Upper $k$-tuple domination in graphs

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For a positive integer $k$, a $k$-tuple dominating set of a graph $G$ is a subset $S$ of $V(G)$ such that $|N[v] \cap S| \geq k$ for every vertex $v$, where $N[v] = \{v\} \cup \{u \in V(G) : uv \in E(G)\}$. The upper $k$-tuple domination number of $G$, denoted by $\Gamma_{\times k}(G)$, is the maximum cardinality of a minimal $k$-tuple dominating set of $G$. In this paper we present an upper bound on $\Gamma_{\times k}(G)$ for $r$-regular graphs $G$ with $r \geq k$, and characterize extremal graphs achieving the upper bound. We also establish an upper bound on $\Gamma_{\times 2}(G)$ for claw-free $r$-regular graphs. For the algorithmic aspect, we show that the upper $k$-tuple domination problem is NP-complete for bipartite graphs and for chordal graphs.

Keywords: Upper $k$-tuple domination, $r$-regular graph, bipartite graph, split graph, chordal graph, NP-completeness.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. In a graph $G$ with vertex set $V(G)$ and edge set $E(G)$, the open neighborhood of a vertex $v$ is $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood is $N[v] = \{v\} \cup N(v)$. The degree of $v$, denoted by $d(v)$, is the cardinality of $N(v)$. Denote by $\delta(G)$ the minimum degree of a vertex in $G$. A graph is $r$-regular if $d(v) = r$ for all $v \in V$. A stable set (respectively, clique) of $G$ is a subset $S$ of $V(G)$ in which every two vertices are not adjacent to each other.

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(respectively, are adjacent). For two disjoint subsets $A$ and $B$ of $V(G)$, let $e[A, B]$ denote the number of edges between $A$ and $B$.

For $S \subseteq V(G)$, the subgraph induced by $S$ is the graph $G[S]$ with vertex set $S$ and edge set $\{uv \in E(G) : u, v \in S\}$. A bipartite graph is a graph whose vertex set can be partitioned into two sets such that every two distinct vertices may be adjacent only if they are in different sets. A split graph is a graph whose vertex set can be partitioned into a stable set and a clique. A chord of a cycle is an edge joining two vertices on the cycle that are not adjacent on the cycle. A chordal graph is a graph in which every cycle of length at least four has a chord. Split graphs are chordal. A graph $G$ is called claw-free if it does not contain the bipartite complete graph $K_{1,3}$ as an induced subgraph.

For positive integer $k$, a $k$-tuple dominating set of $G$ is a subset $S$ of $V(G)$ such that $|N[v] \cap S| \geq k$ for all $v \in V(G)$. For a $k$-tuple dominating set $S$, any vertex in $N[v] \cap S$ is said to dominate $v$. Notice that a graph has a $k$-tuple dominating set if and only if $\delta(G) \geq k - 1$. The $k$-tuple domination number $\gamma_{k \times k}(G)$ of $G$ is the minimum cardinality of a $k$-tuple dominating set of $G$, while the upper $k$-tuple domination number $\Gamma_{k \times k}$ of $G$ is the maximum cardinality of a minimal $k$-tuple dominating set. A $\Gamma_{k \times k}(G)$-set of $G$ is a minimal $k$-tuple dominating set of $G$ of cardinality $\Gamma_{k \times k}(G)$. An application of $k$-tuple domination for fault tolerance networks is presented in [9, 12]. For more results on $k$-tuple domination, we refer to [13, 14, 15, 16, 17, 18, 19].

In this paper, we first give an upper bound on $\Gamma_{k \times k}$ for $r$-regular graphs, and characterize the extremal graphs achieving the upper bound. We also establish a sharp upper bound on $\Gamma_{k \times 2}(G)$ for claw-free $r$-regular graphs. Finally, we show that the upper $k$-tuple domination problem is NP-complete for bipartite graphs and chordal graphs.

2 Upper $k$-tuple domination for $r$-regular graphs

This section establishes a sharp upper bound for upper $k$-tuple domination on $r$-regular graphs.

First, a $k$-tuple dominating set $S$ is minimal if and only if every vertex in $S$ is not avoidable, that is, it has a closed neighbor that is dominated by exactly $k$ vertices in $S$. Hence, we have the following property.

**Lemma 1** In a graph $G$ with $\delta(G) \geq k - 1$, a $k$-tuple dominating set $S$ is minimal if and only if each vertex in $S$ has some closed neighbor $u$ with $|N[u] \cap S| = k$.

