Discrete Mathematics and Theoretical Computer Science DMTCS vol. **21**:4, 2019, #18

Power domination in maximal planar graphs

Paul Dorbec[1] Antonio González[2] Claire Pennarun[3]

[1] *Normandie Univ, UNICAEN, ENSICAEN, CNRS, GREYC, France*
[2] *Departamento de Didáctica de las Matemáticas, Universidad de Sevilla, Spain*
[3] *LaBRI, Univ. Bordeaux, France*

received 27th Jan. 2019, *revised 25th Oct. 2019*, *accepted 27th Oct. 2019*.

Power domination in graphs emerged from the problem of monitoring an electrical system by placing as few measurement devices in the system as possible. It corresponds to a variant of domination that includes the possibility of propagation. For measurement devices placed on a set S of vertices of a graph G, the set of monitored vertices is initially the set S together with all its neighbors. Then iteratively, whenever some monitored vertex v has a single neighbor u not yet monitored, u gets monitored. A set S is said to be a power dominating set of the graph G if all vertices of G eventually are monitored. The power domination number of a graph is the minimum size of a power dominating set. In this paper, we prove that any maximal planar graph of order $n \geq 6$ admits a power dominating set of size at most $\frac{n-2}{4}$.

Keywords: power domination, propagation, maximal planar graph

1 Introduction

The notion of *power domination* arose in the context of monitoring an electrical network (Baldwin et al. (1993); Mili et al. (1990); Phadke et al. (1986)), i.e., knowing the state of each component (e.g. the voltage magnitude at loads) by measuring some variables such as currents and voltages. The measurements are done by placing Phasor Measurement Units (PMUs) at selected locations. PMUs monitor the state of the adjacent components, then with the use of electrical laws (such as Ohm's and Kirschoff's Laws), it is possible to determine the state of components further away in the network. Since PMUs are costly, it is important to monitor a graph with as few PMU as possible. In this paper, we consider the problem of monitoring maximal planar graphs with few PMUs. Before getting further into technical details, we need the following graph definitions.

Let $G = (V(G), E(G))$ be a finite, simple, and undirected graph of order $n = |V(G)|$. The *open neighborhood* of a vertex $u \in V(G)$ is $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* is $N_G[u] = N_G(u) \cup \{u\}$; the *open* and *closed neighborhood of* a subset of vertices S is $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S] = S \cup N_G(S)$, respectively. The subgraph of G induced by S is written $G[S]$ (the subscript G is dropped from the notations when no confusion may arise).

The electrical network monitoring problem was transposed into graph-theore-tical terms by Haynes et al. (2002). Originally, the definition of power domination ensured the monitoring of the edges as well as of the vertices, and contained many propagation rules. Here, we consider an equivalent definition

ISSN 1365–8050

from Brueni and Heath (2005) that only requires monitoring the vertices. Given a graph G and a set $S \subseteq V(G)$, we build a set $M_G(S)$ (or simply $M(S)$ when the graph G is clear from the context) as follows: at first, $M_G(S) = N[S]$, and then iteratively a vertex u is added to $M_G(S)$ if u has a neighbor v in $M_G(S)$ such that u is the only neighbor of v not in $M_G(S)$ (we say that v *propagates* to u). At the end of the process, we say that $M_G(S)$ is the set of vertices *monitored* by S; the *non-monitored* vertices are those of the set $V(G) \setminus M_G(S)$. We say that G is *monitored* by S when $M_G(S) = V(G)$ and, in that case, S is said to be a *power dominating set* of G. The minimum cardinality of such a set is the *power domination number* of G, denoted by $\gamma_{\mathrm{P}}(G)$.

The decision problem POWER DOMINATING SET naturally associated to power domination (i.e., "Given a graph G and an integer k, does G have a power dominating set of order at most k?") was proven NP-complete, by a reduction from the 3-SAT problem (Haynes et al. (2002); Liao and Lee (2005)) (giving NP-completeness of the problem on bipartite graphs, chordal graphs and split graphs). A reduction from DOMINATING SET was also given (Guo et al. (2008); Kneis et al. (2006)), that implies the NP-completeness when restricted to planar graphs or circle graphs. However, polynomial algorithms were proposed to compute the power domination number of trees (Haynes et al. (2002); Guo et al. (2008)), block graphs (Xu et al. (2006)), interval graphs (Liao and Lee (2005)), and circular-arc graphs (Liao and Lee (2005, 2013)).

Concerning the parameter $\gamma_{\mathrm{P}}(G)$, tight upper bounds are also known for particular classes: $\gamma_{\mathrm{P}}(G) \leq \frac{n}{3}$ if G is connected (Zhao et al. (2006)) or a tree (Haynes et al. (2002)), whereas cubic graphs satisfy $\gamma_{\mathrm{P}}(G) \leq \frac{n}{4}$ (Dorbec et al. (2013)). Furthermore, the exact value of $\gamma_{\mathrm{P}}(G)$ has been determined for regular grids and their generalizations: square grid (Dorfling and Henning (2006)) and other products of paths (Dorbec et al. (2008)), hexagonal grids (Ferrero et al. (2011)), as well as cylinders and tori (Barrera and Ferrero (2011)). The only known results for general planar graphs concern graphs with diameter two or three (Zhao and Kang (2007)).

A graph G is a *planar graph* if it admits a crossing-free embedding in the plane. When the addition to G of any edge would result in a non-planar graph, G is said to be a *maximal planar graph*. A planar graph G together with a crossing-free embedding on the plane is called a *plane graph*, or a *triangulation* when G is a maximal planar graph. For any subset $S \subseteq V(G)$, the graph $G[S]$ can be viewed as a plane graph with the embedding inherited from the embedding of G. The only unbounded face is called the *outer face* of G, and the vertices of G are called respectively *exterior* or *interior* depending on whether they belong to the outer face or not. We say that a subgraph of a triangulation G is *facial* if all of its faces but the outer face are also faces of G. In particular, we denote by $[uvw]$ a facial triangle formed by vertices u, v and w in G. Note that power dominating sets are independent of the embedding of the graph as they only depend on vertex adjacencies. We only make use of the embedding of the graph in the proofs.

The main result of this paper consists in the following theorem:

Theorem 1.1 *If G is a maximal planar graph of order $n \geq 6$, then $\gamma_{\mathrm{P}}(G) \leq \frac{n-2}{4}$.*

The bound of Theorem 1.1 is tight for graphs on six vertices. We also know of one graph on ten vertices for which this bound is tight, the *triakis tetrahedron* drawn in Figure 1. To propose a general family of maximal planar graphs that have large power domination number, we use the configurations of Figure 2. Observe that if one of these configurations H is a facial subgraph of G, then any power dominating set of G contains one of the vertices of H. Otherwise, even if all the exterior vertices are monitored, they can not propagate to any of the interior vertices of H. Thus $\gamma_{\mathrm{P}}(G)$ is at least the number of disjoint facial special

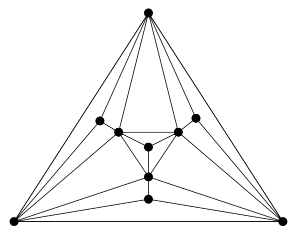

Fig. 1: The triakis tetrahedron, having ten vertices and power domination number two.

configurations in G. Taking many disjoint copies of the two first configurations (that have six vertices) and then completing the graph into a triangulation by arbitrarily adding edges between external vertices of the configurations (see Figure 3), we obtain a family of graphs that have power domination number $\frac{n}{6}$. Note that this construction is similar to the construction for classical domination given in Matheson and Tarjan (1996) reaching the bound $\gamma(G) = \frac{n}{4}$. As a consequence, and thanks to Theorem 1.1, we also get the following result:

Theorem 1.2 *For $n \geq 6$, every maximal planar graph with n vertices has a power dominating set containing at most $\alpha(n)$ vertices, with $\frac{n}{6} \leq \alpha(n) \leq \frac{n-2}{4}$.*

Determining the best possible value of $\alpha(n)$ remains an open problem.

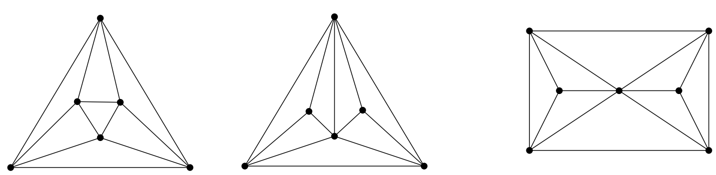

Fig. 2: The good, the bad and the ugly configurations in a triangulation.

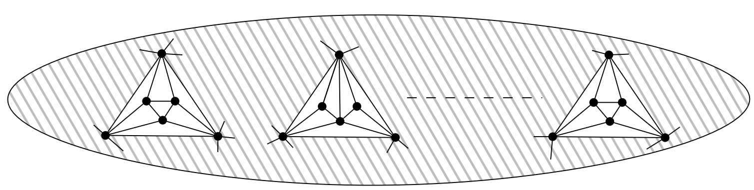

Fig. 3: A class of maximal planar graphs for which $\gamma_{\mathrm{P}}(G) = \frac{n}{6}$. The hatched area is triangulated arbitrarily.

The proof of Theorem 1.1 is done in three distinct steps, each of them described in a separate algorithm in Section 3. The first algorithm deals with the special configurations formed by overlapping configurations from Figure 2. These special configurations are characterized in Section 2. The end of the proof relies on a final Lemma that is proved in Section 4.

2 Identifying bad guys

Our algorithm deals first with some special configurations, that are the possible intersections of the configurations from Figure 2. We here characterize these special configurations. Note that a facial octahedron (third configuration of Figure 2) may only share vertices of its outer face with other configurations. We thus focus on the other two configurations.

We call *3-vertex* a vertex of G with degree 3, and therefore whose neighborhood induces a K_4. A *b-vertex* is any vertex $u \in V(G)$ with exactly two 3-neighbors v and v', and such that $N[u] = N[v] \cup N[v']$. Note that b-vertices have degree at most six and their neighborhood necessarily induces one of the subgraphs of Figure 4. In all figures of this section, b-vertices are depicted with blue squares, and 3-vertices are drawn white.

