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Bipartite Powers of k-chordal Graphs

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Let k be an integer and $k \geq 3$. A graph G is *k-chordal* if G does not have an induced cycle of length greater than k. From the definition it is clear that 3-chordal graphs are precisely the class of chordal graphs. Duchet proved that, for every positive integer m, if G^m is chordal then so is G^{m+2}. Brandstädt et al. in [Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of HHD-free graphs. *Discrete Mathematics*, 177(1-3):9-16, 1997.] showed that if G^m is k-chordal, then so is G^{m+2}.

Powering a bipartite graph does not preserve its bipartitedness. In order to preserve the bipartitedness of a bipartite graph while powering Chandran et al. introduced the notion of *bipartite powering*. This notion was introduced to aid their study of boxicity of chordal bipartite graphs. The m-th bipartite power $G^{[m]}$ of a bipartite graph G is the bipartite graph obtained from G by adding edges (u, v) where $d_G(u, v)$ is odd and less than or equal to m. Note that $G^{[m]} = G^{[m+1]}$ for each odd m.

In this paper we show that, given a bipartite graph G, if G is k-chordal then so is $G^{[m]}$, where k, m are positive integers with $k \geq 4$.

Keywords: k-chordal graph, hole, chordality, graph power, bipartite power.

1 Introduction

A *hole* is a chordless (or an induced) cycle in a graph. The *chordality* of a graph G, denoted by $\mathcal{C}(G)$, is defined to be the size of a largest hole in G, if there exists a cycle in G. If G is acyclic, then its chordality is taken as 0. A graph G is *k-chordal* if $\mathcal{C}(G) \leq k$. In other words, a graph is k-chordal if it has no holes with more than k vertices in it. Chordal graphs are exactly the class of 3-chordal graphs and chordal bipartite graphs are bipartite, 4-chordal graphs. k-chordal graphs have been studied in the literature in [2], [5], [6], [8], [9] and [16]. For example, Chandran and Ram [5] proved that the number of minimum cuts in a k-chordal graph is at most $\frac{(k+1)n}{2} - k$. Spinrad[16] showed that $(k-1)$-chordal graphs can be recognized in $O(n^{k-3}M)$ time, where M is the time required to multiply two n by n matrices.

Powering and its effects on the chordality of a graph has been a topic of interest. The m-th power of a graph G, denoted by G^m, is a graph with vertex set $V(G^m) = V(G)$ and edge set $E(G^m) = \{(u, v) \mid u \neq v, d_G(u, v) \leq m\}$, where $d_G(u, v)$ represents the distance between u and v in G. Balakrishnan and

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Paulraja [1] proved that odd powers of chordal graphs are chordal. Chang and Nemhauser [7] showed that if G and G^2 are chordal then so are all powers of G. Duchet [10] proved a stronger result which says that if G^m is chordal then so is G^{m+2}. Brandstädt et al. in [3] showed that if G^m is k-chordal then so is G^{m+2}, where $k \geq 3$ is an integer. Studies on families of graphs that are closed under powering can also be seen in the literature. For instance, it is known that interval graphs, proper interval graphs [14], strongly chordal graphs [13], circular-arc graphs [15][12], cocomparability graphs [11] etc. are closed under taking powers.

Subclasses of bipartite graphs, like chordal bipartite graphs, are not closed under powering since the m-th power of a bipartite graph need not be even bipartite. Chandran et al. in [4] introduced the notion of *bipartite powering* to retain the bipartitedness of a bipartite graph while taking power. The *m-th bipartite power* $G^{[m]}$ of a bipartite graph G is the bipartite graph obtained from G by adding edges (u, v) where $d_G(u, v)$ is odd and less than or equal to m. Note that $G^{[m]} = G^{[m+1]}$ for each odd m. It was shown in [4] that the m-th bipartite power of a tree is chordal bipartite. The intention there was to construct chordal bipartite graphs of high boxicity. The fact that the chordal bipartite graph under consideration was obtained as a bipartite power of a tree was crucial for proving that its boxicity was high. Since trees are a subclass of chordal bipartite graphs, a natural question that came up was the following: is it true that the m-th bipartite power of every chordal bipartite graph is chordal bipartite? In this paper we answer this question in the affirmative. In fact, we prove a more general result.

