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Cyclic partitions of complete nonuniform hypergraphs and complete multipartite hypergraphs

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A *cyclic q-partition* of a hypergraph (V, E) is a partition of the edge set E of the form $\{F, F^\theta, F^{\theta^2}, \ldots, F^{\theta^{q-1}}\}$ for some permutation θ of the vertex set V. Let $V_n = \{1, 2, \ldots, n\}$. For a positive integer k, $\binom{V_n}{k}$ denotes the set of all k-subsets of V_n. For a nonempty subset K of V_{n-1}, we let $\mathcal{K}_n^{(K)}$ denote the hypergraph $\left(V_n, \bigcup_{k\in K} \binom{V_n}{k}\right)$. In this paper, we find a necessary and sufficient condition on n, q and k for the existence of a cyclic q-partition of $\mathcal{K}_n^{(V_k)}$. In particular, we prove that if p is prime then there is a cyclic p^α-partition of $\mathcal{K}_n^{(V_k)}$ if and only if $p^{\alpha+\beta}$ divides n, where $\beta = \lfloor \log_p k \rfloor$. As an application of this result, we obtain two sufficient conditions on $n_1, n_2, \ldots, n_t, k, \alpha$ and a prime p for the existence of a cyclic p^α-partition of the complete t-partite k-uniform hypergraph $\mathcal{K}^{(k)}_{n_1,n_2,\ldots,n_t}$.

Keywords: Self-complementary hypergraph, Uniform hypergraph, t-complementing permutation, Cyclically t-complementary hypergraph.

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1 Introduction

1.1 Definitions

For a finite set V and a positive integer k, let $\binom{V}{k}$ denote the set of all k-subsets of V. A *hypergraph* is a pair (V, E) in which V is a finite set of *vertices* and E is a collection of subsets of V called *edges*. A hypergraph (V, E) is called k-*uniform* (or a k-*hypergraph*) if $E \subseteq \binom{V}{k}$. The *order* of a hypergraph is the cardinality of its vertex set, and the *rank* of a hypergraph is the maximum cardinality of an edge. Thus a k-uniform hypergraph has rank k. The vertex set and the edge set of a hypergraph X will often be denoted by $V(X)$ and $E(X)$, respectively.

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Throughout the paper we let $V_n = \{1, 2, \ldots, n\}$ and we always assume that the vertex set of a hypergraph of order n is equal to V_n. The complete k-uniform hypergraph of order n is $\left(V_n, \binom{V_n}{k}\right)$ and is denoted by $\mathcal{K}_n^{(k)}$. For a nonempty subset K of V_{n-1}, the *complete K-hypergraph* is $\left(V_n, \bigcup_{k\in K} \binom{V_n}{k}\right)$ and denoted by $\mathcal{K}_n^{(K)}$. Let θ be a permutation of V_n. If $\{E, E^\theta, E^{\theta^2}, \ldots, E^{\theta^{q-1}}\}$ is a partition of $\bigcup_{k\in K} \binom{V_n}{k}$, we call it a *cyclic q-partition* of $\mathcal{K}_n^{(K)}$ and θ is called a *(q, K)-complementing permutation*, or a *permutation of cyclic q-partition of* $\mathcal{K}_n^{(K)}$. If $K = \{k\}$ we abbreviate $\mathcal{K}_n^{(\{k\})}$ to $\mathcal{K}_n^{(k)}$ and we call θ a (q, k)-complementing permutation. We denote the set E^{θ^i} by E_i. It is easy to see that $E_i^\theta = E_{i+1(\bmod\ q)}$ for $i = 0, 1, \ldots, q-1$. A cyclic q-partition of $\mathcal{K}_n^{(K)}$ corresponds to a family of q isomorphic hypergraphs $\{X_i = (V_n, E_i) : i = 0, 1, \ldots, q-1\}$ with edge ranks in K which decompose $\mathcal{K}_n^{(K)}$, and which are permuted cyclically under the action of θ. Each of the hypergraphs X_i in this decomposition are sometimes called *q-complementary*, and θ is called a *q-antimorphism* of X_i.