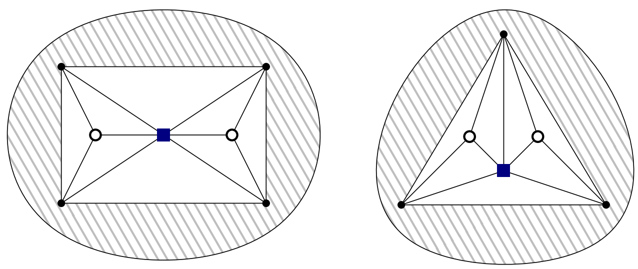

Fig. 4: The two possible neighborhoods of a b-vertex v.

Observation 2.1 *Any two b-vertices $u, u' \in V(G)$ are adjacent if and only if there exists a 3-vertex $v \in N(u) \cap N(u')$.*

Proof: By definition of b-vertices, if u' is adjacent to u, it is also adjacent to a 3-vertex v adjacent to u, and so u and u' have a common neighbor of degree 3. Moreover, if v has degree 3, all its of v are pairwise adjacent, and thus two b-vertices u and u' that have v as a common neighbor are adjacent. □

Lemma 2.2 *If G contains two 3-vertices v_1, v_2 with two common b-neighbors, then G is isomorphic to one of the graphs depicted in Figure 5.*

Proof: Either v_1 and v_2 have three common neighbors (inducing the first subgraph), or they have distinct third neighbors (inducing the second subgraph). In the first subgraph, all triangles are incident to a 3-vertex, so they are facial and there is no possibility for more vertices in the graph. In the second subgraph, the only faces not incident to a 3-vertex are incident to a b-vertex, which can not have other neighbors. Again, all triangles must then be facial. □

Lemma 2.3 *If all the neighbors of a 3-vertex are b-vertices, then G is isomorphic to a graph depicted in Figure 5 or these vertices belong to a facial triakis tetrahedron as depicted in Figure 6.*

Proof: Let v be a 3-vertex adjacent to three b-vertices u_1, u_2 and u_3, which necessarily form a triangle. By definition of a b-vertex, each u_i has another 3-neighbor v_i. If the vertices v_i are not all distinct, then

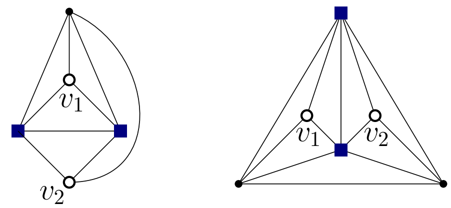

Fig. 5: The two possibilities for G if two 3-vertices v_1 and v_2 have two common b-neighbors. In both cases, $\gamma_{\mathrm{P}}(G) = 1$.

there exist two b-vertices sharing two adjacent 3-vertices, and Lemma 2.2 concludes. So assume the v_i are distinct. Let w_1 and w_2 be the neighbors of v_1 distinct from u_1 (which are both adjacent to u_1). Since u_1 may not have any other neighbor, we infer without loss of generality that w_2 is adjacent to u_2 (and w_1 to u_3), and therefore that w_2 is also adjacent to v_2 (and w_1 to v_3). Similarly, v_2 and v_3 must have some vertex w_3 as a common neighbor, also adjacent to u_2 and u_3. Now, since the neighborhoods of 3-vertices and b-vertices are fully determined, we get a facial triakis tetrahedron, as depicted in Figure 6. □

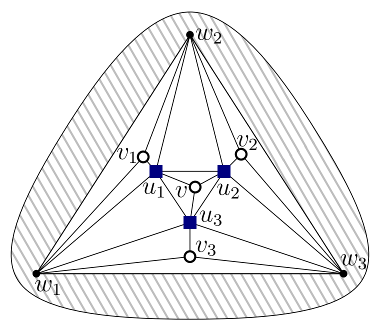

Fig. 6: A facial triakis tetrahedron.

Observe in particular that if a b-vertex has three adjacent b-vertices, then by Observation 2.1, we are in the case of Lemma 2.3 (and the graph is a triakis tetrahedron).

Lemma 2.4 *If G contains three b-vertices forming a three cycle, then either Lemma 2.3 applies, or G contains the first configuration depicted in Figure 7, or it is isomorphic to one of the last two graphs depicted in Figure 7.*

Proof: Let u_1, u_2, u_3 be three b-vertices forming a cycle. If they have a common 3-neighbor, then Lemma 2.3 applies, so assume they do not. By Observation 2.1, every two of these vertices have a 3-vertex as a common neighbor, and they are distinct by hypothesis. Let v_1, v_2, v_3 be the 3-vertices adjacent respectively to u_1 and u_2, u_2 and u_3, and u_1 and u_3, and let z_1, z_2, z_3 be the (not necessarily distinct) third neighbors of respectively v_1, v_2 and v_3. Suppose two z_i are distinct, say z_1 and z_3, and observe that the neighbors of u_1 are exactly $\{u_2, u_3, v_1, v_3, z_1, z_3\}$. Therefore, if $(u_1u_2u_3)$ separates z_1 from z_3, then z_3 is adjacent to u_2 and z_1 is adjacent to u_3. Now, since u_2 is a b-vertex, then v_2 is adjacent to z_3 and z_1, a contradiction. So $[u_1u_2u_3]$ does not separate any two z_i and is facial. Moreover, if say z_1 and z_3 are distinct, then they must be adjacent since u_1 has no other neighbor. So depending on whether the z_i are distinct or not, G contains the first configuration depicted in Figure 7, or is isomorphic to one of the last two graphs depicted in Figure 7 (note that all faces incident to a 3-vertex or a b-vertex in these drawings are facial). □

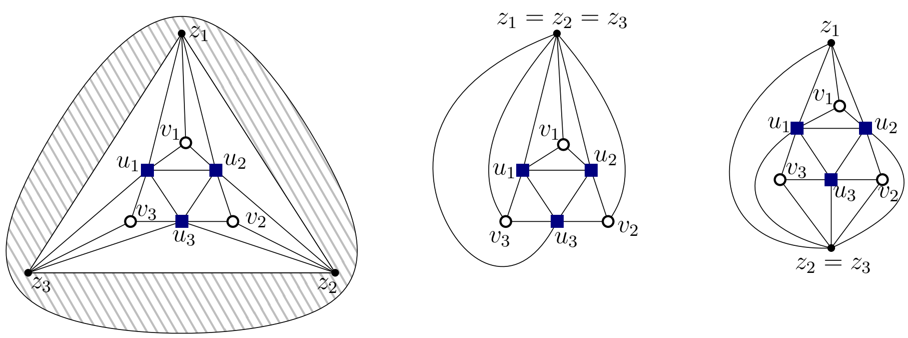

Fig. 7: The possible configurations of G if there is a face composed of b-vertices. The last two graphs, that satisfy $\gamma_P(G) = 1$, are contracts of the first configuration.

Property 2.5 *Let (u_1, u_2, u_3) be a path on three b-vertices. Let v_1 be the 3-vertex adjacent to u_1 and u_2 and let v_2 be the 3-vertex adjacent to u_2 and u_3. If u_1 is not adjacent to u_3, then there exist distinct vertices x and x' such that $\{u_1, u_2, u_3, v_1\} \subseteq N(x)$, $\{u_1, u_2, u_3, v_2\} \subseteq N(x')$, and $[xu_2u_3]$ and $[x'u_1u_2]$ are facial (see Figure 8).*

Proof: Since v_1 and v_2 are 3-vertices, then there exist two vertices x, x' such that $\{u_1, u_2, v_1\} \in N(x)$ and $\{u_2, u_3, v_2\} \in N(x')$. Since u_2 is a b-vertex, we have that $x \neq x'$ (otherwise u_1 and u_3 would be adjacent as the second configuration of Figure 4 shows). Thus, u_2 is a b-vertex corresponding to the first configuration of Figure 4, and so x is adjacent to u_3, x' is adjacent to u_1, and $[xu_2u_3]$ and $[x'u_1u_2]$ are facial. □

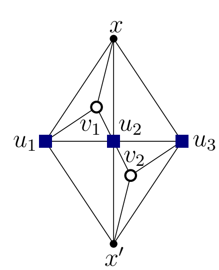

Fig. 8: There are two distinct vertices both adjacent to $\{u_1, u_2, u_3\}$. All triangles are facial.

Observe that the above property together with Lemmas 2.3 and 2.4 covers all possibilities of three connected b-vertices. We now consider the cases when b-vertices form paths and cycles of length at least four.

Corollary 2.6 *Suppose a set of $k \geq 3$ b-vertices form a path $(u_1, \dots, u_k)$ (where u_1 and u_k may be adjacent when $k > 3$). Let $v_1, \dots, v_{k-1}$ be 3-vertices with v_i being adjacent to u_i and u_{i+1}, and let v_0 be the 3-vertex adjacent to u_1 but not to u_2. Then there exists a vertex x adjacent to all u_i, $1 \leq i \leq k$ and to v_0 and v_2. (see Figure 9).*

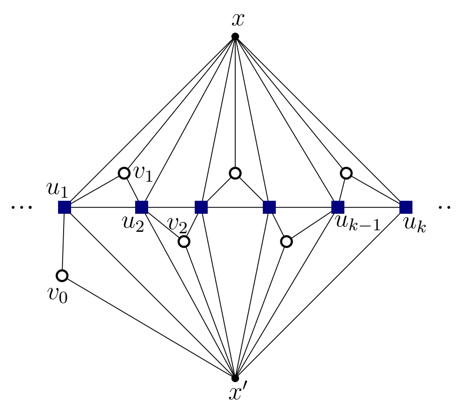

Fig. 9: There are two vertices universal to the path $(u_1, u_2, u_3, \dots, u_k)$. Vertex x' is also adjacent to v_0 and v_2.