For integers $r \geq k \geq 1$, let $\mathcal{H}_{r,k}$ be the family of $r$-regular graphs $H$ whose vertex set is the disjoint union $F_1 \cup F_2 \cup F_3$, where $F_1$ induces an $(r - 1)$-regular graph of which each vertex has exactly one neighbor in $F_2$, $F_2$ is a stable set of which each vertex has exactly $k - 1$ neighbors in $F_1$, and exactly $r + 1 - k$ neighbors in $F_3$, and $F_3$ is a stable set of which each vertex has exactly $r$ neighbors in $F_2$, see Figure 1. Since $(r + 1 - k)|F_2| = e[F_2, F_3] = r|F_3|$, there is some integer $m \geq 1$ such that $|F_2| = rm/g$ and $|F_3| = (r + 1 - k)m/g$, where $g = gcd(r + 1 - k, r) = gcd(r, k - 1)$. And then $|F_1| = (k - 1)rm/g$. The total number of vertices in $H$ is $n = (kr + r + 1 - k)m/g$. According to Lemma 1, $F_1 \cup F_2$ is a minimal $k$-upper dominating set of $H$ and so $\Gamma_{k \times k}(H) \geq \frac{krn}{kr + r + 1 - k}$.

**Theorem 2** If $G$ is a $r$-regular graph of order $n$ with $r \geq k \geq 2$, then $\Gamma_{k \times k}(G) \leq \frac{krn}{kr + r + 1 - k}$ with equality if and only if $G \in \mathcal{H}_{r,k}$.

**Proof:** Let $S$ be a $\Gamma_{k \times k}(G)$-set of $G$ and $S' = V(G) \setminus S$. For $k \leq i \leq r + 1$, we define

$$S_i = \{u \in S : |N[u] \cap S| = i\} \quad \text{and} \quad S_i' = \{u \in S' : |N[u] \cap S| = i\}.$$
Notice that $S_{r+1}' = \emptyset$ and $S_r'$ is stable as every vertex of $S_r'$ has neighbors only in $S$. Since every vertex has at least $k$ closed neighbors in $S$, it is the case that $S = \bigcup_{i=k}^{r+1} S_i$ and $S' = \bigcup_{i=k}^{r} S_i'$ are disjoint unions. Therefore, $|S| = \sum_{i=k}^{r+1} |S_i|$ and $|S'| = \sum_{i=k}^{r} |S_i'|$.

According to Lemma 1, every vertex in $S_{r+1}$ has at least one neighbor in $S_k$, while every vertex in $S_k$ has at most $k - 1$ neighbors in $S_{r+1}$. Therefore,

$$|S_{r+1}| ≤ e[S_{r+1}, S_k] ≤ (k - 1)|S_k|$$  \hspace{1cm} (1)

or equivalently

$$|S_{r+1}| - (k - 1)|S_k| ≤ 0.$$  \hspace{1cm} (2)

By Lemma 1 again, every vertex in $X = \bigcup_{i=k+1}^{r+1} S_i$ has at least one neighbor in $S_k \cup S_r'$ and every vertex in $S_k$ (respectively, $S_k'$) has at most $k - 1$ (respectively, $k$) neighbors in $X$. Therefore,

$$\sum_{i=k+1}^{r+1} |S_i| ≤ e[X, S_k] + e[X, S_{r}'] ≤ (k - 1)|S_k| + k|S_r'|$$  \hspace{1cm} (3)

or equivalently

$$\sum_{i=k+1}^{r+1} |S_i| - (k - 1)|S_k| ≤ k|S_r'|.$$  \hspace{1cm} (4)

We then use a double counting of $e[S, S']$ to get

$$\sum_{i=k}^{r} (r + 1 - i)|S_i| = e[S, S'] = \sum_{i=k}^{r} i|S_i'|.$$  \hspace{1cm} (5)

Let $p = \min\left(\frac{r+1-k}{k}, 1\right)$ and $q = \max\left(\frac{r+1-k}{k}, 1\right) - 1$. Then, $p > 0$, $q ≥ 0$ and $q + 1 ≥ \frac{r+1-k}{k} = p + q$. Consequently, $(r + 1 - k) - (p + q)(k - 1) = p + q$ and $q + r + 1 - i ≥ q + 1 ≥ p + q$ for $k + 1 ≤ i ≤ r$, which gives (6). Adding $p$ times (2), $q$ times (4) and (5) gives the first inequality in (7).