Our Result

Let m, k be positive integers with $k \geq 4$. Let G be a bipartite graph. If G is k-chordal, then so is $G^{[m]}$. Note that the special case when $k = 4$ gives us the following result: chordal bipartite graphs are closed under bipartite powering.

2 Graph Preliminaries

Throughout this paper we consider only finite, simple, undirected graphs. For a graph G, we use $V(G)$ to denote the set of vertices of G. Let $E(G)$ denote its edge set. For every $x, y \in V(G)$, $d_G(x, y)$ represents the distance between x and y in G. For every $u \in V(G)$, $N_G(u)$ denotes its *open neighborhood* in G, i.e. $N_G(u) = \{v \mid (u, v) \in E(G)\}$. A path P on the vertex set $V(P) = \{v_1, v_2, \dots, v_n\}$ (where $n \geq 2$) has its edge set $E(P) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$. Such a path is denoted by $v_1 v_2 \dots v_n$. If $v_i, v_j \in V(P)$, $v_i P v_j$ is the path $v_i v_{i+1} \dots v_j$. The length of a path P is the number of edges in it and is denoted by $||P||$. A cycle C with vertex set $V(C) = \{v_1, v_2, \dots, v_n\}$, and edge set $E(C) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$ is denoted as $C = v_1 v_2 \dots v_n v_1$. We use $||C||$ to denote the length of cycle C.

3 Holes in Bipartite Powers

Let H be a bipartite graph. Let $\mathcal{B}(H)$ be a family of graphs constructed from H in the following manner: $H' \in \mathcal{B}(H)$ if corresponding to each vertex $v \in V(H)$ there exists a nonempty bag of vertices, say B_v, in H' such that (a) for every $x \in B_u$, $y \in B_v$, $(x, y) \in E(H')$ if and only if $(u, v) \in E(H)$, and (b) vertices within each bag in H' are pairwise non-adjacent. Below we list a few observations about H and every H' (, where $H' \in \mathcal{B}(H)$):

Observation 1. *H' is bipartite.*

Observation 2. *H is an induced subgraph of H'.*

Observation 3. *Let k be an integer such that $k \geq 4$. If H is k-chordal, then so is H'.*

Proof: Any hole of size greater than 4 in H' cannot have more than one vertex from the same bag, say B_v, as such vertices have the same neighborhood. Hence, the vertices of a hole (of size greater than 4) in H' belong to different bags and thus there is a corresponding hole of the same size in H. □

Theorem 4. *Let m, k be positive integers with $k \geq 4$. Let G be a bipartite graph. If G is k-chordal, then so is $G^{[m]}$.*

Proof: We prove this by contradiction. Let p denote the size of a largest induced cycle, say $C = u_0u_1 \ldots u_{p-1}u_0$, in $G^{[m]}$. Assume $p > k$. Then, $p \geq 6$ (since $k \geq 4$ and $G^{[m]}$ is bipartite). Between each u_{i-1} and u_i, where $i \in \{0, \ldots, p-1\}$, there exists a shortest path of length not more than m in G (i). Let P_i be one such shortest path between u_{i-1} and u_i in G.

Let H be the subgraph induced on the vertex set $\bigcup_{i=0}^{p-1} V(P_i)$ in G. As mentioned in the beginning of this section, construct a graph H' from H, where $H' \in \mathcal{B}(H)$, in the following manner: for each $v \in V(H)$, let $|B_v| = |\{P_i \mid 0 \leq i \leq p-1, v \in V(P_i)\}|$ i.e., let B_v have as many vertices as the number of paths in $\{P_0 \ldots P_{p-1}\}$ that share vertex v in H. For each $i \in \{0, \ldots, p-1\}$, let $Q'_i = u_{i-1}Q_i$ be a shortest path between u_{i-1} and u_i in H' such that no two paths Q_i and Q_j (where $i \neq j$) share a vertex (i). From our construction of H' from H it is easy to see that such paths exist. Let $Q_i = v_{i,1}v_{i,2} \ldots v_{i,r_i}u_i$, where $r_i = ||Q_i|| \geq 0$. Thus, $Q'_i = u_{i-1}v_{i,1}v_{i,2} \ldots v_{i,r_i}u_i$. Clearly, $||Q'_i|| = ||P_i|| \leq m$. The reader may also note that the cycle C ($= u_0u_1 \ldots u_{p-1}u_0$) which is present in $G^{[m]}$ will be present in $H^{[m]}$ and thereby in $H'^{[m]}$ too.

