thus $w'(v) \ge 2 - \frac{4}{3} - \frac{1}{3} \times 2 = 0$ by $R6$, $R5$.

If $f_4(v) = 2$ and the 3-face is a $(4, 4, 5^+)$- or $(4, 5, 5^+)$-face, then $n_3(v) \le 1$ for the reason that G contains no configuration H_{17}. First, we assume $n_3(v) = 1$. Since G contains no configurations H_{18},

H_{19} and by $R5$, we have that $f_4^{\frac{1}{2}}(v) = 0$. So $w'(v) \geq 2 - 1 - \frac{1}{3} \times 2 - \frac{1}{3} = 0$ by $R6$, $R5$ and $R3$. Now, we assume that $n_3(v) = 0$. Thus $w'(v) > 2 - 1 - \frac{1}{2} \times 2 = 0$ by $R6$ and $R5$.

If $f_4(v) = 2$ and the 3-face is a $(4, 6^+, 6^+)$-face, then $f_3^{\frac{2}{3}}(v) = 0$ by $R6$. Furthermore, as $n_3(v) \leq 2$, we have that $w'(v) > 2 - \frac{1}{2} \times 2 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6} > 0$ by $R5$, $R3$ and $R2$. By Fact 1, this concludes the case where $f_4(v) = 2$.

Case 1.2.2 $f_4(v) = 1$.

If $f_4(v) = 1$ and the 3-face is a $(4, 4, 4)$-face, then $n_3(v) = 0$ for the reason that G contains no configurations H_{16}, H_{20} and H_{21}. Thus $w'(v) \geq 2 - \frac{4}{3} - \frac{1}{2} = \frac{1}{6} > 0$ by $R6$ and $R5$.

If $f_4(v) = 1$ and the 3-face is a $(4, 4, 5^+)$- or $(4, 5, 5^+)$-face, then $n_3(v) = 1$ and $n_3^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configurations H_{17}, H_{22} and by $R2$, $R1$. Thus $w'(v) \geq 2 - 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$ by $R6$, $R5$, $R3$.

If $f_4(v) = 1$ and the 3-face is a $(3, 4, 4^+)$-face, then $n_3(v) = 1$ and $n_3^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configurations H_{13} and H_{14} and by $R2$, $R1$. Thus $w'(v) \geq 2 - \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = \frac{1}{2} > 0$ by $R6$, $R5$ and $R3$.

If $f_4(v) = 1$ and the 3-face is a $(4, 6^+, 6^+)$-face, then $f_3^{\frac{2}{3}}(v) = 0$, $n_3(v) \leq 2$ by Fact 2 and $R6$. Thus $w'(v) \geq 2 - \frac{1}{2} - \frac{1}{2} \times 2 = \frac{1}{2} > 0$ by $R5$ and $R2$. By fact 1, this completes this subcase.

Case 1.2.3 $f_4(v) = 0$.

If the 3-face is a $(4, 4, 4)$-face, then $n_3(v) \leq 1$ and $n_3^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configurations H_{17}, H_{22} and by $R2$, $R1$. Thus $w'(v) \geq 2 - \frac{4}{3} - \frac{1}{3} = \frac{1}{3} > 0$ by $R6$ and $R3$.

If the 3-face is a $(3, 4, 4^+)$- or $(4, 4^+, 5^+)$-face, then $n_3(v) \leq 2$ for the reason that G contains no configuration H_{13}. Thus $w'(v) \geq 2 - 1 - 1 = 0$ by Fact 3, $R5$ and $R1$. By Fact 1, this conclude the subcase $f_4(v) = 0$.

Case 1.3 $f_3(v) = 0$. Then $f_4(v) \leq 2$ by Lemma 2.1.

If $f_4(v) = 2$, then $n_3(v) \leq 2$ by Fact 2. Thus $w'(v) \geq 2 - \frac{1}{2} \times 2 - 1 = 0$ by Fact 3, $R6$ and $R1$. If $f_4(v) = 1$, then $n_3(v) \leq 3$ by Fact 2. Thus $w'(v) \geq 2 - \frac{1}{2} - \frac{1}{2} - \frac{1}{3} \times 2 = \frac{1}{3} > 0$ by $R5$, $R2$ and $R3$. Otherwise, $f_4(v) = 0$, then $n_3(v) \leq 4$. Thus $w'(v) \geq 2 - \frac{1}{2} - \frac{1}{3} \times 3 = \frac{1}{2} > 0$ by Fact 3 and $R2$ and $R3$.

Suppose $d(v) = 5$. Then $w(v) = 5$.

Case 1.4 v is weakly incident with a bad face. Clearly, $f_3(v) \leq 2$. Furthermore, if $f_3(v) = 2$, then $f_4(v) \leq 1$ by Lemma 2.1.

If $f_3(v) = 2$ and $f_4(v) = 1$, then one of the two 3-faces must be adjacent to a bad face which is weakly incident with v by Lemma 2.1. Obviously, it is a $(3, 5, 3^+)$-face. In detail, it is a special face (i.e. a $(3, 5, 3)$-face) or a $(3, 5, 4^+)$-face. Since G contains no configuration H_{23}, the other 3-face is neither a $(4, 4, 5)$- nor a $(4, 5, 5)$-face. Thus $w'(v) \geq 5 - \frac{11}{6} - \frac{4}{3} - \frac{2}{3} - \frac{1}{2} = \frac{2}{3} > 0$ or $w'(v) \geq 5 - \frac{11}{6} - \frac{4}{3} - \frac{2}{3} - \frac{1}{2} - \frac{1}{2} = \frac{1}{6} > 0$ by Fact 4, $R6$, $R5$, $R7$ and $R2$.

If $f_3(v) = 2$ and $f_4(v) = 0$, we have that $w'(v) \geq 5 - 2 \times 2 - \frac{1}{2} - \frac{1}{2} = 0$ by $R6$, $R2$ and $R7$.

If $f_3(v) \leq 1$, then $f_4(v) \leq 2$. We have that $w'(v) \geq 5 - 2 - \frac{2}{3} \times 2 - \frac{1}{2} - \frac{1}{2} = \frac{2}{3} > 0$ by $R6$, $R5$, $R7$ and $R2$.

Case 1.5 v is not weakly incident with a bad face. Clearly, $f_3(v) \leq 2$.

Case 1.5.1 $f_3(v) = 2$. Then $f_4(v) \leq 1$.

If both of the 3-faces are $(4,4,5)$- or $(4,5,5)$-faces, then $n_3(v) = 0$ for the reason that G contains no configuration H_{24}. Thus $w'(v) \geq 5 - 2 \times 2 - \frac{2}{3} = \frac{1}{3} > 0$ by $R6$ and $R5$.

If only one of the 3-faces is a $(4,4,5)$- or $(4,5,5)$-face, then the other 3-face is not a $(3,5,3^+)$-face for the reason that G contains no configuration H_{23}. Thus $n_3(v) \leq 1$ and $n_3^1 = 0$ by the Fact 3. We have that $w'(v) \geq 5 - 2 - \frac{4}{3} - \frac{2}{3} - \frac{1}{2} = \frac{1}{2} > 0$ by $R6$, $R5$ and $R2$.