Proof: Applying Proposition 2.5 to vertices u_1, u_2, u_3, there exist distinct vertices x, x' such that $\{u_1, u_2, u_3, v_1\} \subseteq N(x)$, $\{u_1, u_2, u_3, v_2\} \subseteq N(x')$, and $[xu_2u_3]$ and $[x'u_1u_2]$ are facial. Since x is adjacent to the b-vertex u_3, x must be adjacent to v_3 and thus to u_4. Then u_4 is also adjacent to x' as u_3 has no other neighbor. Iterating this argument, we infer that x and x' are adjacent to all u_i. Now, since x' is adjacent to u_1 but not to v_1, by definition of a b-vertex it is adjacent to v_0, and the corollary follows. □

Lemma 2.7 *If G contains a maximal component of b-vertices isomorphic to P_2, then G contains a facial subgraph isomorphic to one of the graphs of Figure 10.*

Proof: Let u_1, u_2 be b-vertices, and let v_1 be the 3-vertex adjacent to u_1 and u_2, and z the third neighbor of v_1. Let v_0 and v_2 be the second 3-neighbors of respectively u_1 and u_2, which can be assumed distinct by Lemma 2.2. Since v_0 is a 3-vertex and is not adjacent to u_2, v_0 has a neighbor t which is adjacent to u_1 and u_2 (we can see u_1 as the central vertex of any configuration of Figure 4). By definition of b-vertices, v_2 must also be adjacent to t. Let z_1 and z_2 be the third neighbors of respectively v_0 and v_2. If $z_1 = z_2$, then $N[t] \subseteq N[v_0] \cup N[v_2]$, and the vertex t is in fact a b-vertex, contradicting our hypothesis. Thus $z_1 \neq z_2$. Depending on whether z and z_1 are distinct or not, we get one of the configurations of Figure 10 (in both cases, the outer face of the drawing may not be facial). □

Finally, if there is an isolated b-vertex in G, then it belongs to one of the subgraphs depicted in Figure 4. This concludes the proof of the following lemma, that gives a characterization of the possible intersections of the configurations from Figure 2.

The *special configurations* of G are then all the configurations depicted in Figure 11.

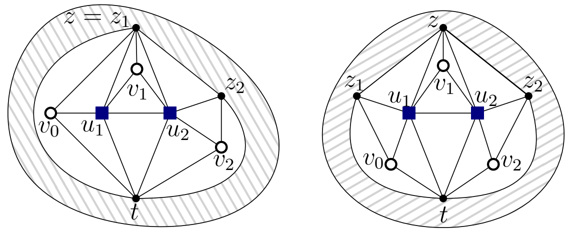

Fig. 10: The possible configurations of G if there is a P_2 component of b-vertices. The outer faces are not necessarily facial. All other triangles of the drawing are facial.

Lemma 2.8 *If G contains a special configuration as facial subgraph, then either G is a small graph (characterized in Lemmas 2.2, 2.3, and 2.4) and $\gamma_P(G) \leq \frac{n-2}{4}$, or each maximal component of b-vertices of G belongs to one of the induced configurations depicted in Figure 11, 1 to 7, or G contains a facial octahedron (configuration 8 in Figure 11).*

Observation 2.9 *If a vertex belongs to two facial subgraphs isomorphic to configurations from Figure 11, then it is a vertex from the outer face for both of them.*

Proof: Let v be a vertex that belongs to two configurations of Figure 11. If v is a b-vertex, then none of the two configurations is an octahedron. Then by maximality of the components of b-vertices in each configuration, the two configurations must rely on the same set of b-vertices, so they are the same configuration. Now suppose v is a 3-vertex. In both configurations it must be an internal vertex, and have an adjacent b-vertex. So the configurations also share a b-vertex and the same argument concludes. Finally, if v is a vertex of degree 4, it is an internal vertex of an octahedron. Since two octahedra cannot intersect on internal vertices and no internal vertex of an octahedron may be adjacent to a 3-vertex, v does not belong to any other configuration. The observation follows. □

3 Constructing the power dominating set

We now describe the process that defines incrementally a power dominating set S of G satisfying the announced bound. In Section 3.1, Algorithm 1 produces a set S_1 monitoring special configurations from Figure 2 with a small number of vertices. Then, Algorithm 2 of Section 3.2 builds a set S_2 by expanding the set S_1 iteratively, while keeping certain properties. If the graph G is not fully monitored after that, we show in Section 3.3 that G has a characterized structure, which guarantees that our last Algorithm 3 maintains the wanted bound while adding some well chosen vertices to S_2 to build the required set S.

During the three algorithms, we ensure the following property on the set of selected vertices, that is necessary for the proof of Lemma 3.5:

Property $(*)$**.** *We say that a subset S of vertices of a plane graph G has* Property $(*)$ *in G whenever, for each induced triangulation $G' \subseteq G$ of order at least 4, if G' is monitored by S then one of the following holds:*

(a) one vertex of the outer face of G' has its closed neighborhood in G monitored by S,

(b) or, we have $|S \cap V(G')| \leq \frac{|V(G')|-2}{4}$.

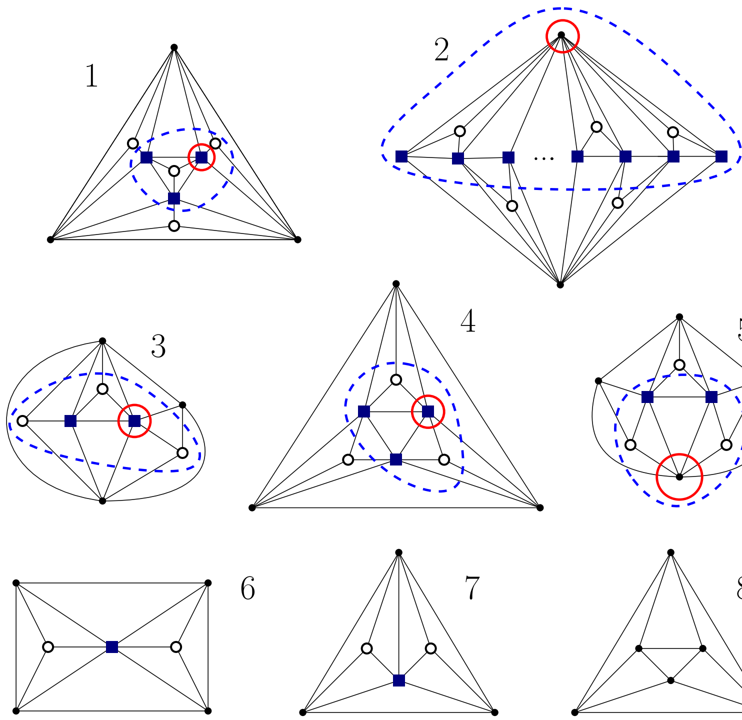

Fig. 11: The different configurations containing a b-vertex, and the octahedron. So-called "special" vertices of configurations 1 to 5 are circled in red. For these configurations, vertices circled with a blue-dashed curve form a set relative to the special vertex of the configuration, and they are called the *circled* vertices of the configuration.

3.1 Monitoring special configurations

The first step of our algorithm is described in Algorithm 1, which takes care of monitoring vertices creating special configurations. In the following, we say a configuration is *monitored* by S_1 when all its interior b-vertices are in $M(S_1)$, or for an octahedron, if all its vertices are in $M(S_1)$.

Note that the output of Algorithm 1 is the empty set whenever G contains neither b-vertices nor facial octahedra. We prove the following lemma:

Lemma 3.1 *Let S_1 be the set obtained by application of Algorithm 1 to G. The following statements hold:*

(i) All b-vertices and all facial octahedra are monitored by S_1.

(ii) If S_1 is not empty, $|S_1| \leq \frac{|M(S_1)|-2}{4}$.

(iii) S_1 has Property $(*)$ in G.

Proof: (i) Every b-vertex in the graph belongs to one of the configurations of Figure 11. The selected vertices in each configuration monitor all the b-vertices of the configuration, and thus the algorithm monitors

Algorithm 1: Monitoring special configurations

Input: A triangulation G of order $n \geq 6$.
Output: A set $S_1 \subseteq V(G)$ monitoring all b-vertices and all vertices of facial octahedra.
$S_1 := \emptyset$
if *G has a vertex u of degree at least $n-2$* **then**
  Label $N[u]$ with u
  Return $\{u\}$
if *G is a triakis tetrahedron (as in Figure 6)* **then**
  u, v: two b-vertices of G at distance 2
  Label u, its 3-neighbors and two of its adjacent b-vertices with u
  Label all other vertices of G with v
  Return $\{u, v\}$
while $\exists$ *a non-monitored configuration H from Figure 11(1,2,3,4,5)* **do**
  u: the special vertex of H
  $S_1 \leftarrow S_1 \cup \{u\}$
  Label u and the circled vertices of H with u
while $\exists$ *non-monitored configurations H, H' from Figure 11(6,7,8) with a common vertex u* **do**
  $S_1 \leftarrow S_1 \cup \{u\}$
  Label u and the interior vertices of H and H' adjacent to u with u
while $\exists$ *a non-monitored configuration H from Figure 11(6,7,8)* **do**
  u: any exterior vertex of H
  $S_1 \leftarrow S_1 \cup \{u\}$
  Label all vertices of H with u
Return S_1

all such vertices. Taking any vertex of a facial octahedron monitors the whole octahedron, thus all facial octahedra are monitored as well.

(ii) If the graph is dealt with by the first *if*s, the statement is straightforward. Otherwise, we first ensure that for each vertex $u \in S_1$, there are indeed at least five vertices labeled with u. For configurations 1, 3, 4 and 5, this is clear by definition of the circled vertices. For configuration 2, the vertex taken plus the (at least) three b-vertices of the path plus at least one 3-vertex make (at least) five labeled vertices. For every vertex u added in the second *while* loop, there are at least two vertices labeled with u in each of the two configurations, which together with u itself makes five vertices. For vertices added in the last *while* loop, at least six vertices are labeled with u each time.

Now, we show that each vertex receives at most one label during Algorithm 1. By Observation 2.9, only vertices on the outer face of some configuration may be labeled several times, and so in only two cases: they may receive their own label when they are themselves added to S_1, or they may receive a label during the last *while* loop if they are in a non-monitored configuration 6, 7 or 8 disjoint from all remaining non-monitored configurations. Since these last configurations are monitored by any vertex of their outer face, all vertices are labeled at most once.