In order to prove the theorem, it is enough to show that there exists an induced cycle of size at least p in H'. Then by combining Observation 3 and the fact that H is an induced subgraph of G, we get $k \geq \mathcal{C}(G) \geq \mathcal{C}(H) \geq \mathcal{C}(H') \geq p$ contradicting our assumption that $p > k$. Hence, in the rest of the proof we show that $\mathcal{C}(H') \geq p$.

Consider the following drawing of the graph H'. Arrange the vertices $u_0, u_1, \ldots, u_{p-1}$ in that order on a circle in clockwise order. Between each u_{i-1} and u_i on the circle arrange the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}$ in that order in clockwise order. Recall that these vertices are the internal vertices of path Q'_i.

Claim 4.1. In this circular arrangement of vertices of H', each vertex has an edge (in H') with both its left neighbor and right neighbor in the arrangement.

Let $x_1, x_2 \in V(H')$, where $x_1 \in V(Q_i)$, $x_2 \in V(Q_j)$. We define the *clockwise distance from x_1 to x_2*, denoted by $clock\_dist(x_1, x_2)$, as the minimum non-negative integer s such that $j = i + s$. Similarly, the *clockwise distance from x_2 to x_1*, denoted by $clock\_dist(x_2, x_1)$, is the minimum non-negative integer s' such that $i = j + s'$. Let $x, y, z \in V(H')$. We say $y <_x z$ if scanning the vertices of H' in clockwise direction along the circle starting from x, vertex y is encountered before z. Let $x \in V(Q_i)$. Vertex y is called the *farthest neighbor of x before z* if $y \in N_{H'}(x)$, $y \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$, $y <_x z$, and for every other $w \in N_{H'}(x)$ either $z <_x w$ or $w \notin V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ or both.

Claim 4.2. There always exists a vertex which is the farthest neighbor of x before z, unless $(x, z) \in E(H')$ and $z \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$.

(i) throughout this proof expressions involving subscripts of u, P, Q, and Q' are to be taken modulo p. Every such expression should be evaluated to a value in $\{0, \ldots, p-1\}$. For example, consider a vertex u_i, where $i < p$ Then, $p + i = i$.

Let $\{A, B\}$ be the bipartition of the bipartite graph H'. We categorize the edges of H' as follows: an edge $(x, y) \in E(H')$ is called an *l-edge*, if $l = \min(clock\_dist(x, y), clock\_dist(y, x))$.

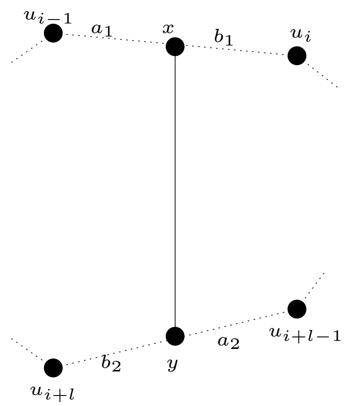

Fig. 1: $x \in V(Q_i), y \in V(Q_{i+l})$ and let $(x, y) \in E(H')$ be an l-edge, where $l > 2$. The dotted line between u_{i-1} and u_i indicate the path Q_i. Similarly, the dotted line between u_{i+l-1} and u_{i+l} indicate the path Q_{i+l}.

Claim 4.3. H' cannot have an l-edge, where $l > 2$.