If both of the 3-faces are $(3,5,3^+)$-faces, then $n_3(v) = 2$, $n_3^{\frac{1}{2}}(v) = 0$ for the reason that G contains no configurations H_{25}, H_{26} and by $R2$, $R1$. Thus $w'(v) \geq 5 - \frac{11}{6} \times 2 - \frac{2}{3} - \frac{1}{3} \times 2 = 0$ by $R6$, $R5$ and $R3$.

If only one of the 3-faces is a $(3,5,3^+)$-face, then $n_3(v) \leq 2$, $n_3^{\frac{1}{2}}(v) \leq 1$ for the reason that G contains no configurations H_{25} and H_{26} and by $R2$, $R1$. Thus $w'(v) \geq 5 - \frac{11}{6} - \frac{4}{3} - \frac{2}{3} - \frac{1}{3} - \frac{1}{2} = \frac{1}{3} > 0$ by $R6$, $R5$, $R3$ and $R2$.

If any of the 3-faces does not belong to $(3,5,3^+)$-, $(4,4,5)$- and $(4,5,5)$-faces, then $n_3(v) \leq 1$. Thus $w'(v) \geq 5 - \frac{4}{3} \times 2 - \frac{2}{3} - 1 = \frac{2}{3} > 0$ by $R6$, $R5$ and $R1$.

Case 1.5.2 $f_3(v) = 1$. Then $f_4(v) \leq 2$, $n_3(v) \leq 4$ (v could be adjacent to five 3-vertices, but at most four of them are simple) by Lemma 2.1. Clearly, $w'(v) \geq 5 - 2 - \frac{2}{3} \times 2 - \frac{1}{2} - \frac{1}{3} \times 3 = \frac{1}{6} > 0$ by Fact 3, $R6$, $R5$, $R2$ and $R3$.

Case 1.5.3 $f_3(v) = 0$. Then $f_4(v) \leq 2$, $n_3(v) \leq 5$ by Lemma 2.1. Clearly, $w'(v) \geq 5 - \frac{2}{3} \times 2 - \frac{1}{2} - \frac{1}{3} \times 4 = \frac{11}{6} > 0$ by Fact 3, $R5$, $R2$ and $R3$.

Suppose $d(v) = 6$. Then $w(v) = 8$.

First, we assume that v is weakly incident with a bad face. Clearly, $f_3(v) \leq 3$. If $f_3(v) = 3$, then $f_4(v) = 0$ by Lemma 2.1. Clearly, $w'(v) \geq 8 - 2 \times 3 - \frac{1}{2} - \frac{1}{2} = 1 > 0$ by Fact 4, $R6$, $R7$ and $R2$. If $f_3(v) \leq 2$, then $f_4(v) \leq 2$. Clearly, $w'(v) \geq 8 - 2 \times 2 - 1 \times 2 - \frac{1}{2} - \frac{1}{2} = 1 > 0$ by Fact 4, $R6$, $R5$, $R7$ and $R2$.

Now we assume that v is not weakly incident with a bad face. Clearly, $f_3(v) \leq 3$. If $f_3(v) = 3$, then $f_4(v) = 0$, $n_3(v) \leq 3$ (a 3-face is incident with at most one simple 3-vertex) by Lemma 2.1. Thus $w'(v) \geq 8 - 2 \times 3 - \frac{1}{2} - \frac{1}{3} \times 2 = \frac{5}{6} > 0$ by Fact 3, $R6$, $R2$ and $R3$. If $f_3(v) = 2$. Then $f_4(v) \leq 2$, $n_3(v) \leq 4$ by Lemma 2.1. Thus $w'(v) \geq 8 - 2 \times 2 - 1 \times 2 - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{1}{2} > 0$ by $R6$, $R5$, $R3$ and $R2$. If $f_3(v) \leq 1$, then $f_4(v) \leq 3$, $n_3(v) \leq 6$ by Lemma 2.1. Clearly, $w'(v) > 8 - 2 - 1 \times 3 - \frac{1}{3} \times 5 - \frac{1}{2} = \frac{5}{6} > 0$ by $R6$, $R5$, $R3$ and $R2$.

Suppose $d(v) = 7$. Then $w(v) = 11$.

First, we assume that v is weakly incident with a bad face. Clearly, $f_3(v) \leq 3$ by Lemma 2.1. Furthermore, $f_3^3(v) \leq 1$ for the reason that G contains no configuration H_{27} and by $R6$. If $f_3(v) = 3$, then $f_4(v) \leq 1$ by Lemma 2.1. Clearly, $w'(v) \geq 11 - 3 - 2 \times 2 - \frac{4}{3} - \frac{1}{2} - \frac{1}{2} = \frac{5}{3} > 0$ by $R6$, $R5$, $R7$ and $R2$. If $f_3(v) \leq 2$, then $f_4(v) \leq 3$. Clearly, $w'(v) \geq 11 - 3 - 2 - \frac{4}{3} \times 3 - \frac{1}{2} - \frac{1}{2} = 1 > 0$ by $R6$, $R5$, $R7$ and $R2$.

Now we assume that v is not weakly incident with a bad face. Clearly, we have $f_3(v) \leq 3$. Since G contains no configuration H_{27}, there exists at most one $(3,4,7)$-face which is incident with v. If $f_3(v) = 3$, then $f_4(v) \leq 1$, $n_3(v) \leq 4$ by Lemma 2.1. Thus $w'(v) \geq 11 - 3 - 2 \times 2 - \frac{4}{3} - \frac{1}{3} \times 3 - \frac{1}{2} = \frac{7}{6} > 0$ by Fact 3, $R6$, $R5$, $R3$ and $R2$. If $f_3(v) = 2$, then $f_4(v) \leq 3$, $n_3(v) \leq 5$ by Lemma 2.1. Thus $w'(v) \geq 11 - 3 - 2 - \frac{4}{3} \times 3 - \frac{1}{3} \times 4 - \frac{1}{2} = \frac{1}{6} > 0$ by Fact 3, $R6$, $R5$, $R3$ and $R2$. If $f_3(v) \leq 1$, then $f_4(v) \leq 3$, $n_3(v) \leq 7$ by Lemma 2.1. Thus $w'(v) \geq 11 - 3 - \frac{4}{3} \times 3 - \frac{1}{3} \times 6 - \frac{1}{2} = \frac{3}{2} > 0$ by Fact 3, $R6$, $R5$, $R3$ and $R2$.

Suppose $d(v) \geq 8$. Then $w(v) = 3d(v) - 10$.
In any case, whether v is weakly incident with a bad face or not, we have

$$f_3(v) + f_4(v) \leq \frac{3}{4}d(v) \tag{3}$$

by Lemma 2.1. Moreover,

$$f_3^3(v) \leq 1 \tag{4}$$

for the reason that G contains no configuration H_{27} and by $R6$. Since a 3-face has at most one simple 3-vertex,

$$n_3(v) \leq f_3(v) + d(v) - 2f_3(v) = d(v) - f_3(v). \tag{5}$$

It follows from (3) and (5) that $f_4(v) \leq \frac{3}{4}d(v) - f_3(v)$ and $n_3(v) \leq d(v) - f_3(v)$, respectively. Thus $w'(v) \geq 3d(v) - 10 - 3 - 2(f_3(v) - 1) - \frac{4}{3}f_4(v) - \frac{1}{2} - \frac{1}{3}(n_3(v) - 1) - \frac{1}{2} \geq 3d(v) - 10 - 3 - 2f_3(v) + 2 - d(v) + \frac{4}{3}f_3(v) - \frac{1}{2} - \frac{1}{3}d(v) + \frac{1}{3}f_3(v) + \frac{1}{3} - \frac{1}{2} = \frac{5}{3}d(v) - \frac{1}{3}f_3(v) - \frac{70}{6}$ by $R6$, $R5$, $R2$, $R3$ and $R7$. Since

$$f_3(v) \leq \frac{1}{2}d(v),$$

we obtain $w'(v) \geq \frac{3}{2}d(v) - \frac{70}{6} \geq \frac{1}{3} > 0$.