1.2 History and the statement of the main results

The 2-complementary 2-hypergraphs are the self-complementary graphs, which have been well studied due to their connection to the graph isomorphism problem and to large sets of combinatorial designs [3, 7, 8, 9, 10, 11, 12, 13]. A good reference on self-complementary graphs and their generalizations is found in [4].

For a prime power $q = p^\alpha$, the cycle types of the (q, k)-complementing permutations have been determined in [6] and independently in [14], while the case for $k = 2$ originally appeared in [1]. We state this result in Theorem 1.1 using the notation of [14], which we now introduce. A necessary and sufficient condition for a permutation to be (q, k)-complementing of $\mathcal{K}_n^{(k)}$ has been given in [14] for arbitrary q and k ($1 \le k \le n-1$). For integers n and d, $d > 0$, the symbol $r(n, d)$ denotes the remainder when n is divided by d. Thus $n \equiv r(n, d) \pmod d$. For a positive integer k and a prime p, the symbol $C_p(k)$ denotes the largest nonnegative integer c such that p^c divides k. Thus if $k = \sum_{i\ge 0} k_i p^i$ is the base-p representation of k, where $0 \le k_i < p$ for all integers $i \le 0$, then $C_p(k)$ is the smallest integer i such that $k_i > 0$. For a finite nonempty set A, we abbreviate $C_p(|A|)$ to $C_p(A)$. The symbol $\mathbb{N}$ denotes the set of nonnegative integers.

Theorem 1.1 [6, 14] *Let n, k, p and α be positive integers such that $k < n$ and p is prime. Let θ be a permutation on V_n with orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_m$. Then θ is a (p^α, k)-complementing permutation of $\mathcal{K}_n^{(k)}$ if and only if there is a nonnegative integer ℓ such that $\sum_{i:C_p(\mathcal{O}_i)<\ell+\alpha} |\mathcal{O}_i| < r(k, p^{\ell+1})$.*

The following corollary to Theorem 1.1 gives necessary and sufficient conditions on the order n of a cyclic p^α-partition of $\mathcal{K}_n^{(k)}$. The equivalence between statements (1) and (2) was noted in both [6] and [14], while the equivalence between (2) and (3) was established in [14].

Corollary 1.2 [6, 14] *Let n, k, p and α be positive integers such that $k < n$ and p is prime. Suppose that $k = \sum_{i\ge 0} k_i p^i$ and $n = \sum_{i\ge 0} n_i p^i$, where $0 \le k_i < p$ and $0 \le n_i < p$ for $i \in \mathbb{N}$. Then the following three statements are equivalent.*

1. *There exists a cyclic p^α-partition of $\mathcal{K}_n^{(k)}$.*

2. *There exists $\ell \in \mathbb{N}$ such that $r(n, p^{\ell+\alpha}) < r(k, p^{\ell+1})$.*

3. *There exist $r, \ell \in \mathbb{N}$ with $r \leq \ell$ such that $n_r < k_r$, $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$, and $n_i = 0$ for $\ell < i < \ell + \alpha$.*

The *general complete hypergraph* is $\mathcal{K}_n^{(V_{n-1})}$ and denoted by $\mathcal{K}_n^*$, for short. The integers p and n for which there exists a cyclic p-partition of $\mathcal{K}_n^*$ were determined in [14], and the cycle types of their p-antimorphisms were characterized.