And $(q+1)k = \max\{r+1-k, k\} ≤ r$ with equality only when $r = k$, which gives the second inequality in (7).

$$(p + q)|S| ≤ ((r + 1 - k) - (p + q)(k - 1))|S_k| + (p + q)|S_{r+1}| + \sum_{i=k+1}^{r} (q + r + 1 - i)|S_i|$$  \hspace{1cm} (6)
By Lemma 1, it has a neighbor in $S$ in the theorem. These give the first inequality in (10), while (8) and (9) give the second inequality in (10).

Lemma 1, it has a neighbor in $S$. Then $|S| = (r + p + q)|S| \leq rn$ and so $\Gamma_{xk}(G) \leq \frac{kn}{k r + r + 1 - k}$, which proves the first part of the theorem.

Suppose $\Gamma_{xk}(G) = \frac{kn}{k r + r + 1 - k}$. By the proof above, the inequalities in (11), the inequalities in (8) when $q > 0$, and the inequalities in (6) and (7) are equalities. The equality in (5) gives that $S = S_k \cup S_{r+1}$ when $p + q < q + 1$ and $S = S_k \cup S_{r+1} \cup S_r$ when $p + q \geq q + 1$, while the second equality in (7) gives that $S' = S'_r$ which is stable. The first equality in (11) gives that every vertex in $S_{r+1}$ has exactly one neighbor in $S_k$, while the second equality gives that every vertex in $S_k$ has exactly $k - 1$ neighbors in $S_{r+1}$ and so exactly $r + 1 - k$ neighbors in $S'$. We claim that $p + q \geq q + 1$ is impossible, for otherwise $p \geq 1$ implying $\frac{r + 1 - k}{k} \geq 1$ and so $r > k$. Then by Lemma 1, every vertex in $S_r$ is adjacent to some vertex in $S_k \setminus \emptyset$ as $S' = S'_r$, a contradiction. Hence, $p + q < q + 1$ and so $S = S_k \cup S_{r+1}$. Letting $F_1 = S_{r+1}$, $F_2 = S_k$ and $F_3 = S_r$, we have that $G$ is in $H_{r,k}$. On the other hand, any graph $G$ in $H_{r,k}$ satisfies $\Gamma_{xk}(G) \geq \frac{kn}{k r + r + 1 - k}$ and so $\Gamma_{xk}(G) = \frac{kn}{k r + r + 1 - k}$.

The extremal graphs in $H_{r,k}$ for Theorem 2 contain claws. For claw-free $r$-regular graphs, the upper bound in Theorem 2 for upper 2-tuple domination can be improved.

**Theorem 3** If $G$ is a claw-free $r$-regular graph of order $n$ with $r \geq 3$, then $\Gamma_{x2}(G) \leq \frac{2n}{r}$. 

**Proof:** Let $S, S', S_i$ and $S'_i$ be defined as in the proof of Theorem 2 with $k = 2$. For $2 \leq i \leq r + 1$, we further write

$$S_{2,i} = \{u \in S_2 : |N(u) \cap S_i| = 1\} \quad \text{and} \quad S_{1,2} = \{u \in S_1 : |N(u) \cap S_2| \geq 1\}.$$ 

Then $|S_2| = \sum_{i=2}^{r+1} |S_{2,i}|$. By the definition of $S_{2,i}$ and $S_{1,2}$, for $3 \leq i \leq r + 1$,

$$|S_{1,2}| \leq e(S_{1,2}, S_{2,i}) = |S_{2,i}|.$$  

(8)

By Lemma 1, $S_{r+1,2} = S_{r+1}$ and so

$$|S_{r+1}| \leq |S_{2,r+1}| \leq |S_2|.$$  

(9)

We then consider $e(S'_r, S)$. Since $S'_r$ is stable and $G$ is claw-free, every vertex in $S$ has at most 2 neighbors in $S'_r$. First, every vertex $u \in S_{2,r+1}$ has no neighbor in $S'_r$. Otherwise, if such a neighbor $v$ exists, then $u$ has another neighbor $w$ in $S'$ and a neighbor $x$ in $S_{r+1}$. Since $x$ has only neighbors in $S_r$, it is not adjacent to $v$ nor to $w$. Also, $v$ is not adjacent to $w$ since it has only neighbors in $S$. Hence a claw occurs at $u$, a contradiction. Second, for $3 \leq i \leq r$, every vertex $u \in S_i \setminus S_{i,2}$ has no neighbor in $S_2$ and hence, by Lemma 1, it has a neighbor in $S'_r$. Thus, if $u$ is in $S_r \setminus S_{i,2}$, it has at least one neighbor in $S'_r$, or it would be the center of a claw. Finally, by definition, every vertex in $S_{r+1}$ has no neighbor in $S'_r$. These give the first inequality in (10), while (8) and (9) give the second inequality in (10).