Now we consider $f \in F(G)$. **Suppose** $d(f) = 3$. Then $w(f) = -4$. By Fact 1, we only discuss the following situations. If f is a special face $(3, 3, 5^+)$-face, then we have that $w'(f) \geq -4 + \frac{11}{6} + \frac{1}{3} = -\frac{11}{6} > -2$ by Lemma 2.2, $R6$ and $R4$. If f is a $(3, 4, 4)$-, $(3, 4, 5)$- or $(3, 4, 6)$-face, we have that $w'(f) \geq -4 + \frac{2}{3} \times 2 + \frac{1}{3} = -\frac{7}{3}$ by Lemma 2.2, $R6$ and $R4$. If f is a $(3, 4, 7^+)$-face, then $w'(f) \geq -4 + \frac{2}{3} + 3 + \frac{1}{3} = 0$ by Lemma 2.2, $R6$ and $R4$. If f is a $(3, 5^+, 5^+)$-face, then $w'(f) \geq -4 + \frac{11}{6} \times 2 + \frac{1}{3} = 0$ by Lemma 2.2, $R6$ and $R4$. If f is a $(4, 4, 4)$-face, then $w'(f) \geq -4 + \frac{4}{3} \times 3 = 0$ by $R6$. If f is a $(4, 4, 5^+)$-face, then $w'(f) \geq -4 + 1 \times 2 + 2 = 0$ by $R6$. If f is a $(4, 5, 5^+)$-face, we have $w'(f) \geq -4 + 1 + 2 \times 2 = 1 > 0$ by $R6$. If f is a $(4, 6^+, 6^+)$-face, then $w'(f) \geq -4 + 2 \times 2 = 0$ by $R6$. If f is a $(5, 5, 5)$-face, we have that $w'(f) \geq -4 + \frac{4}{3} \times 3 = 0$ by $R6$. If f is a $(5^+, 5^+, 6^+)$-face, we have that $w'(f) \geq -4 + 1 \times 2 + 2 = 0$ by $R6$.

Suppose $d(f) = 4$. Then $w(f) = -2$. If f is a $(3, 3, 5, 5^+)$-face, then it is a bad face. Thus $w'(f) \geq -2 + \frac{1}{2} \times 2 + \frac{2}{3} \times 2 = \frac{1}{3} > 0$ by $R5$ and $R7$. If f is a $(3, 3, 6^+, 6^+)$-face, then $w'(v) \geq -2 + 1 \times 2 = 0$ by $R5$. If f is a $(3, 4, 4, 4)$- or $(3, 4, 4, 5)$-face, then it is a bad face. Thus $w'(f) \geq -2 + \frac{1}{2} + \frac{1}{2} \times 2 + \frac{1}{2} = 0$ by $R5$ and $R7$. If f is a $(3, 4, 4, 6)$-face, then it is a bad face. Thus $w'(f) \geq -2 + \frac{1}{2} + \frac{1}{3} \times 2 + 1 = \frac{1}{6} > 0$ by $R5$ and $R7$. If f is a $(3, 4, 4, 7^+)$-face, then we have that $w'(v) \geq -2 + \frac{1}{3} \times 2 + \frac{4}{3} = 0$ by $R5$. If f is a $(3, 4, 5, 5)$-face, then it is a bad face. Thus $w'(f) \geq -2 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} \times 2 = \frac{1}{3} > 0$ by $R5$ and $R7$. If f

is a $(3,4,5,6)$-face, then it is a bad face. Thus $w'(f) \geq -2 + \frac{1}{2} + \frac{1}{3} + \frac{2}{3} + 1 = \frac{1}{2} > 0$ by $R5$ and $R7$. If f is a $(3,4,5,7^+)$-face, then $w'(f) \geq -2 + \frac{1}{3} + \frac{2}{3} + \frac{4}{3} = \frac{1}{3} > 0$ by $R5$. If f is a $(3,4,6^+,6^+)$-face, then $w'(f) \geq -2 + \frac{1}{3} + 1 \times 2 = \frac{1}{3} > 0$ by $R5$. If f is a $(3,5^+,5^+,5^+)$-face, then $w'(f) \geq -2 + \frac{2}{3} \times 3 = 0$ by $R5$. If f is a $(4,4,4,4^+)$-face, then $w'(f) \geq -2 + \frac{1}{2} \times 4 = 0$ by $R5$. If f is a $(4^+,4^+,5^+,5^+)$-face, then $w'(f) \geq -2 + \frac{1}{3} \times 2 + \frac{2}{3} \times 2 = 0$ by $R5$.

Suppose $d(f) = 5$. Then $w'(f) = w(f) = 0$.

Suppose $d(f) \geq 6$. Then $w'(f) \geq w(f) - \frac{1}{3} \times d(f) = 2d(f) - 10 - \frac{1}{3} \times d(f) \geq 0$ by $R4$.

From the above discussion, if x is neither a special vertex nor a special face, then $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$. Let w'_s denote the total new charge of the special 3-vertices and the special 3-faces. Since the new charge of the special 3-vertices is -1 (see the case "$d(v) = 3$") and since the new charge of the special face is at least -2 if it is a $(3,3,5^+)$-face and at least $-\frac{7}{3}$ if it is a $(3,4,4)$-, a $(3,4,5)$-, or a $(3,4,6)$-face (see the case "$d(f) = 3$"), Claim 1 implies that $w'_s \geq \min\{-2-1-1, -\frac{7}{3} - 1\} = -4$. So we obtain that

$$\sum_{x \in V(G) \cup F(G)} w'(x) \geq -4, \tag{6}$$

a contradiction to Equation 2.

Case 2. $\delta(G) = 2$ and there are at most two 2-vertices in G.

Since G contains no structure isomorphic to the configuration H_5, the 3-faces which are incident with 2-vertices may be $(2,3,5)$- or $(2,4^+,4^+)$-faces. Since G contains no structure isomorphic to the configuration H_6, the 4-faces which are incident with 2-vertices may be $(2,3^-,5^+,5^+)$- or $(2,4^+,4^+,4^+)$-faces.