Theorem 1.3 [14] *Let n and p be positive integers with $p < n$. The general complete hypergraph $\mathcal{K}_n^*$ has a cyclic p-partition if and only if p is prime and n is a power of p. Moreover, every (p, V_{n-1})-complementing permutation is cyclic.*

The goal of this paper is to establish the existence of cyclic q-partitions of $\mathcal{K}_n^{(K)}$ for various subsets K of V_{n-1}. Some existence results for cyclic 2-partitions of $\mathcal{K}_n^{(K)}$ were established in [5] for sets K consisting of integers of the form 2^ℓ or $2^\ell + 1$, and sets K consisting of integers which are sums of consecutive powers of 2. In this paper we are mainly interested in existence results for cyclic partitions of $\mathcal{K}_n^{(K)}$ for sets K of the form $\{1, 2, \ldots, k\}$. Such results have applications to cyclic decompositions of complete multipartite k-uniform hypergraphs, which will be discussed in Section 3. The main results of this paper are stated in Theorems 1.4, 1.7 and 1.8, and Corollaries 1.5 and 1.6 below. The proofs of the theorems are given in Section 2.

Theorem 1.4 *Let n, k, p and α be positive integers such that p is prime and $k < n$. A permutation σ of V_n is (p^α, V_k)-complementing if and only if the cardinality of any orbit of σ is congruent to 0 (mod $p^{\alpha+\beta}$), where $\beta = \lfloor \log_p k \rfloor$.*

Theorem 1.4 implies the following.

Corollary 1.5 *Let n, k, p and α be positive integers such that p is prime and $k < n$. There is a cyclic p^α-partition of $\mathcal{K}_n^{(V_k)}$ if and only if $p^{\alpha+\beta}$ divides n, where $\beta = \lfloor \log_p k \rfloor$.*

It is clear that if $K \subset K' \subset V_n$ and a permutation σ of V_n is (q, K')-complementing then σ is also (q, K)-complementing. Hence, and by Corollary 1.2(3), we have the following.

Corollary 1.6 *Let p, n and α be positive integers where p is prime. If $p^{\alpha+\beta}$ divides n, then there is a cyclic p^α-partition of $\mathcal{K}_n^{(K)}$ for any nonempty subset K of $\{m : 1 \leq m \leq k \text{ or } n - k \leq m \leq n - 1\}$, for any k such that $p^\beta \leq k < p^{\beta+1}$. In particular, there is a cyclic p^α-partition of $\mathcal{K}_n^{(V_k)}$ for these k.*

Theorem 1.7 *Let n, k and q be positive integers such that $q = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_s^{\alpha_s}$, where $p_1, p_2, \ldots, p_s$ are mutually distinct primes, $k < n$, and $\beta_j = \lfloor \log_{p_j} k \rfloor$ for every $j = 1, 2, .., s$. A permutation σ of V_n is (q, V_k)-complementing if and only if the cardinality of any orbit of σ is congruent to 0 (mod $p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdot \ldots \cdot p_s^{\alpha_s+\beta_s}$).*

Theorem 1.8 *Let p, n and α be positive integers where p is prime, and let $K \subseteq V_{n-1}$. Suppose that $n = \sum_{i \geq 0} n_i p^i$ and $k = \sum_{i \geq 0} k_i p^i$ for each $k \in K$, where $0 \leq n_i < p$ and $0 \leq k_i < p$ for all $i \in \mathbb{N}$. Let $L(n, \alpha) = \{\ell \geq 0 : n_{\ell+1} = n_{\ell+2} = \cdots = n_{\ell+\alpha-1} = 0\}$.*

If there is $\ell \in L(n, \alpha)$ such that, for each $k \in K$, there is $r \leq \ell$ for which $n_r < k_r$ and $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$, then there exists a cyclic p^α-partition of $\mathcal{K}_n^{(K)}$.

In Section 3 we will look at an application of Corollary 1.6 to cyclic p^α-partitions of complete t-partite k-uniform hypergraphs.