$$r|S'_r| = e(S'_r, S) \leq \sum_{i=2}^{r} 2|S_{2,i}| + \sum_{i=3}^{r}|S_{1,2}| + \sum_{i=3}^{r-1}|S_i| \leq 3|S_2| - 3|S_{r+1}| + \sum_{i=3}^{r-1}|S_i|.$$  

(10)
Upper k-tuple domination in graphs

Formula (5) with \( k = 2 \) gives
\[
\sum_{i=2}^{r} (r + 1 - i)|S_i| = \sum_{i=2}^{r} i|S'_i|.
\] (11)

Formula (3) with \( k = 2 \) gives
\[
\sum_{i=3}^{r+1} |S_i| \leq |S_2| + 2|S'_2|.
\] (12)

Adding \( \frac{r-3}{2} (9), \; \frac{1}{2} (10), \; (11) \) and \( \frac{r-3}{2} (12) \) gives
\[
\frac{r-1}{2}(|S_2| + |S_r| + |S_{r+1}|) + \sum_{i=3}^{r-1} \left( \frac{3r-1}{2} - i - \frac{1}{2} \right)|S_i| \leq (r - 1)(|S'_2| + |S'_r|) + \sum_{i=3}^{r-1} i|S'_i|
\] (13)

The left side is bounded below by \( \frac{r-1}{2} |S| \), the right above by \( (r - 1)|S'| \). Thus we get \( |S| \leq 2|S'| \) and finally, \( \Gamma_{\times 2}(G) \leq \frac{2n}{3} \).

3 Complexity results

The upper domination problem was shown to be NP-complete by Cheston, Fricke, Hedetniemi and Jacobs [6]. However, Cockayne, Favaron, Payan and Thomason [7] proved that \( \Gamma(G) = \beta_0(G) \) for any bipartite graph \( G \), and so \( \Gamma(G) \) can be computed for bipartite graphs in polynomial time. It was also shown by Jacobson and Peters [11] that \( \Gamma(G) = \beta_0(G) \) for any chordal graph \( G \), and so \( \Gamma(G) \) can be computed for chordal graphs in polynomial time. Besides, Hare, Hedetniemi, Laskar, Peters and Wimer [10] also established a polynomial algorithm for determining \( \Gamma(G) \) on generalized series-parallel graphs.

On the other hand, we shall prove that the \( k \)-tuple domination problem, with \( k \geq 2 \) fixed, is NP-complete for bipartite graphs and for chordal graphs. The proofs are separated into the cases of \( k = 2 \) and of \( k \geq 3 \). We consider the decision problem version as follows.

Upper \( k \)-tuple domination problem (U\( k \)TD)

**Instance:** A graph \( G = (V, E) \) and a positive integer \( s \leq |V| \).

**Question:** Does \( G \) have a minimal upper \( k \)-dominating set of cardinality at least \( s \)?

To show that U2TD is NP-complete, we will make use of the well-known NP-complete problem 3-SAT [8].

One-in-three 3SAT (OneIn3SAT)

**Instance:** A set \( U = \{ u_1, \ldots, u_n \} \) of \( n \) variables and a collection \( \mathcal{C} = \{ c_1, \ldots, c_m \} \) of \( m \) clauses over \( U \) such that each clause \( c \in \mathcal{C} \) has \( |c| = 3 \) and no clause contains a negated variable.

**Question:** Is there a truth assignment \( \mathcal{A}: U \rightarrow \{ \text{true}, \text{false} \} \) for \( U \) such that each clause in \( \mathcal{C} \) has exactly one true literal?
**Theorem 4** The upper 2-tuple domination problem is NP-complete even restricted on bipartite graphs or on split graphs, and hence also on chordal graphs.

**Proof.** Obviously, U2TD is in NP. We shall show the NP-completeness of U2TD for bipartite graphs by reducing OneIn3SAT to it in polynomial time. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be an instance of OneIn3SAT. We transform $I$ to the instance $(G_I, s)$ of U2TD in which $s = 3n + m$ and $G_I$ is the bipartite graph formed as follows.

Corresponding to each variable $u_i$ we associate a cycle $C_i = u_i, x_i, y_i, z_i, u_i$. Corresponding to each 3-element clause $c_j$ we associate a vertex named $w_j$. Joining the vertex $u_i$ to the vertex $w_j$ if and only if the literal $u_i$ belongs to the clause $c_j$. According to the above construction, it is easy to see that $G_I$ is a bipartite graph and the construction is accomplished in polynomial time. The graph $G_I$ associate with $(u_1 \lor u_2 \lor u_3) \land (u_2 \lor u_3 \lor u_4)$ is shown in Figure 2.