Proof: Suppose H' has an l-edge, where $l > 2$, between $x \in Q_i$ and $y \in Q_{i+l}$ (see Fig. 1). Let $a_1 = ||u_{i-1}Q'_i x||$, $b_1 = ||xQ'_i u_i||$, $a_2 = ||u_{i+l-1}Q'_{i+l}y||$ and $b_2 = ||yQ'_{i+l}u_{i+l}||$. We consider the following two cases:
Case 1*:* l is even
In this case u_{i-1} and u_{i+l-1} will be on the same side of the bipartite graph H'. Without loss of generality, let $u_{i-1}, u_{i+l-1} \in A$. Then, $u_i, u_{i+l} \in B$. We know that, for every $w_1, w_2 \in V(H'^{[m]})$ with $w_1 \in A$ and $w_2 \in B$, if $(w_1, w_2) \notin E(H'^{[m]})$ then $d_{H'}(w_1, w_2) \geq m + 2$ (recalling m and $d_{H'}(w_1, w_2)$ are odd integers). Therefore, we have $a_1 + 1 + b_2 \geq d_{H'}(u_{i-1}, u_{i+l}) \geq m + 2$. Similarly, $b_1 + 1 + a_2 \geq d_{H'}(u_i, u_{i+l-1}) \geq m + 2$. Summing up the two inequalities we get, $(a_1 + b_1) + (a_2 + b_2) \geq 2m + 2$. This implies that either $||Q'_i||$ or $||Q'_{i+l}||$ is greater than m which is a contradiction.
Case 2*:* l is odd
(proof is similar to the above case and hence omitted).

Hence we prove the claim. □

We find a cycle $C' = z_0 z_1 \dots z_q z_0$ in H' using Algorithm 3.1 [(i)]. Please read the algorithm before proceeding further. .

[(i)] throughout this proof expressions involving subscripts of z are to be taken modulo $q + 1$. Every such expression should be evaluated to a value in $\{0, \dots, q\}$. For example, consider a vertex z_a, where $a < q + 1$. Then, $q + 1 + a = a$.

Algorithm 3.1 Finding Cycle C' in H' such that $||C'|| \geq ||C||$

1. $l \leftarrow \max_{l'}(H'$has an l'-edge$)$. Without loss of generality assume that this l-edge is between a vertex in Q_0 and a vertex in Q_l
2. Scan the vertices of Q_0 in clockwise direction to find the first vertex z_0, where $z_0 \in V(Q_0)$, which has an l-edge to a vertex in Q_l.
3. Scan the vertices of Q_l in clockwise direction to find the last vertex in Q_l which is a neighbor of z_0 in H'. Call it z_1.
4. Find the farthest neighbor of z_1 before z_0. Call it z_2. /* refer proof of Claim 4.4 for a proof of existence of such a z_2*/
5. $s \leftarrow 2$.

while $(z_s, z_0) \notin E(H')$ **do**

  6. Find the farthest neighbor of z_s before z_0. Call it z_{s+1}. /* such a neighbor exists by Claim 4.2*/
  7. $s \leftarrow s + 1$.

end while

8. $q \leftarrow s$.
9. Return cycle $C' = z_0 z_1 \ldots z_q z_0$.

Claim 4.4. There always exists a farthest neighbor of z_1 before z_0.

Proof: Note that $z_0 \in Q_0$ and $z_1 \in Q_l$, where $l \leq 2$ (by Claim 4.3). Recalling that $||C|| = p \geq 6$, we have $z_0 \notin V(Q_l) \cup V(Q_{l+1}) \cup V(Q_{l+2})$. Hence by Claim 4.2, the claim is true. □

Claim 4.5. The while loop in Algorithm 3.1 terminates after a finite number of iterations.

Proof: From Claim 4.1, we know that each vertex has an edge (in H') with both its left neighbor and right neighbor in the circular arrangement. Each time when Step 6 of Algorithm 3.1 is executed, a vertex z_{s+1} is chosen such that z_{s+1} is the farthest neighbor of z_s before z_0. Since H' is a finite graph, there will be a point of time in the execution of the algorithm when in Step 6 it picks a z_{s+1} such that $(z_{s+1}, z_0) \in E(H')$. □

From Claim 4.5, we can infer that C' is a cycle.

Claim 4.6. C' is an induced cycle in H'.