2 Proofs of the main results

Given a (q,k)-complementing permutation θ, we can use a simple algorithm described in [6] and [14] to generate all of the cyclic q-partitions of $\mathcal{K}_n^{(k)}$ with q-antimorphism θ. Hence given a (q,K)-complementing permutation θ, we can construct a cyclic q-partition of $\mathcal{K}_n^{(K)}$ with q-antimorphism θ by taking the union of $|K|$ cyclic q-partitions of complete uniform hypergraphs, one of rank k for each $k \in K$, each of which is constructed using the algorithm in [6]. We obtain the following lemma.

Lemma 2.1 *There exists a cyclic q-partition of $\mathcal{K}_n^{(K)}$ if and only if there exists a (q,K)-complementing permutation.* □

Proof of Theorem 1.4.
Note that $\beta = \lfloor \log_p k \rfloor$ if and only if $p^\beta \le k < p^{\beta+1}$. Let σ be a permutation of the set V_n with orbits $O_1, O_2, ..., O_m$.
Suppose first that the permutation σ is such that every orbit of σ has the cardinality divisible by $p^{\alpha+\beta}$ and let $k' \in V_k$. We shall prove that σ is a (p^α, k')-complementing permutation of $\mathcal{K}_n^{(k')}$. For every orbit O of the permutation σ we have $C_p(O) \ge \alpha + \beta$, hence $\sum_{i:C_p(O_i)<\alpha+\beta} |O_i| = 0$. Moreover, since $0 < k' \le k < p^{\beta+1}$, we have $r(k', p^{\beta+1}) = k'$ and σ is (p^α, k')-complementing by Theorem 1.1, and hence (p^α, V_k)-complementing, by Lemma 2.1.
Suppose now that the permutation σ of the set V_n is (p^α, V_k)-complementing. We shall prove that for every $j = 1, 2, ..., m$ the cardinality of O_j is divisible by $p^{\alpha+\beta}$.
By Theorem 1.1, for every $k' \in V_k$ there is a nonnegative integer ℓ such that

$$\sum_{i:C_p(O_i)<\ell+\alpha} |O_i| < r(k', p^{\ell+1}) \tag{1}$$

Applying (1) for $k' = 1$, it is very easy to check that p^α divides $|O_j|$, for every $j = 1, 2, ..., m$.
Suppose that for an integer γ, $1 \le \gamma \le \beta$ we have proved that $p^{(\gamma-1)+\alpha}$ divides the cardinality of every orbit $O_1, O_2, ..., O_m$.
Since the permutation σ is (p^α, p^γ)-complementing, by Theorem 1.1 there is an integer ℓ such that

$$\sum_{i:C_p(O_i)<\ell+\alpha} |O_i| < r(p^\gamma, p^{\ell+1}). \tag{2}$$

Note that $r(p^\gamma, p^{\ell+1}) = \begin{cases} 0 & \text{if} \quad \ell < \gamma \\ p^\gamma & \text{if} \quad \ell \ge \gamma \end{cases}$. Hence $\ell \ge \gamma$.
Moreover, if $\sum_{i:C_p(O_i)<\ell+\alpha} |O_i| < p^\gamma$ for an ℓ such that $\ell \ge \gamma$, then

$$\sum_{i:C_p(O_i)<\gamma+\alpha} |O_i| < p^\gamma.$$

By consequence $\sum_{i:C_p(O_i)<\gamma+\alpha} |O_i| \in \{0, 1, ..., p^\gamma - 1\}$. But since every orbit of the permutation σ has cardinality divisible by $p^{(\gamma-1)+\alpha}$, we have $\sum_{i:C_p(O_i)<\gamma+\alpha} |O_i| = 0$. This means that $p^{\alpha+\gamma}$ divides $|O_j|$ for every $j = 1, 2, ..., m$. The theorem follows. □

Lemma 2.2 ([14]) *Let k, n and q be positive integers, $k < n$. A permutation σ of the set V_n is (q,k)-complementing if and only if $\sigma^s(e) \neq e$ for any subset $e \subset V_n$ of cardinality k and $s \not\equiv 0$ (mod q).*

Lemma 2.2 implies easily the following.