If S_1 contains two or more vertices at the end of the algorithm, the statement is proved. If S_1 is reduced to a singleton, since the chosen vertex is of degree at least five, the statement holds.

(iii) Let G' be an induced triangulation of G monitored after Algorithm 1. If $|S_1 \cup V(G')| = 0$, then Property $(*)$.(b) holds. Assume then $|S_1 \cup V(G')| > 0$. If there is a vertex $v \in S_1$ such that some vertices labeled with v are not in G', then v is a vertex of the outer face of G' and Property $(*)$.(a) holds. Otherwise, for every vertex $v \in S_1 \cap V(G')$, all vertices with label v (which are at least five as said above) are in G'. If $|S_1 \cap V(G')| \geq 2$, then this is sufficient to deduce that Property $(*)$.(b) holds. Otherwise, we observe that the set of vertices bearing a same label u either does not form an induced triangulation or is of size at least six, so G' contains at least six vertices and the statement also holds. □

In the following, S_1 denotes the output of Algorithm 1 applied to the graph G. Note that we can now forget the labels put on vertices during Algorithm 1.

3.2 *Expansion of* S_1

The next step consists in selecting greedily any vertex that increases the set of monitored vertices by at least four. We first make a small observation.

In the following, the graphs of the form P_2+P_k (i.e., formed by two vertices both adjacent to all vertices of a path P_k) for some $k \geq 1$ are called *tower graphs*. We remark that the only maximal planar graphs of order $n \leq 6$ are the complete graphs K_3 and K_4, the graphs $P_2 + P_3$ and $P_2 + P_4$, the octahedron, and the flip-octahedron (see Figure 12).

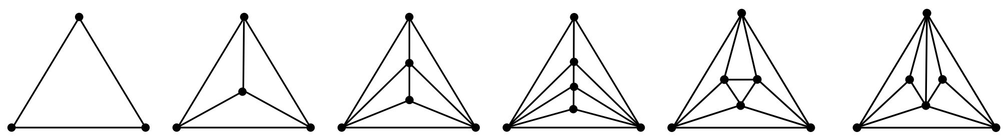

Fig. 12: The maximal planar graphs of order $n \leq 6$.

Observation 3.2 *Let G be a triangulation. Unless G is an octahedron or a tower graph $P_2 + P_k$ (for some $k \geq 1$), one interior vertex of G has degree at least 5.*

Proof: Suppose G is not an octahedron or a tower graph. If G is a flip-octahedron (last configuration of Figure 12), then one of its interior vertices has degree five. Otherwise, by the preceding observation, G contains at least seven vertices. Suppose by way of contradiction that all interior vertices of G have degree at most 4. Denote by u the exterior vertex of G with maximum degree, v, w the other two exterior vertices of G, and $u_1, \ldots, u_k$ the interior neighbors of u ($k \geq 2$ or G is K_4), so that $(vu_1 \ldots u_k w)$ form a cycle. Without loss of generality, we assume that v is adjacent to no less vertices among $u_1, \ldots, u_k$ than w is.

Let ℓ be the maximum integer such that for all $i \leq \ell$, u_i is adjacent to v. Since G is not a tower graph, $\ell < k$. Observe that since v is not adjacent to $u_{\ell+1}$, u_ℓ and v have a common neighbor t (that is neither $u_{\ell-1}$ nor u) to make another face on the edge vu_ℓ. If $\ell > 1$, then v, t, u, $u_{\ell-1}$ and $u_{\ell+1}$ make five neighbors to u_ℓ, a contradiction.

So $\ell = 1$ and since u_2 is not adjacent to v, $t \neq u_2$. (Note that $t \neq w$ or v would have only one neighbor among $u_1, \ldots, u_k$ while w has at least two, contradicting our assumption.) Thus u_1 has at least four neighbors: u, v, u_2 and t. By our initial assumption, $[u_1 u_2 t]$ is a facial triangle. Now if $k \geq 3$, u_2 also already has four neighbors so $[u_2 u_3 t]$ form a facial triangle. But then t and u_3 are already of degree four, so it is not possible to form another facial triangle containing the edge tu_3, a contradiction. So $k = 2$, and $[u_2wt]$ is facial. But then we get an induced octahedron where the only non facial triangle is $[vtw]$, in which adding a vertex would raise the degree of t to more than 4, a contradiction. This concludes the proof. □

Let us now proceed with the second part of the algorithm defining a power dominating set. Assume that after Algorithm 1, $M(S_1) \neq V(G)$. We now apply Algorithm 2 that builds a set of vertices $S_2 \subset V(G)$ by iteratively expanding S_1 in such a way that each addition of a vertex increases by at least four the number of monitored vertices. Moreover, at each round, the vertex added to S_2 has maximal degree in G among all candidate vertices.

Algorithm 2: Greedy selection of vertices to expand S_1

Input: A triangulation G of order $n \geq 6$
Output: A set $S_2 \subseteq V(G)$ with $|S_2| \leq \frac{|M(S_2)|-2}{4}$
$S_2 :=$ Algorithm $1(G)$
$M := M(S_2)$
while $\exists\, u$ *in* $V(G) \setminus S_2$ *such that* $|M(S_2 \cup \{u\})| \geq |M| + 4$ **do**
    Select such a vertex u of maximum degree in G.
    $S_2 \leftarrow S_2 \cup \{u\}$
    $M \leftarrow M(S_2)$
Return S_2

We now prove the following lemma:

Lemma 3.3 *Let S_2 be the output of Algorithm 2 applied to G. The following statements hold:*

(i) $|S_2| \leq \frac{|M(S_2)|-2}{4}$.

(ii) *S_2 has Property $(*)$ in G.*

Proof:
(i) Let ℓ denote the number of rounds of Algorithm 2 (i.e., the number of vertices added during the "while" loop). For $0 \leq i \leq \ell$, we denote by $S_2^{(i)}$ the set of selected vertices after the i-th round of Algorithm 2 (where $S_2^{(0)}$ denotes the result of Algorithm 1 applied to G), and by $M^{(i)}$ the set $M(S_2^{(i)})$ of vertices monitored by $S_2^{(i)}$. The algorithm ensures that for all $0 \leq i \leq \ell-1$, we have that if $S_2^{(i)}$ is not the empty set, then $|S_2^{(i+1)}| = |S_2^{(i)}| + 1$ and $|M^{(i+1)}| \geq |M^{(i)}| + 4$. So, provided we can establish a base case (either for $i = 0$ or $i = 1$), Statement (i) holds by induction on ℓ. If $S_2^{(0)}$ is not the empty set, then $1 \leq |S_2^{(0)}| \leq \frac{|M^{(0)}|}{6}$, and thus $|S_2^{(0)}| \leq \frac{|M^{(0)}|-2}{4}$. Otherwise, by Observation 3.2, the first vertex added to S_2 is of degree at least 5 so $|M^{(1)}| \geq 6$. Thus this time $\frac{|M^{(1)}|-2}{4} \geq 1 = |S_2^{(1)}|$, and the desired result follows by induction.
(ii) Let $G' \subseteq G$ be an induced triangulation monitored by S_2 after Algorithm 2. First assume G' is isomorphic to a tower graph. Note that no vertex selected during Algorithm 1 is an interior vertex of a tower graph. Observe that for each interior vertex v of a tower graph, there exists an exterior vertex v'

such that $N[v] \subseteq N[v']$ and $d(v') > d(v)$. Then at any given round i, Algorithm 2 would rather select v' instead of any interior vertex v, and thus no interior vertex of G' is in S_2. Since G' is monitored, then at least one of the exterior vertices of G' (say u) is in S_2 or has propagated to an interior vertex of G', so $N[u] \subseteq M$ and Statement (a) of Property $(*)$ holds for S_2 in G. If G' is isomorphic to the octahedron or to the flip-octahedron, then one of the exterior vertices of G' is in S_1 thanks to Algorithm 1, and Property $(*)$.(a) also holds.

Assume now that $|V(G')| \geq 6$, and suppose that Property $(*)$.(a) does not hold for G'. Then some vertices of G' belong to S_2, and vertices of $V(G') \cap S_2$ only monitor vertices of G' (no propagation may occur from a vertex of the outer face of G'). Then the same proof as for (i) above restricted to G' shows that Property $(*)$.(b) holds. This proves that Property $(*)$ holds for S_2 in G. □

In the following, S_2 denotes the output of Algorithm 2 on the graph G.

3.3 Monitoring the remaining components

After Algorithm 2, some vertices of the graph may still remain non-monitored. Algorithm 3 thus completes the set S_2 into a power dominating set of G, while keeping the wanted bound. In order to succeed, we need to have a better understanding of the structure of the graph around these non-monitored vertices. More precisely, we show that the graph can be described in terms of *splitting structures*, (see Figure 13): they are structures composed of a set $C = \{u_1, u_2, u_3\}$ of three non-monitored vertices and of two *associated triangulations* G_1 and G_2 whose exterior vertices are monitored.

We make use of the following lemma, that is proved in Section 4. We denote by $\overline{M}_G(S)$ the set of vertices not monitored by S in G (i.e., $V(G) \setminus M_G(S)$).

Lemma 3.4 *Let G be a triangulation, S a subset of vertices of G monitoring all b-vertices and facial octahedra. Let G' an induced triangulation of G. If $\overline{M}_G(S) \cap V(G') \neq \emptyset$, and for any $v \in V(G)$, $|M_G(S \cup \{v\})| \leq |M_G(S)| + 3$ (i.e., Algorithm 2 stopped), then G' corresponds to one of the configurations depicted in Figure 13.*

Observe that the triangulations associated to a splitting structure may contain non-monitored vertices, in which case we can again apply the above lemma and deduce that they are in turn isomorphic to a splitting structure.

If $M(S_2) = V(G)$, then by Lemma 3.3, S_2 is a power dominating set of G with at most $\frac{n-2}{4}$ vertices. Otherwise, Algorithm 3 recursively goes down to splitting structures whose associated triangulations are completely monitored, in which case it adds a vertex to S to monitor the remaining vertices.