Proof: Suppose C' is not an induced cycle. Then there exists a chord (z_a, z_b) in C'. Since (z_a, z_b) is a chord, we have $b \neq a - 1$ or $b \neq a + 1$. Let $l = \max_{l'}(H'$ has an l'-edge$)$. Let $z_a \in V(Q_i)$, $z_b \in V(Q_j)$. We know that $\min(clock\_dist(z_a, z_b), clock\_dist(z_b, z_a)) \leq l$. Without loss of generality, assume $clock\_dist(z_a, z_b) \leq l \leq 2$ (from Claim 4.3). That is, $j - i \leq l \leq 2$ and (z_a, z_b) is a $(j-i)$-edge. If $z_a = z_0$, then $z_b \neq z_1$ and the algorithm exits from the while loop, when $q = b$, thus returning a cycle $z_0 \ldots z_b z_0$. But in such a cycle (z_b, z_0) is not a chord. Therefore, $z_a \neq z_0$. Similarly, $z_b \neq z_0$. We know that $z_{a+1} \neq z_b$, $z_{a+1} <_{z_a} z_b$, and $z_{a+1} \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$. Since $j - i \leq 2$, $z_b \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$. If $z_b <_{z_a} z_0$, then it contradicts the fact that z_{a+1} is the farthest neighbor of z_a before z_0. Therefore, $z_0 <_{z_a} z_b$. Then, either $z_b = z_1$ or $z_1 <_{z_a} z_b$. Recall that $l = \max_{l'}(H'$has an l'-edge$)$, and (z_0, z_1) is an l-edge with $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$. Since (i) (z_a, z_b) is a $(j-i)$-edge, where $j - i \leq l$, (ii) $z_0 <_{z_a} z_b$, and (iii) $z_b = z_1$ or $z_1 <_{z_a} z_b$, we have $l \geq j - i = clock\_dist(z_a, z_b) \geq clock\_dist(z_0, z_b) \geq clock\_dist(z_0, z_1) = l$. Hence, $j - i = l$ and

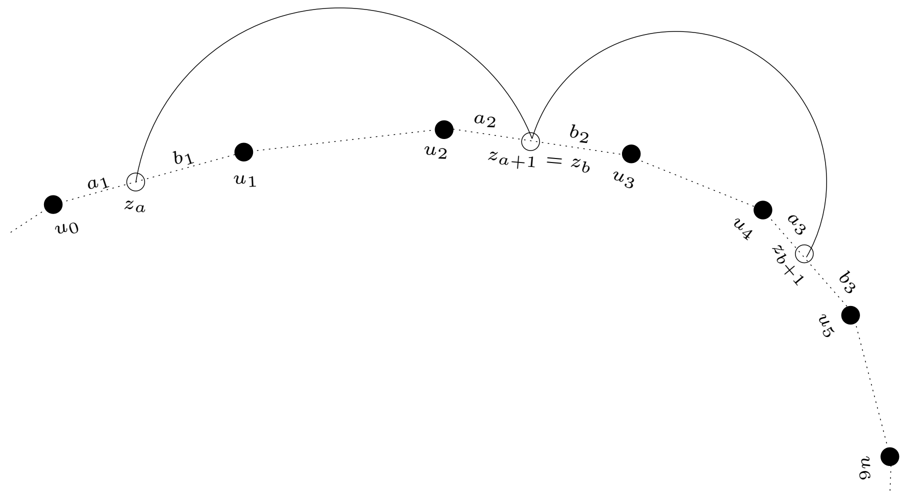

Fig. 2: Figure illustrates the case when path P defined in Claim 4.8 is a trivial path. The dotted lines between each u_{i-1} and u_i indicate the path Q'_i. Each continuous arc corresponds to an edge in the cycle $C' = z_0 \dots z_q z_0$.

(z_a, z_b) is an l-edge. We know that (z_0, z_1) is also an l-edge with $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$. Since $z_0 <_{z_a} z_b$ and $z_b = z_1$ or $z_1 <_{z_0} z_b$, we get $z_a \in V(Q_0)$ and $z_b \in V(Q_l)$. From Step 2 of the algorithm we know that z_0 is the first vertex (in a clockwise scan) in Q_0 which has an l-edge to a vertex in Q_l. This implies that, since $z_0 <_{z_a} z_b$, $z_a = z_0$ which is a contradiction. Hence we prove the claim. $\square$

What is left now is to show that $q + 1 \geq p$, i.e., $||C'|| \geq ||C||$, where $C' = z_0 \dots z_q z_0$ and $C = u_0 \dots u_{p-1} u_0$. In order to show this, we state and prove the following claims.

Claim 4.7. For every $j \in \{0, \dots, p-1\}$, $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') \neq \emptyset$.