![Figure 2: The graph $G_I$ for $(u_1 \lor u_2 \lor u_3) \land (u_2 \lor u_3 \lor u_4)$.](image)

We next show that $I$ has a satisfying truth assignment if and only if $G_I$ has a minimal upper 2-tuple dominating set of cardinality at least $s = 3n + m$.

Suppose first $I$ has a satisfying truth assignment $\mathcal{A}$. A minimal upper 2-tuple dominating set $S$ of $G_I$ of cardinality $s$ is constructed as follows. Let $w_j$ belong to $S$ for all $1 \leq j \leq m$. For each $1 \leq i \leq n$, if $\mathcal{A}(u_i) = \{\text{true}\}$, then let $u_i, x_i$ and $z_i$ be in $S$; otherwise, let $x_i, y_i$ and $z_i$ be in $S$. Clearly, $|N[v] \cap S| \geq 2$ for each vertex $v \in V(G_I)$ and $S$ satisfies the conditions of Lemma 1 and so $S$ is a minimal upper 2-tuple dominating set of $G_I$ with cardinality $s = 3n + m$.

On the other hand, assume that $S$ is a minimal upper 2-tuple dominating set of $G_I$ with cardinality at least $s = 3n + m$. It follows from Lemma 1 that $|S \cap V(C_i)| \leq 3$ for $1 \leq i \leq n$. Further, $|S \cap \{w_1, w_2, \ldots, w_m\}| \leq m$ and so $|S| \leq 3n + m$. Notice that $|S| \geq 3n + m$ by the assumption. Hence, $|S \cap V(C_i)| = 3$ for $1 \leq i \leq n$ and $\{w_1, w_2, \ldots, w_m\} \subseteq S$. Let $\{u_{j_1}, u_{j_2}, u_{j_3}\}$ be the open neighborhood of $w_j$ in $G_I$ for $1 \leq j \leq m$. Since $S$ is a minimal upper 2-tuple dominating set, $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| \geq 1$. We claim that $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| = 1$. Otherwise, $|S \cap \{u_{j_1}, u_{j_2}, u_{j_3}\}| \geq 2$. Then the degree of $w_j$ in $G_I[S]$ is more than 1. However, $w_j$ does not satisfy the conditions of Lemma 1 because $|S \cap V(C_i)| = 3$ for $1 \leq i \leq n$, a contradiction. Let $\mathcal{A}: U \rightarrow \{\text{true}, \text{false}\}$ be defined by $\mathcal{A}(u_i) = \{\text{true}\}$ if $u_i \in S$ and $\mathcal{A}(u_i) = \{\text{false}\}$ if $u_i \notin S$. By the construction of $G_I$, we have each clause $c_j$ of $I$ contains only one variable $u_i$ belonging to $S$. So $\mathcal{A}$ is a satisfying truth assignment for $I$. Consequently, $I$ has a satisfying truth assignment if and only if $G_I$ has a minimal upper 2-tuple dominating set of cardinality at least $s = 3n + m$. This completes the proof for bipartite graphs.
To deal with split graphs, we add edges to make \(\{u_1, y_1, u_2, y_2, \ldots, u_n, y_n\}\) a clique. The same arguments hold, and show the NP-completeness of the upper 2-tuple domination problem for split graphs.

Theorem 5 For any fixed integer \(k \geq 3\), the \(k\)-tuple domination problem is NP-complete for bipartite graphs and for chordal graphs.

Proof: For any bipartite graph \(G\) on \(n\) vertices, consider the bipartite graph \(G'\) obtained by the following process. For each vertex \(v\) of \(G\), we add a copy of \(K_{k-1,k-1}\), denoted \(G_v\) with bipartition denoted \((X_v, Y_v)\). Then we link by an edge the vertex \(v\) to \(k-2\) vertices in \(X_v\). The widget added to each vertex \(v\) is drawn in Figure 3.

The NP-completeness of the upper \(k\)-tuple domination problem for bipartite graphs then follows from the NP-completeness of the upper 2-tuple domination problem for bipartite graphs.

To deal with the chordal case, we start with a chordal graph \(G\) and for each \(v \in V(G)\) we add to \(G_v\) edges joining any pair of vertices in \(X_v\), forming a clique. The same arguments hold, and show the NP-completeness of the upper \(k\)-tuple domination problem for chordal graphs as a consequence of the NP-completeness of the upper 2-tuple domination problem for chordal graphs.

\[\square\]
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