We now prove that after the addition of vertices during Algorithm 3, the wanted bound still holds.

Lemma 3.5 *Let G' be an induced triangulation of G, and S a subset of vertices of G monitoring all b-vertices and facial octahedra. Let C be a splitting structure in G' with G_1 and G_2 its associated triangulations. Let u be a vertex of C, and let S' denote the set $S \cap V(G_1)$ and S'' the set $S \cap V(G_2)$. If G_1 and G_2 are monitored by S and S' and S'' have Property $(*)$ respectively in G_1 and G_2, then $S' \cup S'' \cup \{u\}$ has Property $(*)$ in G', and G' is monitored.*

Proof: First recall that after application of Algorithm 2, any vertex in $M_G(S)$ has at most three non-monitored neighbors. Therefore, in the induced triangulation G', a vertex adjacent to a vertex in C may not be adjacent to vertices from another configuration C' in G, or it would have two non-monitored

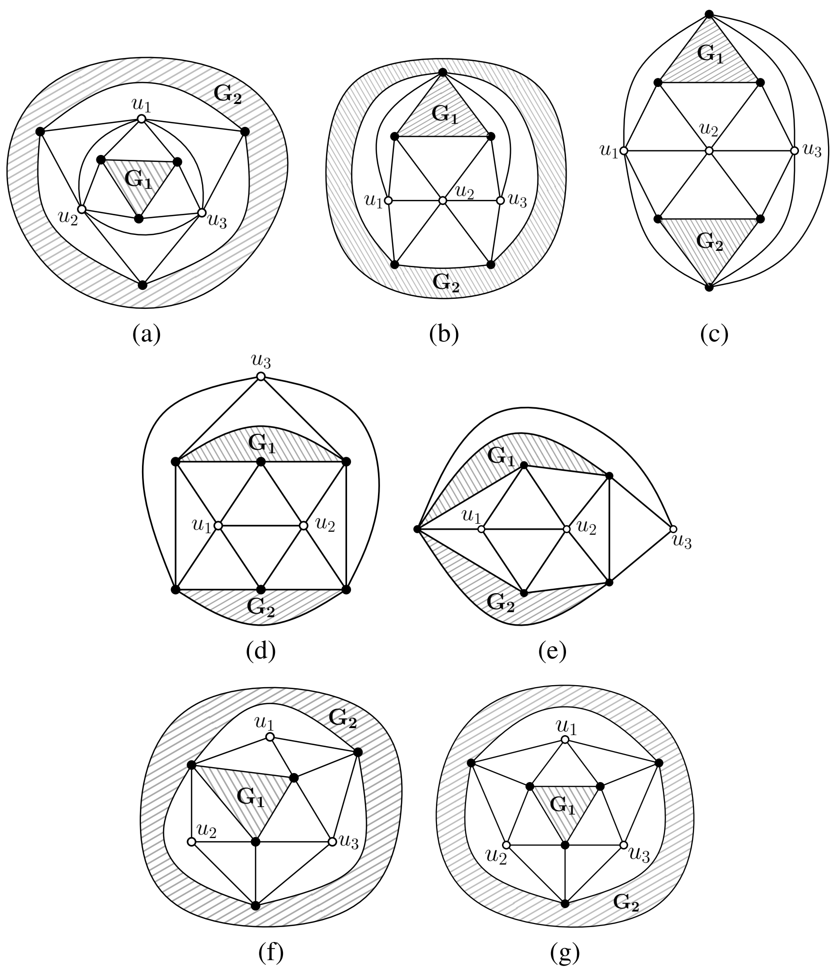

Fig. 13: The seven different splitting structures and their associated triangulations G_1 and G_2. White vertices are non monitored. All triangles are facial except for G_1 and G_2.

Algorithm 3: Monitoring the last vertices

Input: A triangulation G of order $n \geq 6$ and an induced triangulation $G' \subseteq G$
Output: A set $S \subseteq V(G')$ monitoring G' and such that $|S| \leq \frac{|V(G')|-2}{4}$
$S \leftarrow V(G') \cap$Algorithm 2(G)
if $\exists\, u \notin M_G(S)$ **then**
  $G_1, G_2 \leftarrow$ triangulations associated to the splitting structure of G' containing u
  $S' \leftarrow$ Algorithm 3(G, G_1)
  $S'' \leftarrow$ Algorithm 3(G, G_2)
  $S \leftarrow S' \cup S'' \cup \{u\}$
Return S

neighbors in C and two in C', a contradiction. Thus if a vertex can propagate in G', then it can also propagate in G.

We know that S' and S'' have Property $(*)$ in repectively G_1 and G_2, and so G_1 and G_2 both satisfy either Property $(*)$.(a) or (b). Since all exterior vertices of G_1 and G_2 have non-monitored neighbors, then in fact, G_1 and G_2 satisfy Property $(*)$.(b). Thus $|S'| \leq \frac{|V(G_1)|-2}{4}$ and $|S''| \leq \frac{|V(G_2)|-2}{4}$. We remark that $|V(G')| \geq |V(G_1)| + |V(G_2)| + 2$ in every splitting structure. After adding a vertex $u \in C$, we have:

$$|S' \cup S'' \cup \{u\}| \leq \frac{|V(G_1)|-2}{4} + \frac{|V(G_2)|-2}{4} + 1 = \frac{|V(G_1| + |V(G_2)|}{4} \leq \frac{|V(G')|-2}{4}.$$

Moreover, the exterior vertices of induced triangulations of G' all have only monitored neighbors (the exterior vertices of G' excepted) and thus $S' \cup S'' \cup \{u\}$ has Property $(*)$ in G'.

To prove that the addition of one vertex of C is sufficient to monitor G', we consider different cases depending on the splitting structure.

- For splitting structures (a), (b) and (c), adding u_2, then u_1 and u_3 are monitored by adjacency.
- For splitting structures (d) and (e), adding u_1, then u_2 is monitored by adjacency, and then any vertex of the outer face of G_1 or G_2 propagates to u_3.
- For splitting structures (f) and (g), adding u_1, then two exterior vertices of G_1 propagate independently to the other two vertices of C.

Thus G' is monitored, which concludes the proof. □

We can now use Lemma 3.5 to prove by direct induction on the splitting structures that at the end of Algorithm 3, Property $(*)$ holds for S in G. Moreover, the proof of Lemma 3.5 shows that for the set S, G satisfies Property $(*)$.(b). Thus the output S of Algorithm 3 satisfies the wanted bound and the graph is completely monitored. We thus get the following corollary that concludes the proof of Theorem 1.1.

Corollary 3.6 *At the end of Algorithm 3, $M(S) = V(G)$ and $|S| \leq \frac{|V(G)|-2}{4}$.*

In the following section, we finally prove Lemma 3.4.

4 Defining splitting structures

This section is dedicated to the proof of Lemma 3.4. In the following, we work under the assumption of the lemma, i.e., we assume that the set S monitors all octahedra and b-vertices, and that the addition of any vertex v to S would extend the set of vertices monitored by S by at most three. Any vertex contradicting the second part of the assumption is called a *contradicting vertex*. For simplicity, when G and S are clear from context, we denote $M = M_G(S)$ and $\overline{M} = V(G) \setminus M_G(S)$.

As a direct consequence of the definition of power domination, we get the following observation:

Observation 4.1 *Let S be a set of vertices of G such that for every vertex $v \in V(G)$, $|M_G(S \cup \{v\})| \leq |M_G(S)| + 3$. The following properties hold:*

(i) Each vertex of M has either zero, two or three non-monitored neighbors.

(ii) Each vertex of $\overline{M}$ has at most 2 neighbors in $\overline{M}$.

(iii) For every vertex $u \in M \setminus S$, there exists $v \in M \cap N(u)$ such that $N[v] \subset M$ (that propagated to u).

We now make the following statement.

Lemma 4.2 *If v is of degree at least five, then for every two neighbors u_1 and u_2 of v, there exists a neighbor w of v adjacent to u_1 or u_2, but not both, and the corresponding triangle $[vu_iw]$ is facial.*

Proof: We partition the set of neighbors of v into two paths from u_1 to u_2: a path $(w'_1, \ldots, w'_k)$ of length at least three (i.e., $k \geq 2$) and another path $(w_1, \ldots, w_\ell)$, possibly empty. We have $w'_1 \neq w'_k$. By way of contradiction, assume both w'_1 and w'_k are adjacent to both u_1 and u_2. Contracting the path $(w'_1, \ldots, w'_{k-1})$ into w'_1 and the path $(u_1, w_1, \ldots, w_\ell)$ into u_1, we get that u_1, u_2, v, w'_1, w'_k induce a K_5 in the resulting graph, contradicting planarity of G. Thus w'_1 is not adjacent to u_2 (and $[w'_1u_1v]$ is facial) or w'_k is not adjacent to u_1 (and $[w'_ku_2v]$ is facial). □

We remark that Lemma 4.2 also holds when v is in M with at least two neighbors in $\overline{M}$. Indeed, by Observation 4.1, v has a neighbor v' that propagated to it. Then v' only has monitored neighbors, and two of them are also adjacent to v. Thus v has degree at least five, which is the hypothesis of Lemma 4.2.

Lemma 4.3 *Components of $G[\overline{M}]$ are of order at most three.*

Proof:
Let C be a component of $G[\overline{M}]$. By Observation 4.1, each vertex of $\overline{M}$ has degree at most two in $\overline{M}$, so C is a path or a cycle. Then adding any vertex of C to S would monitor all of C. Since we work under the assumption of Lemma 3.4, C is of order at most three. □

Thus each component of $G[\overline{M}]$ is isomorphic to either K_3, P_3, P_2, or K_1. Lemmas 4.4, 4.5 and 4.6 deal successively with the first three cases, whereas Lemma 4.7 goes through the case where $\overline{M}$ is an independent set in the induced triangulation considered.