Proof: Suppose the claim is not true. Find the minimum j that violates the claim. Clearly, $j \neq 0$ as $z_0 \in V(Q_0)$. We claim that $z_q \in V(Q_{j-1})$. Suppose $z_q \notin V(Q_{j-1})$. Let $a = \max\{i \mid z_i \in V(Q_{j-1})\}$ (note that, since $j \neq 0$, by the minimality of j, $(V(Q_{j-1}) \cup V(Q_j)) \cap V(C') \neq \emptyset$ and therefore $V(Q_{j-1}) \cap V(C') \neq \emptyset$). Since $z_a \neq z_q$, by the maximality of a, we have $z_{a+1} \notin V(Q_{j-1})$. From our assumption, $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') = \emptyset$ and therefore $z_{a+1} \notin V(Q_{j-1}) \cup V(Q_j) \cup V(Q_{j+1})$. Thus $z_a \neq z_q$ and z_{a+1} is not the farthest neighbor of z_a before z_0. This is a contradiction to the way z_{a+1} is chosen by Algorithm 3.1. Hence, $z_q \in V(Q_{j-1})$. We know that $(z_q, z_0) \in E(H')$ with $z_q \in V(Q_{j-1})$ and $z_0 \in V(Q_0)$. Since $l = \max_{l'}(H'$ has an l'-edge), we have $\min(clock\_dist(z_q, z_0), clock\_dist(z_0, z_q)) \leq l$. That is, $j \geq p + 1 - l$ or $j \leq 1 + l$. As $l \leq 2$ (by Claim 4.3), we have $j = p - 1$ or $j \leq 1 + l$. Since $z_0 \in V(Q_0)$, $(V(Q_{p-1}) \cup V(Q_0)) \cap V(C') \neq \emptyset$ and hence $j \neq p - 1$. Therefore, $j \leq 1 + l$. Since $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$ (recall $l \leq 2$), we get $j = 1 + l$. We know that, for every $z_a, z_b \in V(C')$, if $a < b$ then $z_a <_{z_0} z_b$. Therefore, $z_1 <_{z_0} z_q$. We have $z_1 \in V(Q_l)$. Since $j = 1 + l$, we also have $z_q \in V(Q_l)$. Thus, we have $z_1, z_q \in V(Q_l)$ and $z_1 <_{z_0} z_q$. But this contradicts the fact that z_1 is the last vertex in Q_l encountered in a clockwise scan that has z_0 as its neighbor. $\square$

Claim 4.8. Let $(z_a, z_{a+1}), (z_b, z_{b+1}) \in E(C')$ be two 2-edges, where $a < b$. Let P, P' denote the clockwise $z_{a+1} - z_b$, $z_{b+1} - z_a$ paths respectively in C'. Both P and P' contain at least one 0-edge.

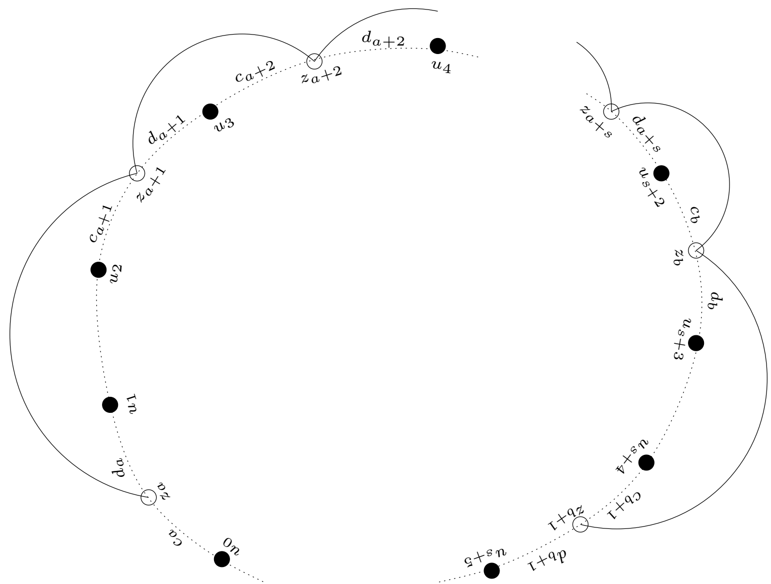

Fig. 3: Figure illustrates the case when path P defined in Claim 4.8 is $P = z_{a+1}z_{a+2}\dots z_{a+1+s}$, where $s \geq 1$ and $z_{a+1+s} = z_b$. The dotted lines between each u_{i-1} and u_i indicate the path Q'_i. Each continuous arc corresponds to an edge in the cycle $C' = z_0 \dots z_q z_0$.