Lemma 2.3 *Let n and q be positive integers and let $K \subset V_{n-1}$, $K \neq \emptyset$. A permutation σ of the set V_n is (q,K)-complementing if and only if $\sigma^s(A) \neq A$ for any subset $A \subset V_n$ such that $|A| \in K$ and $s \not\equiv 0$ (mod q).*

Proof of Theorem 1.7.
By Lemma 2.3, a permutation σ of the set V_n is (q, V_k)-complementing if and only if $\sigma^s(A) = A$ implies $s \equiv 0$ (mod q) for any subset A of V_n such that $|A| \in V_k$. But this implication holds if and only if $\sigma^s(A) = A$ implies $s \equiv 0$ (mod $p_j^{\alpha_j}$) for all $j = 1, 2, ..., s$ and all such sets A, which holds if and only if the permutation σ is $(p_j^{\alpha_j}, V_k)$-complementing for every $j = 1, 2, \dots, s$, by Lemma 2.3. Now Theorem 1.4 guarantees that σ is $(p_j^{\alpha_j}, V_k)$-complementing for every j if and only if the cardinality of every orbit of σ is divisible by $p_j^{\alpha_j+\beta_j}$ for every j, which holds if and only if every orbit of σ has cardinality divisible by $p_1^{\alpha_1+\beta_1} \cdot p_2^{\alpha_2+\beta_2} \cdot \dots \cdot p_s^{\alpha_s+\beta_s}$, as claimed. □

Since permutations θ of V_n map edges of rank k onto edges of rank k, it follows that each cyclic q-partition $\{E, E^\theta, E^{\theta^2}, \dots, E^{\theta^{q-1}}\}$ of $\mathcal{K}_n^{(K)}$ gives rise to a cyclic q-partition $\{E \cap \binom{V_n}{k}, E^\theta \cap \binom{V_n}{k}, E^{\theta^2} \cap \binom{V_n}{k}, \dots, E^{\theta^{q-1}} \cap \binom{V_n}{k}\}$ of $\mathcal{K}_n^{(k)}$ for each $k \in K$. Hence we can view a cyclic q-partition of $\mathcal{K}_n^{(K)}$ as a union of $|K|$ cyclic q-partitions of complete uniform hypergraphs, one of rank k for each $k \in K$, which all share a common q-antimorphism θ. Moreover, the complements of the edges in a cyclic q-partition of $\mathcal{K}_n^{(k)}$ with q-antimorphism θ form a cyclic q-partition of $\mathcal{K}_n^{(n-k)}$ with the same q-antimorphism θ. The following lemma follows easily from these observations.

Lemma 2.4 *Let n and q be positive integers and let $K \subseteq V_{n-1}$.*

1. *A permutation θ is (q,K)-complementing if and only if it is (q,k)-complementing for all $k \in K$.*
2. *There exists a cyclic q-partition of $\mathcal{K}_n^{(M)}$ with q-antimorphism θ if and only if there exists a cyclic q-partition of $\mathcal{K}_n^{(K)}$ with q-antimorphism θ for all nonempty subsets K of M .*
3. *There exists a cyclic q-partition of $\mathcal{K}_n^{(K)}$ if and only if there exists a cyclic q-partition of $\mathcal{K}_n^{(K \cup \hat{K})}$, where $\hat{K} = \{n - k : k \in K\}$.* □

Proof of Theorem 1.8:
Suppose that there is $\ell \in L(n, \alpha)$ such that, for each $k \in K$, there is $r \leq \ell$ for which $n_r < k_r$, and $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$. Then Corollary 1.2 guarantees that $r(n, p^{\ell+\alpha}) < r(k, p^{\ell+1})$ and there exists a cyclic p^α-partition of $\mathcal{K}_n^{(k)}$ for each $k \in K$. By Lemma 2.4(2), we must show that there exists such a set of $|K|$ cyclic p^α-partitions with a common p^α-antimorphism θ.