The discharging rules are the same as the rules in Case 1 except for the charge which is given to a 3- or 4-face which is incident with 2-vertices. For each $v \in V(G)$, if $d(v) \geq 4$, then v gives charge $\frac{2}{3}$ to its incident $(2,x,y)$-face f; and v gives charge $\frac{1}{3}$ to its incident $(2,x,y,z)$-face f only if the face f is not adjacent to other 4-faces which are incident with v, otherwise, v gives charge $\frac{1}{3}$ to only one of the adjacent 4-faces. Clearly, the charge which is given to a $(2,x,y)$- (resp. $(2,x,y,z)$)-face is not greater than that which is given to $(3,x,y)$- (resp. $(3,x,y,z)$)-faces. For each $v \in V(G)$, the number of $(2,x,y)$- (resp. $(2,x,y,z)$)-faces which is incident with and accept charge from v is not greater than that of $(3,x,y)$- (resp. $(3,x,y,z)$)-faces which is incident with v. So we can guarantee the new charge of each element $x \in V(G) \cup F(G)$ is larger than or equal to zero except for the special 3-vertices, the special 3-faces, the 2-vertices and the 3- or 4-faces which are incident with the 2-vertices. For convenience, let w'_{t1} (resp. w'_{t2}) denote the total new charge of one 2-vertex (resp. two 2-vertices) and the faces which are incident with the 2-vertex (resp. the two 2-vertices).

Suppose that there exists only one 2**-vertex in** G. If the 2-vertex is incident with one 3-face, then it will be not incident with any 4-face by Lemma 2.1. Since G contains no configuration H_5, the 3-face is a $(2,3^+,5^+)$- or $(2,4^+,4^+)$-face, thus $w'_{t1} \geq -4 - 4 + \frac{2}{3} = -\frac{22}{3}$ or $w'_{t1} \geq -4 - 4 + \frac{2}{3} \times 2 = -\frac{20}{3}$. If the 2-vertex is incident with a 4-face, then it may be incident with two 4-faces. Furthermore, the 4-face is a $(2,3^+,5^+,5^+)$- or a $(2,4^+,4^+,4^+)$-face for the reason that G contains no configuration H_6. Clearly, $w'_{t1} \geq -2 - 2 - 4 + \frac{1}{3} \times 2 = -\frac{22}{3}$ or $w'_{t1} \geq -2 - 2 - 4 + \frac{1}{3} \times 3 = -7$. From the above discussion, we obtain that

$$w'_{t1} \geq \min\{-7, -\frac{20}{3}, -\frac{22}{3}\} = -\frac{22}{3}. \quad (7)$$

By (6), we have that $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -4 + w'_{t1} \geq -4 - \frac{22}{3} = -\frac{34}{3}$, a contradiction to Equation 2.

Suppose that there exist two 2**-vertices in** G. If the two 2-vertices are incident with a same 3-face, then f is a $(2, 2, 5^+)$-face for the reason that G contains no configuration H_5. Thus $w'_{t2} \geq -4 \times 2 - 4 + \frac{2}{3} = -\frac{34}{3}$. If the two 2-vertices are incident with a same 4-face, then the 4-face is a $(2, 2, 5^+, 5^+)$-face for the reason that G contains no configuration H_6. Since each of the two 2-vertices may be incident with another 4-face, we have that $w'_{t2} \geq -2 - 2 - 2 - 4 - 4 + \frac{1}{3} \times 2 = -\frac{40}{3}$. If the two 2-vertices are not incident with a same face, then the discussion is similar to the situation when there exists only one 2-vertex in G. By (7), we have $w'_{t2} \geq -\frac{22}{3} \times 2 = -\frac{44}{3}$. From the above discussion, we have $w'_{t2} \geq \min\{-\frac{44}{3}, -\frac{34}{3}, -\frac{40}{3}\} = -\frac{44}{3}$. By (6), we have that $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -4 - \frac{44}{3} = -\frac{56}{3}$, a contradiction to Equation 2.

Case 3. $\delta(G) = 2$ and there are at least three 2-vertices in G.

Since G contains no configurations $H_{28} \ldots H_{35}$, G has the following properties.

Fact 5 *Any vertex v is adjacent to at most one 2-vertex.*

Fact 6 *No two 2-vertices are adjacent to each other.*

Fact 7 *For each $v \in V(G)$ with $d(v) \geq 4$, if v is adjacent to a 2-vertex, then it is not incident with any 3-face that is incident with a 3-vertex.*

Fact 8 *If v is adjacent to a 3-vertex, then it is not incident with any 3-face that is incident with a 2-vertex.*

Fact 9 *Every 3-face in G that is incident with a 2-vertex is a $(2, 6^+, 6^+)$-face.*

Fact 10 *If a vertex is adjacent to a 2-vertex, then it is not adjacent to any 3-vertex that is adjacent to another 3^{--}-vertex.*

Fact 11 *There is at most one 2-vertex which is adjacent to a k-vertex ($3 \leq k \leq 4$) in G.*

Fact 12 *Any 4-face that is incident with a 2-vertex in G is a $(2, 3^+, 7^+, 7^+)$- or $(2, 6^+, 6^+, 6^+)$-face.*

For convenience, we call a 2-vertex a $special$ 2-$vertex$ if it is adjacent to a k-vertex ($3 \leq k \leq 4$), otherwise a $simple$ 2-$vertex$. By Fact 11, there is at most one special 2-vertex. Let $n_2(v)$ denote the number of simple 2-vertices which are adjacent to v. Obviously, $n_2(v) \in \{0, 1\}$ by Fact 5.

Now redistribute the charge according to the following discharging rules.

For each $x \in V(G) \bigcup F(G)$, if x is neither a 2-vertex nor a face which is not incident with any 2-vertex, then the discharging rules are the same as those in Case 1. Otherwise, the following discharging rules are abided.

- $R8$**. Transfer** 2 **from each** 5^+**-vertex to every adjacent** 2**-vertex.**
- $R9$**. Transfer** 2 **from each** 6^+**-vertex to every incident** 3**-face.**
- $R10$**. If** f **is a** 4**-face which is incident with a** 2**-vertex and** v**, then** v **gives** 0 **to** f **if** $d(v) = 3$**,** 4 **or** 5**;** $\frac{2}{3}$ **if** $d(v) = 6$**;** $\frac{4}{3}$ **if** $d(v) \geq 7$**.**

By Fact 10, $R1$ and $R2$, we have the following fact.

Fact 13 *For each $v \in V(G)$, if $n_2(v) = 1$, then $n_3^{\frac{1}{2}}(v) = 0$.*

In the following, let us check the new charge of each element $x \in V(G) \bigcup F(G)$.

Consider any vertex $v \in V(G)$, suppose $d(v) = 2$. Then $w(v) = -4$, $n_2(v) = 0$ by Fact 6. Since G contains no structure H_9, v is not weakly incident with any bad face. If v is a simple 2-vertex, then $w'(v) = -4 + 2 \times 2 = 0$ by $R8$. Otherwise, v is a special 2-vertex. We have $w'(v) = w(v) = -4$.

Suppose $d(v) \geq 3$. If $n_2(v) = 0$, then the discussion is similar to the one of the corresponding situation in Case 1. In the following, we only focus on the situation $n_2(v) = 1$.

Since G contains no configurations H_7 and H_8, we have the following fact.

Fact 14 *For each $v \in V(G)$, if $n_2(v) = 1$, then v is not weakly incident with any bad face.*

Suppose $d(v) = 3$. By Fact 7, v is a simple 3-vertex. Since G contains no configuration H_{10}, v is adjacent to at least one 5^+-vertex or is adjacent to at least two 4^+-vertices. We have $w'(v) = -1 + 1 = 0$ by $R1$, or $w'(v) = -1 + \frac{1}{2} \times 2 = 0$ by $R2$.