Proof: Consider the path P (proof is similar in the case of path P'). Path P is a non-trivial path only if $z_{a+1} \neq z_b$. Suppose $z_{a+1} = z_b$ (see Fig. 2). Let $z_a \in V(Q_f)$. For the sake of ease of notation, assume $f = 1$ (the same proof works for any value of f). Let $a_1 = ||u_0Q'_1z_a||$, $b_1 = ||z_aQ'_1u_1||$, $a_2 = ||u_2Q'_3z_b||$, $b_2 = ||z_bQ'_3u_3||$, $a_3 = ||u_4Q'_5z_{b+1}||$, and $b_3 = ||z_{b+1}Q'_5u_5||$. We know that, for every $w_1, w_2 \in V(H'^{[m]})$ with $w_1 \in A$ and $w_2 \in B$, if $(w_1, w_2) \notin E(H'^{[m]})$ then $d_{H'}(w_1, w_2) \geq m + 2$. Since $(u_0, u_3) \notin E(H'^{[m]})$, $(u_1, u_4) \notin E(H'^{[m]})$ and $(u_2, u_5) \notin E(H'^{[m]})$, we have $a_1 + b_2 \geq m + 1$, $b_1 + a_3 \geq m$, and $a_2 + b_3 \geq m + 1$. Adding the three inequalities and by applying an easy averaging argument we can infer that either $a_1 + b_1 = ||Q_1|| > m$, $a_2 + b_2 = ||Q_3|| > m$, or $a_3 + b_3 = ||Q_5|| > m$ which is a contradiction. Therefore P is a non-trivial path i.e., $z_{a+1} \neq z_b$. Assume P does not contain any 0-edge. Let $P = z_{a+1}z_{a+2}\dots z_{a+1+s}$, where $s \geq 1$, $a + 1 + s = b$, and $(z_{a+1}, z_{a+2})\dots(z_{a+s}, z_{a+1+s})$ are 1-edges (see Fig. 3). Since $(u_0, u_3) \notin E(H'^{[m]})$, $(u_1, u_4) \notin E(H'^{[m]})$, we have $c_a + d_{a+1} \geq m + 1$ and $d_a + d_{a+2} \geq m$ (please refer Fig. 3 for knowing what $c_a, d_a, \dots, c_{b+1}, d_{b+1}$ are). Summing up the two inequalities, we get $d_{a+1} + d_{a+2} \geq 2m + 1 - (c_a + d_a)$. We know that, for each $i \in \{0, \dots p - 1\}$, $||Q'_i|| \leq m$. Therefore, we have $c_a + d_a \leq m$. Hence, $d_{a+1} + d_{a+2} \geq m + 1$. Since $(c_{a+1} + d_{a+1}) + (c_{a+2} + d_{a+2}) \leq 2m$, we get

$$c_{a+1} + c_{a+2} \leq m - 1 \tag{1}$$

Since $(u_{s+2}, u_{s+5}) \notin E(H'^{[m]})$, $(u_{s+1}, u_{s+4}) \notin E(H'^{[m]})$, we have,

$$\begin{aligned} c_b + d_{b+1} &\geq m + 1 \\ c_{a+s} + c_{b+1} &\geq m \end{aligned}$$