Lemma 4.4 *Let G and M satisfy the assumption of Lemma 3.4. If an induced triangulation G' contains a component of $G[\overline{M}]$ isomorphic to K_3, then G' is isomorphic to the configuration depicted in Figure 14.*

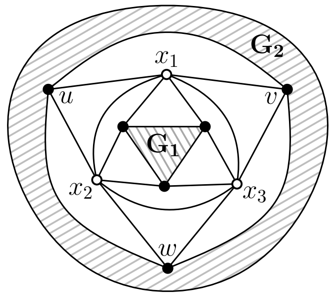

Fig. 14: The configuration for G' containing a non-monitored component isomorphic to K_3. G_1 and G_2 are triangulations. All other triangles of the drawing are facial.

Proof: Let C be a component of $G[\overline{M}]$ isomorphic to K_3 with $V(C) = \{x_1, x_2, x_3\}$. Let u be a vertex of M, adjacent to at least one of the vertices of C.

We first consider the case when u has neighbors in $\overline{M} \setminus C$. If u is adjacent to two vertices in C, then by Observation 4.1, u has exactly one neighbor in $\overline{M} \setminus C$, say v. Then $M(S \cup \{u\}) \supseteq M(S) \cup \{x_1, x_2, x_3, v\}$, and u is a contradicting vertex. So u has only one neighbor in C, say x_1. Within the neighborhood of x_1, the path from x_2 to x_3 going through u must contain at least three interior vertices since u is not adjacent to x_2 or x_3, so x_1 is of degree at least five. Applying Lemma 4.2 on x_1, we get that a neighbor w of x_1 is adjacent to x_2 or x_3 but not both. Since w is adjacent to two vertices in C, it has no other neighbors in $\overline{M}$ or the above case would apply. Hence, adding u to S, all neighbors of u in $\overline{M}$ get monitored, then w propagates to x_2 or x_3 which can in turn propagate to the last vertex of C. So u is a contradicting vertex.

We assume now that any vertex of M adjacent to C has only vertices of C as neighbors in $\overline{M}$. Note that such a vertex must be adjacent to at least two vertices in C. Let u be a common neighbor of x_1 and x_2 such that $[ux_1x_2]$ is facial (u exists since the edge x_1x_2 is contained in exactly two facial triangles). By Lemma 4.2, there is a neighbor v of u that is adjacent to only one of $\{x_1, x_2\}$ (say x_1) and $[uvx_1]$ is facial. The vertex v must have a second non-monitored neighbor, that must be in C, so v is adjacent to x_3. Observe that the triangle $[vx_1x_3]$ must be facial. Otherwise, there is a vertex $t \neq v$ such that $[tx_1x_3]$ is facial and t is separated from x_2 by (vx_1x_3). Then by Lemma 4.2, t has a neighbor t' with only one neighbor among $\{x_1, x_3\}$ also separated from x_2 by (vx_1x_3), and thus with only one non-monitored neighbor, a contradiction. Now, v and x_3 have a common neighbor w outside the triangle $[vx_1x_3]$, such that $[vwx_3]$ is facial. By definition of v, we have $w \neq x_2$. We also have $w \neq u$ or v would be of degree three contradicting Observation 4.1. The cycle (uvx_3x_2) separates w from x_1, so the second non-monitored neighbor of w (different from x_3) must be x_2. Unless an additional edge uw form a facial triangle $[uwx_2]$, there is another neighbor of x_2 that is separated from both x_1 and x_3 by the cycle $(uvwx_2)$, a contradiction. So u is adjacent to w in a facial triangle $[uwx_2]$.

In a similar way that we proved that $[vx_1x_3]$ is facial, we infer that $[wx_2x_3]$ is facial. By construction, $[ux_1x_2]$, $[uvx_1]$, $[vwx_3]$ are facial, and we proved $[vx_1x_3]$, $[uwx_2]$ and $[wx_2x_3]$ also are. If the triangle $[x_1x_2x_3]$ is facial, then the graph induced by the vertices u, v, w, x_1, x_2, x_3 is a facial octahedron, contradicting the assumption of Lemma 3.4. Thus $[x_1x_2x_3]$ is not facial, and applying the same line of reasoning as above inside $[x_1x_2x_3]$ shows that G' is isomorphic to the configuration depicted in Figure 14. □

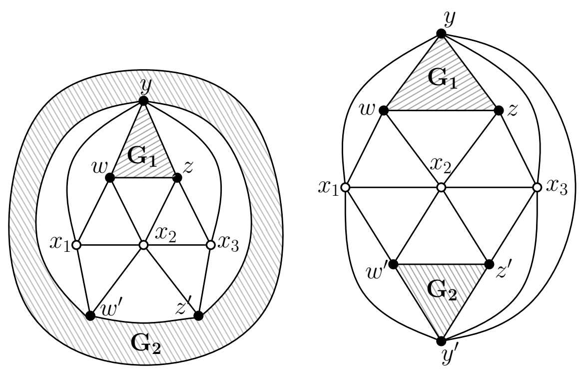

Fig. 15: The two possible configurations for G' containing a non-monitored component isomorphic to P_3. G_1 and G_2 are triangulations. All other triangles of the drawing are facial.

Lemma 4.5 *Let G and M satisfy the assumption of Lemma 3.4. If an induced triangulation G' contains a component of $G[\overline{M}]$ isomorphic to P_3, then G' is isomorphic to one of the splitting structures depicted in Figure 15.*

Proof: Let C be a component of $G[\overline{M}]$ isomorphic to P_3 with $V(C) = \{x_1, x_2, x_3\}$.

Let u be a vertex adjacent to C. We first prove that all neighbors of u in $\overline{M}$ are vertices of C. By Observation 4.1, u has at most two neighbors in $\overline{M} \setminus V(C)$. If u has exactly one neighbor u_1 in $\overline{M} \setminus V(C)$, then x_2 is a contradicting vertex, since u propagates to u_1 once x_2 is added to S. Assume then that u has two neighbors in $\overline{M} \setminus V(C)$ and thus only one neighbor in C. If u is adjacent to x_1 or x_3, then u is a contradicting vertex. Suppose that u is adjacent to x_2 only, which must then be of degree at least five. We apply Lemma 4.2 on x_2 and get a neighbor v of x_2 adjacent to x_1 or x_3 but not both. Taking u in S, v then propagates (and then x_2 propagates to x_3) so u is a contradicting vertex. Thus neighbors of C may not be adjacent to vertices in $\overline{M}$ that are not in C.

We now prove that there is no vertex of M adjacent to all vertices of C. Suppose by way of contradiction that u is a vertex in M adjacent to x_1, x_2 and x_3. By Lemma 4.2, u has a neighbor $z \in M$ with exactly one neighbor in $\{x_1, x_3\}$ (say x_1) and $[uzx_1]$ is facial. Note that by the above statement, z is also adjacent to x_2.

Again, we can apply Lemma 4.2 to find a neighbor z' of z adjacent to x_1 or x_2 but not both. Vertex z' must have a second non-monitored neighbor, namely x_3. So z' cannot be adjacent to x_1 which is separated from x_3 by (ux_2z), so z' is adjacent to x_2 and x_3 and $[x_2zz']$ is facial. Now u is necessarily adjacent to z' forming a facial triangle $[ux_3z']$ (otherwise some vertex would have a single neighbor in C).

Observe that the triangle $[zx_1x_2]$ must be facial. Otherwise, there is a vertex $t \neq z$ such that $[tx_1x_2]$ is facial and t is separated from x_3 by (zx_1x_2). Then by Lemma 4.2, t has a neighbor t' with only one neighbor among $\{x_1, x_2\}$ also separated from x_3 by (zx_1x_2), and thus with only one non-monitored neighbor, a contradiction. With a similar argument, we get that $[z'x_2x_3]$, $[ux_1x_2]$ and $[ux_2x_3]$ are facial. But then x_2 is a b-vertex (as in the bad configuration of Figure 2), a contradiction.

Let $w, w', z, z' \in M$ such that $[x_1x_2w]$, $[x_1x_2w']$, $[x_2x_3z]$, $[x_2x_3z']$ are faces. By the above statement, all these vertices are distinct. Suppose that there is a neighbor u of x_2 different from the above vertices.

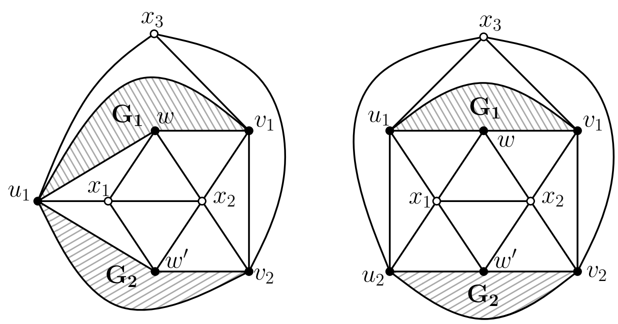

Fig. 16: The two possible configurations of an induced triangulation G' if $G[\overline{M}]$ has a component isomorphic to P_2. G_1 and G_2 are triangulations. All other triangles of the graph are facial.

Vertex u has a second neighbor in C, say x_1. The cycle (ux_1x_2) separates w or w' from x_3, say w. By Lemma 4.2, w has a neighbor with exactly one neighbor in $\{x_1, x_2\}$, and that cannot be adjacent to x_3, a contradiction. Thus x_2 has no other neighbor. Renaming vertices if necessary, we suppose $[x_2wz]$ and $[x_2w'z']$ are facial triangles.

Note that x_1 or x_3 must have another neighbor. Otherwise, w is adjacent to w' and$z$ is adjacent to z', which implies that x_2 is a b-vertex (as in the ugly configuration of Figure 2), a contradiction. Let $y \in M$ be a neighbor of x_1 such that $[x_1wy]$ is facial. The second neighbor of y in C is necessarily x_3. Similarly, z has a neighbor z_1 such that $[x_3z_1z]$ is facial and adjacent to x_1 and x_3. Note that $z_1 \neq w$ or w would be adjacent to three vertices in C. Then $y = z_1$ or the cycle (x_3ywz) would separate z_1 from x_1. We prove with similar arguments that there is a vertex y' such that $[x_1w'y']$ and $[x_3z'y']$ are facial.

