

# The 3-way flower intersection problem for Steiner triple systems

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The flower at a point  $x$  in a Steiner triple system  $(X, \mathcal{B})$  is the set of all triples containing  $x$ . Denote by  $J_F^3(r)$  the set of all integers  $k$  such that there exists a collection of three  $STS(2r+1)$  mutually intersecting in the same set of  $k+r$  triples,  $r$  of them being the triples of a common flower. In this article we determine the set  $J_F^3(r)$  for any positive integer  $r \equiv 0, 1 \pmod{3}$  (only some cases are left undecided for  $r = 6, 7, 9, 24$ ), and establish that  $J_F^3(r) = I_F^3(r)$  for  $r \equiv 0, 1 \pmod{3}$  where  $I_F^3(r) = \{0, 1, \dots, \frac{2r(r-1)}{3} - 8, \frac{2r(r-1)}{3} - 6, \frac{2r(r-1)}{3}\}$ .

**Keywords:** Steiner triple system, 3-way intersection, 3-way flower intersection, Pairwise balanced design, Group divisible design, Latin square.

## 1 Introduction

A Steiner system  $S(2, k, v)$  is a pair  $(X, \mathcal{B})$  where  $X$  is a  $v$ -set and  $\mathcal{B}$  is a family of  $k$ -subset of  $X$  called blocks, such that each 2-subset of  $X$  is contained in exactly one block of  $\mathcal{B}$ . If  $k = 3$ , then the Steiner system  $S(2, 3, v)$  is called *Steiner triple system* of order  $v$  or briefly  $STS(v)$ . It is well known that an  $STS(v)$  exists if and only if  $v \equiv 1, 3 \pmod{6}$  Hanani (1975).

A *Kirkman triple system* of order  $v$  (briefly  $KTS(v)$ ) is a Steiner triple system of order  $v$ ,  $(X, \mathcal{B})$  together with a partition  $\mathcal{R}$  of the set of triples  $\mathcal{B}$  into subsets  $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n$  called parallel classes such that each  $\mathcal{R}_i$  (for  $i = 1, 2, \dots, n$ ) is a partition of  $X$ . It is well known that a  $KTS(v)$  exists if and only if  $v \equiv 3 \pmod{6}$  Ray-Chaudhuri and Wilson (1971).

It can be easily checked that the number of triples contained in an  $STS(v)$  (or  $KTS(v)$ ) is  $t_v = v(v-1)/6$ . For each non-negative integer  $n$ , let  $S[n]$  denote the set of non-negative integers less than or equal to  $n$ , with the exception of  $n-1, n-2, n-3$  and  $n-5$ . Let  $I(v) = S[t_v]$ .

Two Steiner (or Kirkman) triple systems  $(X, \mathcal{B}_1)$  and  $(X, \mathcal{B}_2)$  are said to *intersect* in  $k$  triples, provided  $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$ . Denote by  $J(v)$  (or  $J_R(v)$ ) the set of all integer numbers  $k$  such that there exists a pair of  $STS(v)$  (or  $KTS(v)$ ) intersecting in  $k$  triples.

Lindner and Rosa (1975) have completely determined the set  $J(v)$  and proved that  $J(3) = 1, J(7) = \{0, 1, 3, 7\}, J(9) = \{0, 1, 2, 3, 4, 6, 12\}$  and  $J(v) = I(v)$  for  $v \equiv 1, 3 \pmod{6}$  and  $v \geq 13$ . Also Shen (2001), Chang and Lo Faro (1999) have determined the set  $J_R(v)$ .

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For each non-negative integer  $n$ , let  $S^3[n]$  denote the set of non-negative integers less than or equal to  $n$ , with the exception of  $n - 1, n - 2, n - 3, n - 4, n - 5$  and  $n - 7$ . Let  $I^3(v) = S^3[t_v]$ .

Milici and Quattrocchi (1987) generalized the intersection problem for  $STS(v)$ s and determined the set  $J^3(v)$  of all integer numbers  $k$  such that there exists a collection of three  $STS(v)$ s pairwise intersecting in the same set of  $k$  blocks. The following theorem contains their results.

**Theorem A** Milici and Quattrocchi (1987)  $J^3(v) = I^3(v)$  for every  $v \geq 19$ ;  $J^3(7) = \{1, 7\}$ ,  $J^3(9) = \{0, 1, 3, 4, 12\}$ ,  $J^3(13) = I^3(13) \setminus \{14, 15, 16, 17, 18, 20\}$  and  $J^3(15) = I^3(15) \setminus \{24, 25, 26, 27\}$ .