Suppose $d(v) = 4$. Then $w(v) = 2$, $f_3(v) \leq 1$ by Fact 9.

First we assume $f_3(v) = 1$. Then $f_4(v) \leq 2$. If the 3-face is a $(4, 4, 4)$-face, then $f_4(v) \leq 1$ and $n_3(v) = 0$ for the reason that G contains no configuration H_{16}, H_{17} and by Fact 12. Thus $w'(v) \geq 2 - \frac{4}{3} - \frac{1}{2} - 0 = \frac{1}{6} > 0$ by $R6$, $R9$ and $R10$. Otherwise, if the 3-face is not a $(4, 4, 4)$-face, we have $f_4(v) \leq 2$, $f_4^{\frac{1}{2}}(v) \leq 1$ and $n_3(v) \leq 1$ for the reason that G contains no H_{13} and by Fact 12, $R5$, $R10$. Thus $w'(v) \geq 2 - 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0$ by Fact 13, $R6$, $R9$, $R5$ and $R3$.

Now we assume that $f_3(v) = 0$. Then $f_4(v) \leq 2$, $f_4^{\frac{1}{3}} \leq 1$ and $n_3(v) \leq 3$ for the reason that G contains no chordal 6-cycles and by $R10$. Thus $w'(v) \geq 2 - \frac{1}{2} - \frac{1}{3} \times 3 = \frac{1}{2} > 0$ by Fact 13, $R10$ and $R3$.

Suppose $d(v) = 5$. Then $w(v) = 5$, $f_3(v) \leq 2$.

Case 3.1 $f_3(v) = 2$. Then $f_4(v) \leq 1$ for the reason that G contains no chordal 4- and 6-cycles. By Fact 9 and Fact 12, the 4-face which is incident with v is a $(2, 5, 7^+, 7^+)$-face. Thus $f_4^{\frac{2}{3}}(v) = 0$ by $R10$. Additionally, since G contains no configuration H_{36}, we have that $f_3^{\frac{11}{6}}(v) = 0$ by Fact 7 and $R6$. Thus $w'(v) \geq 5 - \frac{4}{3} \times 2 - 2 - 0 = \frac{1}{3} > 0$ by $R6$, $R9$, $R5$ and $R8$.

Case 3.2 $f_3(v) = 1$. Since G contains neither chordal 4- and 6-cycles nor configuration H_{37}, we have $f_4(v) \leq 3$.

Case 3.2.1 $f_4(v) = 3$. Then $n_3(v) \leq 1$ by Fact 7 and Fact 12. Furthermore, since at most one 4-face which is incident with v is not a $(2, 5, 7^+, 7^+)$-face, we have that $f_4^{\frac{2}{3}}(v) \leq 1$ by $R5$ and $R10$. Thus $w'(v) \geq 5 - 2 - \frac{2}{3} - 2 - \frac{1}{3} = 0$ by Fact 13, $R6$, $R9$, $R5$, $R8$ and $R3$.

Case 3.2.2 $f_4(v) = 2$.

Case 3.2.2.1 The 2-vertex which is adjacent to v is not around any of the two 4-faces. If the 3-face which is incident with v is a $(5, 6^+, 6^+)$-face, then $f_4^{\frac{2}{3}}(v) \leq 2$, $n_3(v) \leq 2$ as G contains no configuration H_{37} and by $R5$, Fact 7. Thus we have $w'(v) \geq 5 - 1 - \frac{2}{3} \times 2 - 2 - \frac{1}{3} \times 2 = 0$ by $R6$, $R9$, $R5$, $R8$ and $R3$. Otherwise, the 4-faces which are incident with v are both $(5, 5^+, 5^+, 5^+)$-faces as G contains no configuration H_{37}. Clearly, $n_3(v) = 0$. Thus we have $w'(v) \geq 5 - 2 - \frac{1}{2} \times 2 - 2 = 0$ by $R6$, $R9$, $R5$ and $R8$.

Case 3.2.2.2 The 2-vertex which is adjacent to v is around one of the two 4-faces. Then $f_4^{\frac{2}{3}}(v) \leq 1$, $n_3(v) \leq 1$ as G contains no configuration H_{38} and by $R5$, Fact 7. Thus we have $w'(v) \geq 5 - 2 - \frac{2}{3} - 2 - \frac{1}{3} = 0$ by $R6$, $R9$, $R5$, $R8$ and $R3$.

Case 3.2.2.3 The 2-vertex which is adjacent to v is around the two 4-faces. Then $f_4^{\frac{1}{2}}(v) = 0$, $n_3(v) \leq 1$ by Fact 12. Thus we have $w'(v) \geq 5 - 2 - 2 - \frac{1}{3} = \frac{2}{3} > 0$ by $R6$, $R9$, $R8$ and $R3$.

Case 3.2.3 $f_4(v) = 1$. Then $n_3(v) \leq 2$ by Fact 9 and Fact 10. If $n_3(v) = 2$, then the 4-face is adjacent to the 3-face and the 3-face is a $(5, 6^+, 6^+)$-face as G contains no configuration H_{37} and H_{38}. We have $w'(v) \geq 5 - 1 - \frac{2}{3} - 2 - \frac{1}{3} \times 2 = \frac{2}{3} > 0$ by $R6$, $R5$, $R8$ and $R3$. Otherwise, $n_3(v) \leq 1$. We have $w'(v) \geq 5 - 2 - \frac{2}{3} - 2 - \frac{1}{3} = 0$ by $R6$, $R5$, $R8$ and $R3$.

Case 3.2.4 $f_4(v) = 0$. Then $n_3(v) \leq 2$ by Fact 9 and Fact 10. We have $w'(v) \geq 5 - 2 - 2 - \frac{1}{3} \times 2 = \frac{1}{3} > 0$ by $R6$, $R8$ and $R3$.

Case 3.3 $f_3(v) = 0$. Then $f_4(v) \leq 3$ for the reason that G contains no chordal 6-cycles. Since at most two 4-faces which are incident with v are not $(2, 5, 7^+, 7^+)$-faces, we have $f_4^{\frac{2}{3}}(v) \leq 2$ by $R5$. Furthermore, $n_3(v) \leq 4$. Thus $w'(v) \geq 5 - \frac{2}{3} \times 2 - \frac{1}{3} \times 4 - 2 = \frac{1}{3} > 0$ by $R5$, $R3$ and $R8$.

Suppose $d(v) = 6$. Then $w(v) = 8$, $f_3(v) \leq 3$. If $f_3(v) = 3$, then $f_4(v) = 0$, $n_3(v) = 0$ for the reason that G contains no chordal 4- and 6-cycles and by Fact 7. Thus $w'(v) \geq 8 - 2 \times 3 - 2 = 0$ by $R6$, $R9$ and $R8$. If $f_3(v) = 2$, then $f_4(v) \leq 2$, $n_3(v) \leq 1$ for the reason that G contains no chordal 4- and 6-cycles and by Fact 7, Fact 8. Since G contains no configuration H_{38} and by $R10$, we have that $f_4^1(v) = 0$. Thus $w'(v) \geq 8 - 2 \times 2 - \frac{2}{3} \times 2 - 2 - \frac{1}{3} = \frac{1}{3} > 0$ by Fact 11, $R6$, $R9$, $R10$, $R8$ and $R3$. If $f_3(v) \leq 1$, then $f_4(v) \leq 3$, $n_3(v) \leq 5$. Since G contains no configuration H_{38}, we have that $f_4^1(v) = 0$. Thus $w'(v) \geq 8 - 2 - \frac{2}{3} \times 3 - 2 - \frac{1}{3} \times 5 = \frac{1}{3} > 0$ by Fact 13, $R6$, $R9$, $R10$, $R8$ and $R3$.