Now since $r(n, p^{\ell+\alpha}) < r(k, p^{\ell+1})$ for all $k \in K$, it follows that $n = Mp^{\ell+\alpha} + j$ for some positive integer M and some $j < \min_{k \in K}\{r(k, p^{\ell+1})\}$. Let θ be a permutation on V_n whose orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$ consist of j orbits of length 1 and M orbits of length $p^{\ell+\alpha}$. Fix $k \in K$. We will show that θ is a (p^α, k)-complementing permutation for this k.

Now certainly $\sum_{i: C_p(\mathcal{O}_i) < \ell+\alpha} |\mathcal{O}_i| = j < r(k, p^{\ell+1})$. To apply Theorem 1.1 to show that θ is (p^α, k)-complementing, we need to show that this holds for some ℓ^* with $k_{\ell^*} \neq 0$. By assumption, there is

$r \leq \ell$ for which $n_r < k_r$, and $n_i = k_i$ for $r < i \leq \ell$ whenever $r < \ell$. Let ℓ^* be the largest integer in $\{r, r+1, r+2, \ldots, \ell\}$ such that $k_{\ell^*} \neq 0$. Then $r(k, p^{\ell+1}) = r(k, p^{\ell^*+1})$. Also, we must have $n_i = 0$ for $\ell^* < i < \ell + \alpha$. Since $\ell^* + \alpha \leq \ell + \alpha$, it follows that $\sum_{i:C_p(\mathcal{O}_i)<\ell^*+\alpha} |\mathcal{O}_i| = \sum_{i:C_p(\mathcal{O}_i)<\ell+\alpha} |\mathcal{O}_i|$. Thus $k_{\ell^*} \neq 0$ and

$$\sum_{i:C_p(\mathcal{O}_i)<\ell^*+\alpha} |\mathcal{O}_i| = \sum_{i:C_p(\mathcal{O}_i)<\ell+\alpha} |\mathcal{O}_i| < r(k, p^{\ell+1}) = r(k, p^{\ell^*+1}).$$

Now Theorem 1.1 guarantees that θ is a (p^α, k)-complementing permutation. Since k was an arbitrary element of K, Lemma 2.4(1) implies that θ is a (p^α, K)-complementing permutation. Hence Lemma 2.1 guarantees that there exists a cyclic p^α-partition of $\mathcal{K}_n^{(K)}$, as claimed. □

Proof of Corollary 1.6:
If $p^{\alpha+\beta}$ divides n, then using the notation of Theorem 1.8 we have $\beta \in L(n, \alpha)$. Let k be an integer such that $p^\beta \leq k < p^{\beta+1}$. For each $m \in V_k$ and each integer $i \geq 0$, let m_i denote the coefficient of p^i in the base-p representation of m. Let $r = r(m) = \max\{i : m_i > 0\}$. Then $m_{r(m)} > n_{r(m)} = 0$ and $m_i = n_i = 0$ for $i = r(m)+1, r(m)+2, \ldots, \beta$. Hence Theorem 1.8 guarantees the existence of a cyclic p^α-partition of $\mathcal{K}_n^{(V_k)}$. Now Lemma 2.4(3) implies that there exists a cyclic p^α-partition of $\mathcal{K}_n^{(M)}$ for $M = \{m : 1 \leq m \leq k \text{ or } n-k \leq m \leq n-1\}$, and Lemma 2.4(2) implies that there exists a cyclic p^α-partition of $\mathcal{K}_n^{(K)}$ for every nonempty subset K of M, as claimed. □