If $y = y'$, then G' is isomorphic to the first splitting structure of Figure 15. Otherwise, suppose first that x_1 has another neighbor t. It also has to be adjacent to x_3. Then applying Lemma 4.2 to t, we find a vertex adjacent to only one vertex in C, a contradiction. So y and y' are adjacent, and $[x_1yy']$ and $[x_3yy']$ are facial. Thus G' is isomorphic to the second configuration of Figure 15. □

Lemma 4.6 *Let G and M satisfy the assumption of Lemma 3.4. If an induced triangulation G' contains a non-monitored component isomorphic to P_2, then G' is isomorphic to one of the configurations depicted in Figure 16.*

Proof:

Let $C = \{x_1, x_2\}$ with $x_1x_2 \in E(G)$, and let w and w' be the vertices such that $[x_1x_2w]$ and $[x_1x_2w']$ are facial.

Claim 1. There is exactly one vertex of $\overline{M}$ at distance 2 of C.

Proof. Suppose there is no vertex of $\overline{M}$ at distance 2 from C. By Lemma 4.2, w has a neighbor $t \in M$ adjacent to only one vertex among $\{x_1, x_2\}$. Then t has only one neighbor in $\overline{M}$, which contradicts Observation 4.1. Thus there is a vertex of $\overline{M}$ at distance 2 from C. Suppose that there is $u' \in \overline{M} \setminus V(C)$ neighbor of a vertex $u \in N(C)$ and $v' \in \overline{M} \setminus V(C)$ neighbor of another vertex $v \in N(C)$. Then u is a contradicting vertex (whether it is distinct from v or not). (□)

Let x_3 be the only vertex of $\overline{M}$ at distance 2 of C.

Claim 2. The vertices adjacent to x_3 are exactly the vertices of $(N(x_1) \cup N(x_2)) \setminus \{w, w'\}$.

Proof. Suppose there is a vertex $w'' \in M, w'' \neq \{w, w'\}$ adjacent to x_1 and x_2. The cycle(x_1x_2w'') separates x_3 from either w or w', say w. By Lemma 4.2, there exists a vertex v adjacent to w and to only one vertex among $\{x_1, x_2\}$. By Observation 4.1, v has a second non-monitored neighbor, that cannot be x_3, which contradicts Claim 1. Thus w and w' are the only common neighbors of x_1 and x_2. Therefore, all vertices adjacent to only one of x_1 and x_2 (i.e., in $N(x_1) \cup N(x_2) \setminus \{w, w'\}$) are adjacent to x_3 (and there is at least one such vertex).

Suppose there exists some vertex v adjacent to x_3 but not in $N(x_1) \cup N(x_2)$. Then v is in $\overline{M}$ or it has another neighbor $x_4 \in \overline{M} \setminus \{x_1, x_2, x_3\}$, and v is a contradicting vertex. Thus no vertex $v \in V(G) \setminus (N(x_1) \cup N(x_2))$ is adjacent to x_3.

We now prove that w and w' are not adjacent to x_3. Suppose w is adjacent to x_3. By Lemma 4.2, w has a neighbor u_1 adjacent to only one of $\{x_1, x_2\}$ (say x_1) such that $[u_1x_1w]$ is facial. (Thus u_1 is also adjacent to x_3 and $[wu_1x_3]$ is facial, since it separates x_3 from x_1 and x_2.) Again by Lemma 4.2, u_1 has a neighbor v_1 in M adjacent to only one of $\{x_1, x_3\}$. Suppose first v_1 is adjacent to x_3 (and not to x_1). Then v_1 is also adjacent to x_2. Following Observation 4.1, w has other neighbors in M different from u_1. So there is a vertex t such that $[x_2tw]$ is facial, and since t is separated from x_1 by $(x_2v_1x_3w)$, t is adjacent to x_3. Applying Lemma 4.2 on t, we get a contradiction. So v_1 is adjacent to x_1 but not to x_3, and thus $v_1 = w'$ (and w' is not adjacent to x_3). But x_3 has degree at least three, so there is a vertex v_2 adjacent to x_2 and x_3. Again, $[u_1x_3w]$, $[v_2x_2w]$ and $[v_2x_3w]$ must be facial. But then there is no vertex that may have propagated to w. Thus w and w' are not adjacent to x_3. (□)

Let us now consider the neighbors of x_1 and x_2 in $M \setminus \{w, w'\}$. Let $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_\ell)$ be the paths from w to w' among respectively $N(x_1) \cap M$ and $N(x_2) \cap M$. Since x_3 has degree at least 3, then by Claim 2, $k + \ell \geq 3$. First observe that k and ℓ both are at most 2. Otherwise, say $k \geq 3$, then by Claim 2, each u_i is adjacent to x_3, and the triangles $[u_iu_{i+1}x_3]$ are facial, in particular $[u_1u_2x_3]$ and $[u_2u_3x_3]$. But then u_2 contradicts Observation 4.1.

We thus have two cases:

- $k + \ell = 3$, say u_1 is the only neighbor of x_1 and v_1, v_2 are the only two neighbors of x_2 in $M \setminus \{w, w'\}$. By Claim 2, u_1, v_1 and v_2 are neighbors of x_3. Moreover, since none of $\{w, w'\}$ is adjacent to x_3, u_1 is adjacent to v_1 and v_2. Also by Claim 2, triangles $[u_1v_1x_3]$, $[v_1v_2x_3]$ and $[u_1v_2x_3]$ are facial, and G is isomorphic to the first graph depicted in Figure 16.

- x_1 and x_2 both have exactly two neighbors in $M \setminus \{w, w'\}$. By Claim 2, u_1, u_2, v_1 and v_2 are neighbors of x_3. Again, u_1 is adjacent to v_1 and u_2 is adjacent to v_2 since neither w nor w' is adjacent to x_3. Also by Claim 2, triangles $[u_1v_1x_3]$, $[v_1v_2x_3]$, $[u_1u_2x_3]$ and $[u_2v_2x_3]$ are facial and G is isomorphic to the second graph depicted in Figure 16.

This concludes the proof. □

Lemma 4.7 *Let G and M satisfy the assumptions of Lemma 3.4. If an induced triangulation G' is such that $\overline{M} \cap V(G')$ is an independent set, then G' is isomorphic to one of the splitting structures depicted in Figure 17.*

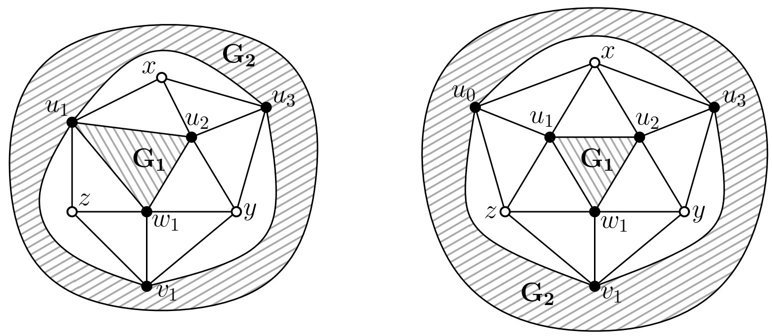

Fig. 17: The two possible configurations of an induced triangulation G' if $\overline{M} \cap V(G')$ is an independent set. G_1 and G_2 are triangulations. All other triangles of the drawings are facial.

Proof: Let G satisfy the assumptions of the lemma. In this proof, we denote M' the set $M \cap V(G')$, and $\overline{M}'$ the set $\overline{M} \cap V(G')$.

Claim 1. There exists a vertex $u \in M'$ with only two neighbors in $\overline{M}'$.

Proof. Suppose by way of contradiction that every vertex in M' with non-monitored neighbors has exactly three neighbors in $\overline{M}'$. Note first that no two vertices in M' have exactly two common neighbors in $\overline{M}'$, or they would be contradicting vertices. Hence, they share either one or three such neighbors.

Suppose first that all vertices in M' with a common neighbor in $\overline{M}'$ have exactly one such common neighbor. We define an auxiliary graph H as follows: the vertices of H are the vertices in $\overline{M}'$, and two vertices in H are adjacent if they have a common neighbor in G'. Observe that from a planar drawing of G', we can easily build a planar drawing of H: we keep the position of the vertices, and for each edge (uv) in H, u and v have a common neighbor x in G' and we can have the edge (uv) follow closely the edges (ux) and (xv) (that would not create crossings since $N_{G'}(x) \cap \overline{M}' = 3$). By our assumption that no two vertices in M' have more than one common neighbor in $\overline{M}'$, the degree of a vertex in H is precisely twice its degree in G'. Since every vertex in $\overline{M}'$ has degree at least 3 and every vertex in M' has three neighbors in $\overline{M}'$, that implies that H has minimum degree at least 6. But this contradicts Euler's formula for planar graphs.

So there are at least two vertices u and v in M' with three common neighbors in $\overline{M}'$, say x_1, x_2 and x_3, forming a subgraph isomorphic to a $K_{2,3}$. Consider such five vertices, such that the subgraph G'' induced by the vertices within the outer face of the $K_{2,3}$ does not contain the same structure. Denote x_1 and x_3 the exterior vertices (i.e., x_2 is inside the cycle (x_1ux_3v)). Since x_1, x_2 and x_3 are pairwise non adjacent, there is another neighbor w of x_2 in M', which has at least two other neighbors in $\overline{M}'$. By minimality of the selected $K_{2,3}$, now all vertices in G'' that belong to M' and share a neighbor in $\overline{M}'$ do share exactly one. Building the graph H on G'' the same way as above, we get a planar graph H where every vertex has degree at least six except for x_1, x_2 and x_3 that have respectively degree at least 2, 4 and 2. Therefore, we get that the sum of the degrees of the vertices in H is at least $6|V(H)| - 10$, again a contradiction with Euler's formula. This concludes the proof. (□)

Claim 2. If a vertex u of M' has degree 2 in $\overline{M}'$, then all the vertices of M' sharing a neighbor in $\overline{M}'$

with u also have degree 2 in $\overline{M}'$.