Suppose $d(v) = 7$. Then $w(v) = 11$, $f_3(v) \leq 3$. By Fact 7, there is no $(3, 4, 7)$-face which is incident with v. If $f_3(v) = 3$, then $f_4(v) \leq 1$, $n_3(v) = 0$ for the reason that G contains no chordal 4- and 6-cycles and by Fact 7. Thus $w'(v) \geq 11 - 2 \times 3 - \frac{4}{3} - 2 = \frac{5}{3} > 0$ by $R6$, $R9$, $R10$ and $R8$. If $f_3(v) = 2$, then $f_4(v) \leq 3$, $n_3(v) \leq 2$ for the reason that G contains no chordal 4- and 6-cycles and by Fact 7, Fact 8. Thus $w'(v) \geq 11 - 2 \times 2 - \frac{4}{3} \times 3 - \frac{1}{3} \times 2 - 2 = \frac{1}{3} > 0$ by Fact 13, $R6$, $R9$, $R10$, $R3$ and $R8$. If $f_3(v) = 1$, then $f_4(v) \leq 4$, $n_3(v) \leq 4$ for the reason that G contains no chordal 6-cycles and by Fact 7, Fact 8. Thus $w'(v) \geq 11 - 2 - \frac{4}{3} \times 4 - \frac{1}{3} \times 4 - 2 = \frac{1}{3} > 0$ by Fact 13, $R6$, $R9$, $R10$, $R3$ and $R8$. If $f_3(v) = 0$, then $f_4(v) \leq 4$, $n_3(v) \leq 6$ for the reason that G contains no chordal 6-cycles. Thus $w'(v) \geq 11 - \frac{4}{3} \times 4 - \frac{1}{3} \times 6 - 2 = \frac{5}{3} > 0$ by Fact 13, $R10$, $R3$ and $R8$.

Suppose $d(v) \geq 8$. Then $w(v) = 3d(v) - 10$. By Fact 7, there is no $(3, 4, 8^+)$-face which is incident with v. Since $n_3(v) + 2f_3(v) + 1 \leq d(v)$, we have that

$$n_3(v) \leq d(v) - 2f_3(v) - 1.$$

Since G contains no chordal 4- and 6-cycles, we have that $f_3(v) + f_4(v) \leq \frac{3}{4}d(v) + 1$. Thus

$$f_4(v) \leq \frac{3}{4}d(v) - f_3(v) + 1.$$

Thus $w'(v) \geq 3d(v) - 10 - 2f_3(v) - \frac{4}{3}f_4(v) - \frac{1}{3}n_3(v) - 2 \geq 3d(v) - 10 - 2f_3(v) - d(v) + \frac{4}{3}f_3(v)) -$

$\frac{4}{3} - \frac{1}{3}d(v) + \frac{2}{3}f_3(v) + \frac{1}{3} - 2 = \frac{5}{3}d(v) - \frac{39}{3} \geq \frac{1}{3} \geq 0$ by Fact 13, $R6$, $R9$, $R10$, $R3$ and $R8$.

Consider $f \in F(G)$. **Suppose** $d(f) = 3$. Then $w(f) = -4$ and $n_2(f) \leq 1$. If $n_2(f) = 1$, then f is a $(2, 6^+, 6^+)$-face by Fact 9. Thus $w'(f) \geq -4 + 2 \times 2 = 0$ by $R9$. Otherwise, the discussion is similar to the corresponding situation when $d(f) = 3$ in Case 1, so it is omitted here.

Suppose $d(f) = 4$. Then $w(f) = -2$, $n_2(f) \leq 1$ by Fact 6.

If $n_2(f) = 1$. Then f is a $(2, 3^+, 7^+, 7^+)$- or a $(2, 6^+, 6^+, 6^+)$-face by Fact 12. Thus $w'(f) \geq -2 + \frac{4}{3} \times 2 = \frac{2}{3} > 0$ or $w'(v) \geq -2 + \frac{2}{3} \times 3 = 0$ by $R10$. If $n_2(f) = 0$, then the discussion is similar to the corresponding situation when $d(f) = 4$ in Case 1, so it is omitted here.

Suppose $d(f) \geq 5$. Then the discussion is similar to the corresponding situation in Case 1 and is omitted here.

From the above discussion, we can obtain that $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$ that is not a special 3-vertex, a special 2-vertex, nor a special face. From (6), we have $w'_s \geq -4 - 4 = -8$ by Claim 1 and Fact 11. So we obtain $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -8$, a contradiction to Equation 2.

Case 4 $\delta(G) = 1$.

Now, the 3-faces in G are $(3^-, 5^+, 5^+)$-faces or $(4^+, 4^+, 4^+)$-faces and any 4-face that is incident with a 2-vertex is a $(2, 5^+, 5^+, 5^+)$-face for the reason that G contains no configurations H_{39} and H_{40}. Then there is neither any special 3-vertex nor any special face in G.

Case 4.1 There is only one 1-vertex in G.

Case 4.1.1 There are at most two 2-vertices in G.

The discharging rules are the same as the rules in Case 1 except for the charge which is given to a 3- or 4-face which is incident with 2-vertices. For each $v \in V(G)$, if $d(v) \geq 5$, then v gives charge 1 to its incident $(2, x, y)$-face f; and v gives charge $\frac{1}{2}$ to its incident $(2, x, y, z)$-face f only if the face f is not adjacent to other 4-faces which are incident with v, otherwise, v gives charge $\frac{1}{2}$ to only one of the adjacent 4-faces. Clearly, the charge which is given to a $(2, x, y)$- (resp. $(2, x, y, z)$)-face is not greater than that which is given to $(3, x, y)$- (resp. $(3, x, y, z)$)-faces. For each $v \in V(G)$, the number of $(2, x, y)$- (resp. $(2, x, y, z)$)-faces which is incident with and accept charge from v is not greater than that of $(3, x, y)$- (resp. $(3, x, y, z)$)-faces which is incident with v. So we can guarantee the new charge of each element $x \in V(G) \cup F(G)$ is larger than or equal to zero except for the 2-vertices and the 3- or 4-faces which are incident with the 2-vertices. For convenience, let w'_{t1} (resp. w'_{t2}) denote the total new charge of one 2-vertex (resp. two 2-vertices) and the faces which are incident to the 2-vertex (resp. the two 2-vertices).

Suppose that there is one 2-vertex in G. If the 2-vertex is incident with one 3-face, then it will be not incident with any 4-face as G contains no chordal 4-cycles. Since the 3-face is a $(2, 5^+, 5^+)$-face, we have that $w'_{t1} \geq -4 - 4 + 1 \times 2 = -6$. If the 2-vertex is incident with some 4-faces, since each such 4-face is a $(2, 5^+, 5^+, 5^+)$-face, we have that $w'_{t1} \geq -2 - 2 - 4 + \frac{1}{2} \times 4 = -6$. From the above discussion, we obtain that

$$w'_{t1} \geq -6. \tag{8}$$

So $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -7 + w'_{t1} \geq -7 - 6 \geq -13$ (a 1-vertex has charge -7), a contradiction to Equation 2.