Summing up the two inequalities, we get

$$c_b + c_{a+s} \geq 2m + 1 - (c_{b+1} + d_{b+1})$$

Since $b = a + s + 1$ and $c_{b+1} + d_{b+1} \leq m$, we get

$$c_{a+s+1} + c_{a+s} \geq m + 1 \tag{2}$$

Substituting for $s = 1$ in Inequality 2, we get $c_{a+2} + c_{a+1} \geq m + 1$. But this contradicts Inequality 1. Hence $s > 1$. Suppose $s = 2$. Since $(u_2, u_5) \notin E(H'^{[m]}))$, we have $c_{a+1} + d_{a+3} \geq m$. Adding this with Inequality 2, we get $c_{a+1} + c_{a+2} \geq (2m + 1) - (c_{a+3} + d_{a+3}) \geq m + 1$. But this contradicts Inequality 1. Hence $s > 2$. Since $(u_s, u_{s+3}) \notin E(H'^{[m]})), \ldots, (u_2, u_5) \notin E(H'^{[m]}))$, we have the following inequalities:-

$$\begin{aligned} c_{a+s-1} + d_{a+s+1} &\geq m \\ \vdots \quad &\vdots \quad \vdots \\ c_{a+1} + d_{a+3} &\geq m \end{aligned}$$

Adding the above set of inequalities and applying the fact that $c_i + d_i \leq m$, $\forall i \in \{0, \ldots q\}$, we get $c_{a+1} + c_{a+2} + d_{a+s} + d_{a+s+1} \geq 2m$. Adding this with Inequality 2, we get $c_{a+1} + c_{a+2} \geq (3m+1) - (c_{a+s+1} + d_{a+s+1}) - (c_{a+s} + d_{a+s}) \geq m + 1$. But this contradicts Inequality 1. Hence we prove the claim. □

Claim 4.9. For every $j, j' \in \{0, \ldots, p-1\}$, where $j < j'$ and $(V(Q_j) \cup V(Q_{j'})) \cap V(C') = \emptyset$, there exist $i, i' \in \{0, \ldots, p-1\}$, where only i satisfies $j < i < j'$, such that $|V(Q_i) \cap V(C')| \geq 2$ and $|V(Q_{i'}) \cap V(C')| \geq 2$.

Proof: By Claim 4.7, (i) $j' \neq j + 1$ or $j' \neq j - 1$, and (ii) there exist $r, r' \in \{0, \ldots, q\}$ such that (z_r, z_{r+1}) is a 2-edge with its endpoints on Q_{j-1} and Q_{j+1} and $(z_{r'}, z_{r'+1})$ is a 2-edge with its endpoints on $Q_{j'-1}$ and $Q_{j'+1}$. By Claim 4.8, we know that if P, P' denote the clockwise $z_{r+1} - z_{r'}$, $z_{r'+1} - z_r$ paths respectively in C', then both P and P' contains at least one 0-edge. This proves the claim. □

In order to show that the size of cycle C' $(= z_0 \ldots z_q z_0)$ is at least p, we consider the following three cases:-

Case $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 0$: In this case, for every $j \in \{0, \ldots p-1\}$, Q_j contributes to $V(C')$ and therefore $||C'|| \geq p = ||C||$.

Case $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 1$: Let Q_j be that only path (among $Q_0 \ldots Q_{p-1}$) that does not contribute to $V(C')$. Then we claim that there exists a $Q_{j'}$, where $j' \neq j$, such that $V(C') \cap V(Q_{j'}) \geq 2$. Suppose the claim is not true then it is easy to see that $||C'|| = p - 1$ which is an odd number thus contradicting the bipartitedness of H'. Hence the claim is true. Now, by applying the claim it is easy to see that $||C'|| = \sum_j |V(C') \cap V(Q_j)| \geq p = ||C||$.

Case $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| > 1$: Scan vertices of H' starting from any vertex in clockwise direction. Claim 4.9 ensures that between every Q_j and $Q_{j'}$, which do not contribute to $V(C')$, encountered there exists a Q_i which compensates by contributing at least two vertices to $V(C')$. Therefore, $||C'|| \geq p = ||C||$. ■

4 Discussion

An interesting open question that naturally follows from our result is the following: given a graph G and positive integers k, m where $k \geq 4$, if $G^{[m]}$ is k-chordal, then is $G^{[m+2]}$ also k-chordal? As mentioned earlier, Brandstädt et al. in [3] showed a similar result in the context of *ordinary* graph powering. They showed that, for every graph G, if G^m is k-chordal, then so is G^{m+2}, where k, m are positive integers with $k \geq 3$. A straightforward extension of their proof technique doesn't seem to work in our context due to the bipartite nature of the powering that we consider.

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