Amjadi and Soltankhah (2017) generalized the intersection problem for  $KTS(v)$ s and for sufficiently large  $v$  determined the set  $J_R^3(v)$  of all integer numbers  $k$  such that there exists a collection of three  $KTS(v)$ s pairwise intersecting in the same set of  $k$  blocks. Just two cases are left undecided.

There are some other results on  $\mu$ -way intersection problem, for example, Rashidi and Soltankhah (2017) solved the 3-way intersection problem of  $S(2, 4, v)$  designs also Milici and Quattrocchi (1986) found some results on the maximum number of  $STS$ s such that any two of them intersect in the same block-set and Gotalizadeh and Soltankhah (2019) found the maximum number of Steiner triple systems which intersect in some special blocks.

The *flower* at a point  $x$  in a Steiner triple system, is the set of all triples containing  $x$ . The *flower intersection problem* for  $STS$ s is the determination for each  $v = 2r + 1 \equiv 1, 3 \pmod{6}$  of the set  $J_F(r)$  of all  $k$  such that there exists a pair of Steiner triple systems on the same  $v$ -set having  $k + r$  triples in common,  $r$  of them being the triples of a common flower. Note that  $r \equiv 0, 1 \pmod{3}$ ; we call such a non-negative  $r$  *admissible*.

For each admissible  $r$ , let  $I_F(r) = S^{\lceil \frac{2r(r-1)}{3} \rceil}$ . Hoffman and Lindner (1987) have completely determined the set  $J_F(r)$ . The following theorem contains their results.

**Theorem B** Hoffman and Lindner (1987) For all admissible  $r$ ,  $J_F(r) = I_F(r)$ , except  $1, 4 \notin J_F(4)$ .

Also Zhang et al. (2014) solved the flower intersection problem of  $S(2, 4, v)$  designs; Milici and Quattrocchi (1990) solved the flower intersection problem for  $S(3, 4, v)$  designs for some values of  $v$  and Chang and Lo Faro (2003) solved the flower intersection problem of  $KTS(v)$ s.

Let  $I_F^3(r) = S^3[\lceil \frac{2r(r-1)}{3} \rceil]$  and  $J_F^3(r)$  be the set of all integer numbers  $k$  such that there exists a collection of three  $STS(2r + 1)$ s pairwise intersecting in the same  $k + r$  blocks,  $r$  of them being the triples of a common flower. Here we determine  $J_F^3(r)$  for all admissible values of  $r$  except for some small values of  $r$ .

The necessary conditions are expressed in the following lemma and it is straightforward.

**Lemma 1** For each admissible  $r$ ,  $J_F^3(r) \subseteq I_F^3(r)$ ,  $J_F^3(r) \subseteq \{a - r \mid a \in J^3(2r + 1), a \geq r\}$  and  $J_F^3(r) \subseteq J_F(r)$ .

## 2 Recursive constructions

In this section we give several recursive constructions for finding the 3-way flower intersection numbers of Steiner triple systems. The concept of PBDs, GDDs and Latin squares play an important role in these constructions. Let us give their formal definition.

Let  $v$  be a positive integer and  $K$  be a set of positive integers. A *pairwise balanced design* (briefly a PBD)  $B(K, \lambda, v)$  is a pair  $(X, \mathcal{A})$  where  $X$  is a  $v$ -set and  $\mathcal{A}$  is a set of subsets (called *blocks*) of  $X$  such

that  $|B| \in K$  for each  $B \in \mathcal{A}$  and each pair of distinct elements of  $X$  is contained in exactly  $\lambda$  blocks of  $\mathcal{A}$ .

Let  $X$  be a finite set containing  $v$  points,  $\mathcal{G}$  a family of distinct subsets of  $X$ , called *groups* which partition  $X$ , and  $\mathcal{A}$  a collection of subsets of  $X$ , called *blocks*. Let  $K$  be a set of positive integers. A design  $(X, \mathcal{G}, \mathcal{A})$  is called a *group divisible design* (GDD)  $K$ -GDD if

1.  $\{|B| : B \in \mathcal{A}\} \subseteq K$ ;
2.  $|G \cap B| \leq 1$  for every  $G \in \mathcal{G}$  and every  $B \in \mathcal{A}$ ;
3. Every pair of points from distinct groups occurs in exactly a unique block of  $\mathcal{A}$ .