Suppose that there are two 2-vertices in G. Since the two 2-vertices are not incident with a same 3- or 4-face, by (8), we have that $w'_{t2} \geq -6 \times 2 = -12$. So $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -7 - 12 = -19$, a contradiction to Equation 2.

Case $4.1.2$ There are at least three 2-vertices in G. The discharging rules are the same as Case 3. It follows from the discussion which is the same as the situation in Case 3 that $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -7 - 4 = -11$, a contradiction to Equation 2.

Case 4.2 There are at least two 1-vertices in G.

If there are two 1-vertices in G, then there is neither a 2-vertex nor a third 1-vertex in G for the reason that G contains no configuration H_{41}. The discharging rules are the same as Case 1. It follows from the discussion which is the same as the situation in Case 1 that $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -7 \times 2 = -14$, a contradiction to Equation 2. □

Lemma 2.9 *(Hajnal and Szemerédi (1970)) Every graph has an equitable k-coloring whenever $k \geq \Delta(G) + 1$.*

Lemma 2.10 *(Pelsmajer (2004); Wang and Lih (2004)) Every graph G with maximum degree $\Delta(G) \leq 3$ is equitably k-choosable whenever $k \geq \Delta(G) + 1$.*

In the following, let us give the proof of the main theorem.

Theorem 2.11 *If G is a planar graph without chordal 4- and 6-cycles, then G is equitably k-colorable where $k \geq \max\{7, \Delta(G)\}$.*

Proof: Let G be a counterexample with fewest vertices. If each component of G has at most four vertices, then $\Delta(G) \leq 3$. Clearly, G is equitably k-colorable by Lemma 2.9. Otherwise, there is at least one component with at least five vertices.

For convenience, we divide all the configurations in Figure 1 and Figure 2 into two classes according to whether it contains the vertex which is labelled x_{k-3} or not. A configuration belongs to C_1 if it contains the vertex labelled x_{k-3}, otherwise, it belongs to C_2.

Suppose that G has one of the configurations of C_1. In the following, we show how to find a set S in order to apply Lemma 2.4. For convenience, let S' be the set of the labelled vertices of this configuration. For example, if G has the configuration H depicted in Figure 1, then let $S' = \{x_k, x_{k-1}, \cdots, x_{k-4}, x_1\}$. By Corollary 2.7, G is 4-degenerate. Thus starting from S', we can find the remaining unspecified vertices to obtain the set S of Lemma 2.4 from highest to lowest indices by choosing a vertex with the minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of G, we have $G - S$ is equitably k-colorable. By Lemma 2.4, we can obtain that G is equitably k-colorable, a contradiction.

Thus G has a configuration of C_2 and $\delta(G) \leq 3$ by Lemma 2.3. Similarly, let S'' be the set of the labelled vertices of this configuration, in which the vertices are labelled as they are in Figure 2. Let $G' = G - S''$. If there exists a vertex $v \in V(G')$ such that $d_{G'}(v) \leq 3$ or there exists a vertex $u \in \{x_1, x_2, x_3\} \cap S''$ such that $d_G(u) \leq 4$, then we label v or u with x_{k-3} and let $S''' = S'' \cup \{x_{k-3}\}$. By Corollary 2.7, G is 4-degenerate. Now starting from S''', we can find the remaining unspecified vertices to obtain the set S of Lemma 2.4 from highest to lowest indices by choosing a vertex with the minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of G, we have $G-S$ is equitably k-colorable. By Lemma 2.4, we can obtain that G is equitably k-colorable, a contradiction.

Thus $\delta(G') \geq 4$ and $d_G(v) \geq 5$ for each vertex $v \in \{x_1, x_2, x_3\} \cap S''$. Clearly, it follows the Fact.

Fact 15 *For each $x \in V(H') - \{x_k, x_{k-1}, x_{k-2}\}$, we have that $d_G(x) \geq 5$ where $H' \in C_2$.*

Now we can easily get that G has only one configuration that belongs to C_2. Otherwise, $\delta(G') \leq 3$. Additionally, by Lemma 2.3, G' contains the configuration H of Figure 1. If G does not contain the configuration H_{41}, then by Fact 15, at most one 1-vertex, at most two 3^--vertices and at most one special face can exist in G simultaneously, i.e. G contains the configuration H_{39}. Let us now show a self-contradictory conclusion by a discharging procedure. The discharging rules are the same as Case 1 in Lemma 2.8. Clearly, we can guarantee that the new charge of each face other than the special face, and each vertex $v \in V(G)$ with $d(v) \geq 4$ is larger than or equal to zero. Hence $\sum_{x \in V(G) \cup F(G)} w'(x) \geq -7 + -4 \times 2 - 4 = -19$, a contradiction to $\sum_{x \in V(G) \cup F(G)} w(x) = -20$.

Thus G contains the configuration H_{41}. Additionally, from the above discussion, we know G has no configuration H, and G' has the configuration H in Figure 1. It is clear that one of the vertices $\{x_k, x_{k-1}, x_{k-2}, x_1\}$ of configuration H_{41} in Figure 2 must be adjacent to one of the vertices $\{x_k, x_{k-1}, x_{k-2}\}$ of configuration H in Figure 1. It is not difficult to find a set $\bar{S}$, starting from which, we can find the remaining unspecified vertices in S of Lemma 2.4 from highest to lowest indices by choosing a vertex with the minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of G, we have that $G-S$ is equitably k-colorable. By Lemma 2.4, we have that G is equitably k-colorable, a contradiction. In the following, we give the detailed steps on how to find the set $\bar{S}$.

For convenience, we use w_1, w_2, w_3 and w_4 to denote the vertices x_k, x_{k-1}, x_{k-2} and x_1 of configuration H_{41} in Figure 2, respectively, and use u_1, u_2 and u_3 to denote the vertices x_k, x_{k-1} and x_{k-2} of configuration H in Figure 1, respectively.

If there exists one 1-vertex which is adjacent to one of the vertices in $\{u_1, u_2, u_3\}$, then the 1-vertex only may be w_2 or w_3 from the above discussion. Without loss of generality, we assume w_2 and u' are adjacent to u for which $\{u, u'\} \subset \{u_1, u_2, u_3\}$. Now we label the vertices w_2, w_1, w_3, u, u' with $x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}$, respectively. We choose $\bar{S} = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}\}$.

Otherwise, if w_3 is adjacent to one of the vertices in $\{u_1, u_2, u_3, u_4\}$ such that $d_G(w_3) = 2$, for convenience, we assume w_3 and u' are adjacent to u for which $\{u, u'\} \subset \{u_1, u_2, u_3\}$. Now we label the vertices $w_1, w_2, w_3, u, u', w_4$ with $x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}, x_1$, respectively. We choose $\bar{S} = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}, x_1\}$.