3 Decomposing complete multipartite uniform hypergraphs

Let $X = (V, E)$ be a hypergraph. A subset $S \subseteq V$ is *stable* in X if S contains no edge of X. The hypergraph X is called *t-partite* if V can be partitioned into t stable and mutually disjoint subsets $A_1, A_2, ..., A_t$. Hence $V = A_1 \cup A_2 \cup ... \cup A_t$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $e \not\subset A_i$ for $i = 1, 2, ..., t$ and any edge $e \in E$. The *complete t-partite k-hypergraph* with vertex set $V = A_1 \cup A_2, ..., A_t$ has edge set $E_k = \{e : e \subset V, |e| = k, e \not\subset A_i \text{ for } i = 1, 2, ..., t\}$, and is denoted by $\mathcal{K}^{(k)}(A_1, A_2, ..., A_t)$ or $\mathcal{K}^{(k)}_{n_1, n_2, ..., n_t}$ when $|A_i| = n_i$ for $i = 1, 2, ..., t$. Let θ be a permutation of the set $A_1 \cup A_2 \cup ... \cup A_t$ such that $A_i^\theta = A_i$ for $i = 1, 2, ..., t$. We say that θ is a *(q, k)-complementing permutation of* $\mathcal{K}^{(k)}(A_1, A_2, ..., A_t)$ if there is a subset E of the set of edges of $\mathcal{K}^{(k)}(A_1, A_2, ..., A_t)$ such that $E \cup E^\theta \cup E^{\theta^2} \cup ... \cup E^{\theta^{q-1}}$ is a partition of $E(\mathcal{K}^{(k)}(A_1, A_2, ..., A_t))$. Such a q-partition of $E(\mathcal{K}^{(k)}(A_1, A_2, ..., A_t)$ is then called a *cyclic q-partition of* $\mathcal{K}^{(k)}(A_1, A_2, ..., A_t)$.

For a nonempty set K of positive integers, let $\mathcal{K}^{(K)}(A_1, A_2, ..., A_t)$ denote the hypergraph with the vertex set $V = A_1 \cup A_2 \cup ... \cup A_t$ and the edge set $\bigcup_{k \in K} E_k$.
A permutation θ of $A_1 \cup A_2 \cup ... \cup A_t$ is said to be a (q, K)-*complementing permutation* of $\mathcal{K}^{(K)}(A_1, A_2, ..., A_t)$ if it is a (q, k)-complementing permutation of $\mathcal{K}^{(k)}(A_1, A_2, ..., A_t)$ for every $k \in K$.

For a permutation θ of V and a subset A of V, let $\theta_{|A}$ denote the restriction of the function θ to the set A. That is $\theta_{|A}(x) = \theta(x)$, for all $x \in A$. If A is an invariant set of θ, then $\theta_{|A}$ is a permutation of A.

Theorem 3.1 *Let k, p, t and α be positive integers such that p is prime. Let $V = A_1 \cup A_2 \cup ... \cup A_t$ where $A_i \cap A_j = \emptyset$ for $i \neq j$ and let θ be a permutation of the set V such that ${A_i}^\theta = A_i$ for every $i = 1, 2, ..., t$. Each of the following two conditions is sufficient for θ to be a (p^α, V_k)-complementing permutation of $\mathcal{K}^{(V_k)}(A_1, A_2, ..., A_t)$.*

1. *For all but at most one* $i \in \{1, 2, ..., t\}$*, the cardinality of all the orbits of* $\theta_{|A_i}$ *are divisible by* $p^{\alpha+\beta}$*, where* $\beta = \lceil \log_p k \rceil - 1$.

2. *For every* $i \in \{1, 2, ..., t\}$*, the cardinalities of all orbits of* $\theta_{|A_i}$ *are divisible by* $p^{\alpha+\gamma}$*, where* $\gamma = \lfloor \log_p k/2 \rfloor$.

Proof: Let B be a subset of V such that $|B| = k$ and there is no i such that $B \subset A_i$. It suffices to show that if $B^{\theta^s} = B$ then $s \equiv 0 \pmod{p^\alpha}$.