Proof. Let u_1 be a vertex of M' with two neighbors v_1, v_2 in $\overline{M}'$. Suppose that there exists a vertex u_2 in M' adjacent to v_1 or v_2 (say v_1) and with degree 3 in $\overline{M}'$. If u_2 is not adjacent to v_2, then u_2 is a contradicting vertex. So assume u_2 is also adjacent to v_2 and let v_3 be the third neighbor of u_2 in $\overline{M}'$. Applying Lemma 4.2 to vertex u_1, let t be a vertex adjacent to only one of v_1 and v_2, say v_1, and such that $[u_1v_1t]$ is facial. There is another vertex adjacent to t in $\overline{M}'$. If this vertex is not v_3, then t is a contradicting vertex (as u_1 propagates to v_2 then u_2 to v_3). So t has only two neighbors in $\overline{M}'$, v_1 and v_3.

Now, every other vertex in the graph is separated from v_1, v_2 or v_3 by one of the three separating cycles $(u_1v_1u_2v_2)$, $(tv_1u_2v_3)$ and $(tu_1v_2u_2v_3)$. The monitored vertex u_2 necessarily has more neighbors (by Observation 4.1). Suppose there is a neighbor of u_2 in the cycle $(tu_1v_2u_2v_3)$. Then there is a neighbor w to u_2 and v_2 forming a face $[u_2v_2w]$. If w is not adjacent to v_3, then w has some extra neighbors in $\overline{M}'$, and is a contradicting vertex (u_1 propagates to v_1 then u_2 to v_3). If w is also adjacent to v_3, by Lemma 4.2 it has a neighbor adjacent to only one of v_2 and v_3, also separated from v_1 by the cycle $(tu_1v_2u_2v_3)$, and the same argument applies. The same arguments apply also if u_2 has neighbors in the other separating cycles. Thus there is no vertex adjacent to v_1 or v_2 with degree 3 in $\overline{M}'$. This concludes the proof. (□)

Let u_1 be a vertex of M' with exactly two neighbors in $\overline{M}'$, denoted x and z. By Lemma 4.2, there is a neighbor of u_1 adjacent to only one of x and z, say u_2 is adjacent to x but not z (and $[xu_1u_2]$ is facial). By Claim 2, u_2 has only one other neighbor in $\overline{M}'$, denote it y. Note that we now have the property **(P)**: any neighbor $v \in M'$ of x, y or z has at least two neighbors in $\{x, y, z\}$ and is not adjacent to any vertex of $\overline{M}' \setminus \{x, y, z\}$. Otherwise v would be a contradicting vertex. Consider the two paths from u_2 to z that partition $N(u_1)$. Let w_1 be the last vertex before z in the path that does not go through x (i.e., $[u_1w_1z]$ is facial and w_1 is not adjacent to x). Since u_2 is not adjacent to z, then $w_1 \neq u_2$. By the above property (P), w_1 is adjacent to y. Moreover, y may not have a neighbor separated from x and z by $(u_1u_2yw_1)$, so $[u_2w_1y]$ is a facial triangle.

Suppose first that x is of degree three, and let u_3 denote its third neighbor, adjacent to both u_1 and u_2. It has one other neighbor among y and z, say y. Observe that $[u_2u_3y]$ is necessarily a facial triangle, and that by Claim 2, u_3 is not adjacent to z. Since z is of degree at least 3, it has a neighbor $v_1 \neq u_3$ such that $[u_1v_1z]$ is facial. By property (P), v_1 is adjacent to y. Now z has no other neighbor within the cycle (v_1yw_1z), or it would be a common neighbor to y and z, but applying Lemma 4.2 would lead to a contradiction. So w_1 is adjacent to v_1, and $[v_1w_1y]$ and $[v_1w_1z]$ are facial triangles. In addition, y cannot have a neighbor separated from x and z by $(u_1u_3yv_1)$ so $[u_3v_1y]$ is a facial triangle. Thus we are in the first configuration of Figure 17.

Assume now that each of x, y and z have degree at least 4. Let u_3 form a facial triangle with x and u_2. If u_3 is adjacent to z, then the fourth neighbor of x is also adjacent to z. By Lemma 4.2, it has a neighbor adjacent to only one of x and z, which is separated from y by (u_1xu_3z), a contradiction. So u_3 is adjacent to y forming a facial triangle $[u_2u_3x]$. By the same argument, we infer the existence of u_0 and v_1, common neighbors of x and z and of y and z respectively, and that the corresponding triangles are facial. If x were of degree 5, then we would get similarly a contradiction applying Lemma 4.2 on u_3 or u_0. By the same reasoning on y and z, we obtain the second configuration of Figure 17, which concludes the proof of Lemma 4.7. □

The results from the four Lemmas 4.4, 4.5, 4.6 and 4.7 conclude the section: after Algorithm 2, each induced triangulation of G is isomorphic to one of the graphs depicted in Figure 13.

References

A. Aazami. Domination in graphs with bounded propagation: algorithms, formulations and hardness results. *Journal of combinatorial optimization*, 19(4):429–456, 2010.

A. Aazami and K. Stilp. Approximation algorithms and hardness for domination with propagation. *SIAM Journal on Discrete Mathematics*, 23(3):1382–1399, 2009.

J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier. Fixed parameter algorithms for dominating set and related problems on planar graphs. *Algorithmica*, 33(4):461–493, 2002. ISSN 0178-4617. doi: 10.1007/s00453-001-0116-5. URL `http://dx.doi.org/10.1007/s00453-001-0116-5`.

D. Archdeacon. Topological graph theory: a survey. *Congr. Numer.*, 115:5–54, 1996. ISSN 0384-9864. Surveys in graph theory (San Francisco, CA, 1995).

T. Baldwin, L. Mili, M. Boisen, and R. Adapa. Power system observability with minimal phasor measurement placement. *IEEE Transactions on Power Systems*, 8(2):707–715, 1993.

R. Barrera and D. Ferrero. Power domination in cylinders, tori, and generalized petersen graphs. *Networks*, 58(1):43–49, 2011.

R. Bowen and S. Fisk. Generation of triangulations of the sphere. *Math. Comp.*, 21:250–252, 1967.

D. J. Brueni and L. S. Heath. The PMU placement problem. *SIAM Journal on Discrete Mathematics*, 19 (3):744–761, 2005.

G. J. Chang, P. Dorbec, M. Montassier, and A. Raspaud. Generalized power domination of graphs. *Discrete Applied Mathematics*, 160(12):1691–1698, 2012.

P. Dorbec and S. Klavžar. Generalized power domination: Propagation radius and Sierpiński graphs. *Acta Applicandae Mathematicae*, 134(1):75–86, 2014.

P. Dorbec, M. Mollard, S. Klavžar, and S. Špacapan. Power domination in product graphs. *SIAM J. Discrete Math.*, 22(2):554–567, 2008. ISSN 0895-4801. doi: 10.1137/060661879. URL `http://dx.doi.org/10.1137/060661879`.

P. Dorbec, M. A. Henning, C. Löwenstein, M. Montassier, and A. Raspaud. Generalized power domination in regular graphs. *SIAM Journal on Discrete Mathematics*, 27(3):1559–1574, 2013.

M. Dorfling and M. A. Henning. A note on power domination in grid graphs. *Discrete Applied Mathematics*, 154(6):1023–1027, 2006.

P. A. Dreyer Jr. *Applications and variations of domination in graphs*. PhD thesis, The State University of New Jersey, 2000.

D. Ferrero, S. Varghese, and A. Vijayakumar. Power domination in honeycomb networks. *Journal of Discrete Mathematical Sciences and Cryptography*, 14(6):521–529, 2011.

J. Guo, R. Niedermeier, and D. Raible. Improved algorithms and complexity results for power domination in graphs. *Algorithmica*, 52(2):177–202, 2008.

T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, and M. A. Henning. Domination in graphs applied to electric power networks. *SIAM Journal on Discrete Mathematics*, 15(4):519–529, 2002.

S. T. Hedetniemi and R. C. Laskar. Bibliography on domination in graphs and some basic definitions of domination parameters. *Annals of Discrete Mathematics*, 48:257–277, 1991.

L. Kang and N. Wormald. Minimum power dominating sets of random cubic graphs. *Journal of Graph Theory*, 85(1):152–171, 2017.

J. Kneis, D. Mölle, S. Richter, and P. Rossmanith. Parameterized power domination complexity. *Information Processing Letters*, 98(4):145–149, 2006.

C.-S. Liao. Power domination with bounded time constraints. *Journal of Combinatorial Optimization*, 31 (2):725–742, 2016.

C.-S. Liao and D.-T. Lee. Power domination problem in graphs. In *Computing and Combinatorics*, pages 818–828. Springer, 2005.

C.-S. Liao and D.-T. Lee. Power domination in circular-arc graphs. *Algorithmica*, 65(2):443–466, 2013.

L. R. Matheson and R. E. Tarjan. Dominating sets in planar graphs. *European J. Combin.*, 17(6):565–568, 1996. ISSN 0195-6698. doi: 10.1006/eujc.1996.0048. URL `http://dx.doi.org/10.1006/eujc.1996.0048`.

L. Mili, T. Baldwin, and R. Adapa. Phasor measurement placement for voltage stability analysis of power systems. In *Proceedings of the 29th IEEE Conference on Decision and Control*, pages 3033–3038, Honolulu, Hawaii, 1990. IEEE.

A. Phadke, J. Thorp, and K. Karimi. State estimation with phasor measurements. *IEEE Transactions on Power Systems*, 1(1):233–238, 1986.

C. Wang, L. Chen, and C. Lu. k-power domination in block graphs. *Journal of Combinatorial Optimization*, 31(2):865–873, 2016.

G. Xu, L.-Y. Kang, E. Shan, and M. Zhao. Power domination in block graphs. *Theoretical Computer Science*, 359(1):299–305, 2006.

M. Zhao and L.-Y. Kang. Power domination in planar graphs with small diameter. *Journal of Shanghai University (English Edition)*, 11:218–222, 2007.

M. Zhao, L.-Y. Kang, and G. J. Chang. Power domination in graphs. *Discrete Math.*, 306(15):1812–1816, 2006. ISSN 0012-365X. doi: 10.1016/j.disc.2006.03.037. URL `http://dx.doi.org/10.1016/j.disc.2006.03.037`.