If w_4 is adjacent to one of the vertices in $\{u_1, u_2, u_3, u_4\}$, for convenience, we assume w_4 and u' are adjacent to u for which $\{u, u'\} \subset \{u_1, u_2, u_3\}$. Now we label the vertices $w_1, w_2, w_3, u, u', w_4$ with $x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}, x_1$, respectively. We choose $\bar{S} = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}, x_1\}$. This completes the proof of Theorem 2.11. □

Corollary 2.12 *Let G be a planar graph without chordal 4- and 6-cycles. If $\Delta(G) \geq 7$, then $\chi_e(G) \leq \Delta(G)$.*

Corollary 2.13 *Let G be a planar graph without chordal 4- and 6-cycles. If $\Delta(G) \geq 7$, then $\chi_e^*(G) \leq \Delta(G)$.*

Theorem 2.14 *If G is a planar graph without chordal 4- and 6-cycles and $k \geq \max\{7, \Delta(G)\}$, then G is equitably k-choosable.*

Proof: Let G be a counterexample with the fewest vertices, i.e. G is a critical graph. If each component of G has at most four vertices, then $\Delta(G) \leq 3$. So G is equitably k-choosable by Lemma 2.10. Otherwise, the proof is similar to the proof of Theorem 2.11 by Lemma 2.8 and Lemma 2.5. □

Corollary 2.15 *Let G be a planar graph without chordal 4- and 6-cycles. If $\Delta(G) \geq 7$, then G is equitably $\Delta(G)$-choosable.*

3 Remarks and perspective

Most of the results on equitable and list equitable colorings on planar graphs are restricted to 3-degenerate graphs. In this paper, we confirm the Conjecture 2 and Conjecture 4 for the planar graphs without chordal 4- and 6-cycles which are not necessarily 3-degenerate. Can a similar conclusion be drawn for 4-degenerate graphs and ordinary planar graphs?

Acknowledgements

We would like to thank the referees for providing some very helpful suggestions for revising this paper.

References

J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. North-Holland, New York, 1976.

O. V. Borodin. Structural properties of plane graphs without adjacent triangles and an application to 3-colorings. *J. Graph Theory*, 12:183–186, 1996.

B. L. Chen and K. W. Lih. Equitable coloring of trees. *J. Combin. Theory Ser. B*, 61:83–87, 1994.

B. L. Chen, K. W. Lih, and P. L. Wu. Equitable coloring and the maximum degree. *European J. Combin.*, 15:443–447, 1994.

B. L. Chen and C. H. Yen. Equitable δ-coloring of graphs. *Discrete Mathematics*, 312(9):1512–1517, 2012.

A. J. Dong, G. J. Li, and G. H. Wang. Equitable and list equitable colorings of planar graphs without 4-cycles. *Discrete Mathematics*, 313(15):1610–1619, 2013.

A. J. Dong, X. Tan, X. Zhang, and G. J. Li. Equitable coloring and equitable choosability of planar graphs without 6- and 7-cycles. *Ars Combinatoria*, 103:333–352, 2012a.

A. J. Dong, X. Zhang, and G. J. Li. Equitable coloring and equitable choosability of planar graphs without 5- and 7-cycles. *Bulletin of the Malaysian Mathematical Sciences Society*, 35:897–910, 2012b.

A. Hajnal and E. Szemerédi. Proof of a conjecture of erdös, in: A. rényi, v.t. sós(eds.), in: Combin theory and its applications. *North-Holland, Amsterdam*, II:601–623, 1970.

S. Janson and A. Ruciński. The infamous upper tail. *Random Structure and Algorithms*, 20:317–342, 2002.

H. A. Kierstead and A. V. Kostochka. Equitable list coloring of graphs with bounded degree. *Journal of Graph Theory*, 2012. doi: 10.1002/jgt.21710.

A. V. Kostochka. Equitable colorings of outerplanar graphs. *Discrete Mathematics*, 258:373–377, 2002.

A. V. Kostochka and K. Nakprasit. Equitable colorings of k-degenerate graphs. *Combin. Probab. Comput.*, 12:53–60, 2003.

A. V. Kostochka and K. Nakprasit. Equitable δ-colorings of graphs with low average degree. *Theoretical Computer Science*, 349:82–91, 2005.

A. V. Kostochka, M. J. Pelsmajer, and D. B. West. A list analogue of equitable coloring. *J. Graph Theory*, 47:166–177, 2003.

Q. Li and Y. H. Bu. Equitable list coloring of planar graphs without 4- and 6-cycles. *Discrete Mathematics*, 309:280–287, 2009.

K. W. Lih and P. L. Wu. On equitable coloring of bipartite graphs. *Discrete Math.*, 151:155–160, 1996.

R. Luo, J. S. Sereni, D. C. Stephens, and G. Yu. Equitable coloring of sparse planar graphs. *SIAM Journal on Discrete Mathematics*, (4):1572–1583, 2010.

W. Meyer. Equitable coloring. *Amer. Math. Monthly*, 80:920–922, 1973.

K. Nakprasit. Equitable colorings of planar graphs with maximum degree at least nine. *Discrete Mathematics*, 312(5):1019–1024, 2012a.

K. Nakprasit. Equitable colorings of planar graphs without short cycles. *Theoretical Computer Science*, 465:21–27, 2012b.

M. F. Pelsmajer. Equitable list coloring for graphs of maximum degree 3. *J. Graph Theory*, 47:1–8, 2004.

S. V. Pemmaraju. Equitable colorings extend chernoff-hoeffding bounds. In *In Proceedings of the 5th International Workshop on Randomization and Approximation Techniques in Computer Science(APPROX-RANDOM 2001)*, pages 285–296, 2001.

X. Tan. Equitable δ-coloring of planar graphs without 4-cycles. In *The Ninth International Symposium on Operations Research and Its Applications (ISORA 10)*, pages 400–405, 2010.

W. F. Wang and K. W. Lih. Equitable list coloring of graphs. *Taiwanese J. Math.*, 8:747–759, 2004.

W. F. Wang and K. M. Zhang. Equitable colorings of line graphs and complete r-partite graphs. *System Sci. Math. Sci.*, 13:190–194, 2000.

J. L. Wu and P. Wang. Equtiable coloring planar graphs with large girth. *Discrete Mathematics*, 308: 985–990, 2008.

Z. Yan and W. Wang. Equitable coloring of kronecker products of complete multipartite graphs and complete graphs. *Discrete Applied Mathematics*, 162:328–333, 2014.

H. P. Yap and Y. Zhang. The equitable δ-coloring conjecture holds for outerplanar graphs. *Bull. Inst. Math. Acad. Sin.*, 25:143–149, 1997.

H. P. Yap and Y. Zhang. Equitable colorings of planar graphs. *J. Combin. Math. Combin. Comput.*, 27: 97–105, 1998.

X. Zhang and J. L. Wu. On equitable and equitable list coloring of series-parallel graphs. *Discrete Mathematics*, 311:800–803, 2011.

J. L. Zhu and Y. H. Bu. Equitable list colorings of planar graphs without short cycles. *Theoretical Computer Science*, 407:21–28, 2008.

J. L. Zhu and Y. H. Bu. Equitable and equitable list colorings of graphs. *Theoretical Computer Science*, 411:3873–3876, 2010.

J. L. Zhu, Y. H. Bu, and X. Min. Equitable list-coloring for c_5-free plane graphs without adjacent triangles. *Graphs and Combinatorics*, 31:1–10, 2013.