(1) Since B has non empty intersection with at least two sets, there exists an index i_0 such that $C = B \cap A_{i_0} \neq \emptyset$ and the cardinality of any orbit of $\theta_{|A_{i_0}}$ is divisible by $p^{\alpha+\beta}$. Note that $C^{\theta^s} = C^{\theta^s_{|A_{i_0}}} = C$. Write $k' = |C|$. We have clearly $0 < k' < k$, hence $k' \in V_{k-1}$. Since $\beta = \lceil \log_p k \rceil - 1$, we have $p^\beta \leq k - 1 < p^{\beta+1}$, and so Theorem 1.4 implies that $\theta_{A_{|i_0}}$ is a (p^α, V_{k-1})-complementing permutation of $\mathcal{K}^{V_k}_{|A_{i_0}|}$. Hence $s \equiv 0 \pmod{p^\alpha}$, and the sufficiency of condition (1) is proved.

(2) The proof of sufficiency of the condition (2) is similar to the proof given above. It suffices to observe that there is an index i_1 such that $C' = B \cap A_{i_1} \neq \emptyset$ and $|C'| \leq k/2$. □

Corollary 3.2 *Let* $n_1, n_2, \ldots, n_t, k, p$ *and* α *be positive integers such that* p *is prime. If at least one of the following two conditions is verified then there is a* (p^α, k)*-complementing permutation of* $\mathcal{K}^{(k)}_{n_1,n_2,\ldots,n_t}$.

1. *For all but at most one* $i \in \{1, 2, ..., t\}$, $p^{\alpha+\beta} | n_i$ *where* $\beta = \lceil \log_p k \rceil - 1$.

2. *For every* $i \in \{1, 2, ..., t\}$, $p^{\alpha+\gamma} | n_i$ *where* $\gamma = \lfloor \log_p k/2 \rfloor$. □

References

[1] L. Adamus, B. Orchel, A. Szymański, P. Wojda and M. Zwonek, A note on t-complementing permutations for graphs. *Inf. Process. Lett.* **110** 44-45 (2009).

[2] J.M. Bernaldez, On k-complementing permutations of cyclically k-complementary graphs, *Discrete Math.* **151**, 67-70 (1996).

[3] M.J. Colbourn and C.J. Colbourn, Graph isomorphism and self-complementary graphs, *SIGACT News* **10** (1), 25-29 (1978).

[4] A. Farrugia, *Self-complementary graphs and generalizations: a comprehensive reference manual.* Master's Thesis, University of Malta, 1999.

[5] S. Gosselin, Self-complementary non-uniform hypergraphs. *Graphs Combin.* **12**, 615-635 (2012).

[6] S. Gosselin, Cyclically t-complementary uniform hypergraphs. *European J. Combin.* **31**, 1629-1636 (2010).

[7] M. Knor and P. Potočnik, A note on 2-subset-regular self-complementary 3-uniform hypergraphs. To appear in *Ars Combin.*

[8] M. Muzychuk, On Sylow subgraphs of vertex-transitive self-complementary graphs, *Bull. London Math. Soc.* **31**, 531-533 (1999).

[9] P. Potočnik and M. Šajna, The existence of regular self-complementary 3-uniform hypergraphs, *Discrete Math.* **309**, 950-954 (2009).

[10] P. Potočnik and M. Šajna, Vertex-transitive self-complementary uniform hypergraphs. *European J. Combin.* **30**, 327-337 (2009).

[11] S.B. Rao, On regular and strongly-regular self-complementary graphs, *Discrete Math.* **54**, 73-82 (1985).

[12] G. Ringel, Selbstkomplementäre Graphen, *Arch. Math. (Basel)* **14**, 354-358 (1963).

[13] H. Sachs, Über Selbstkomplementäre Graphen, *Publ. Math. Debrecen* **9**, 270-288 (1962).

[14] A. Szymański and A.P. Wojda, Cyclic partitions of complete uniform hypergraphs, *Electron. J. Comb.* **17**, 1-12 (2010).