If  $\mathcal{G}$  contains  $t_1$  groups of size  $m_1$ ,  $t_2$  groups of size  $m_2$ ,  $\dots$ , and  $t_s$  groups of size  $m_s$ , we call  $m_1^{t_1} m_2^{t_2} \dots m_s^{t_s}$  the *group type* (or type) of the GDD. A  $K$ -GDD with group type  $m_1^{t_1} m_2^{t_2} \dots m_s^{t_s}$  is actually a pairwise balanced design and it is denoted by  $B(K \cup M, 1, v)$  (or  $(v, K \cup M, 1)$ -PBD), where  $M = \{m_1, m_2, \dots, m_s\}$ . We usually write  $\{k\}$ -GDD as  $k$ -GDD and  $B(\{k\}, 1, v)$  as  $B(k, 1, v)$  (or  $(v, k, 1)$ -BIBD).

A *Latin square* of order  $n$  is an  $n \times n$  array  $L = (\ell_{ij})$  on  $n$  symbols in which every row and every column of  $L$  contains no repeated symbols. Two Latin squares  $L$  and  $L'$  of the same order are *orthogonal* if  $\ell_{ab} = \ell_{cd}$  and  $\ell'_{ab} = \ell'_{cd}$ , implies  $a = c$  and  $b = d$ . A set of Latin squares  $L_1, L_2, \dots, L_m$  are *mutually orthogonal*, or a set of *MOLS*, if for every  $1 \leq i < j \leq m$ ,  $L_i$  and  $L_j$  are orthogonal.

We use the results of 3-way intersection problem for Latin squares to determine  $J_F^3(r)$ . The 3-way intersection problem for Latin squares is the problem of determining, for all orders  $n$ , the set of integers  $k$  for which there exist three Latin squares of order  $n$  having precisely  $k$  identical cells, with their remaining  $n^2 - k$  cells different in all three Latin squares.

Denote by  $J^3(n)$  the set of integers  $k$  for which there exist three Latin squares of order  $n$  which have precisely  $k$  cells where all three squares have identical entries and  $n^2 - k$  cells where all three squares contain different entries. Let for  $n \geq 4$ ,  $I^3(n) = [0, n^2 - 15] \cup \{n^2 - 12, n^2 - 9, n^2\}$  and  $I^3(3) = \{0, 9\}$ . Adams et al. (2002) completely solve the 3-way intersection problem. The following theorem contains their results. Let  $[i, j]$  denote the set of integers  $\{i, i + 1, i + 2, \dots, j - 1, j\}$ , where  $i < j$ .

**Theorem C** Adams et al. (2002)  $J^3(3) = I^3(3) = \{0, 9\}$ ,

$$J^3(4) = I^3(4) \setminus \{7\} = \{0, 1, 4, 16\},$$

$$J^3(5) = I^3(5) \setminus \{8, 9, 13, 16\} = [0, 7] \cup \{10, 25\},$$

$$J^3(6) = I^3(6) \setminus \{20, 21, 24\} = [0, 19] \cup \{27, 36\},$$

$$J^3(7) = I^3(7) \setminus \{31, 34\} = [0, 30] \cup \{37, 40, 49\},$$

$$J^3(n) = I^3(n) \text{ for } n \geq 8.$$

Before expressing the next theorem we need to introduce some notations. For sets of integers  $X$  and  $Y$ , let  $X + Y = \{x + y \mid x \in X, y \in Y\}$ . Also for any integer  $n$ , let  $nX = \{nx \mid x \in X\}$ .

Let  $J'^3(n)$  denotes the set of integers  $k$  for which there exist three Latin squares of order  $n$  with the same constant secondary diagonal which have precisely  $k = k' + n$  cells ( $n$  of these are cells of secondary diagonal) where all three squares have identical entries and  $n^2 - n - k'$  cells where all three squares contain different entries. The following theorem is similar to Lemma 2.2 of Adams et al. (2002).

**Theorem 1**  $J^3(n) + (n + 1)\{[0, n - 3] \cup \{n\}\} + (2n + 1)\{[1, n - 2] \cup \{n + 1\}\} \subseteq J'^3(2n + 1)$ .

**Proof:** The proof is exactly the same as described in the Lemma 2.2 of Adams et al. (2002), just it is enough to use permutations which do not change the secondary diagonal of the Latin square.  $\square$

We use four constructions of  $STSS$ s from smaller  $STSS$ s which are adapted from constructions used in Billington et al. (2008), Hoffman and Lindner (1987) and Lindner and Rosa (1975).

**Construction 1** ( $STSS(3v)$  from  $STSS(v)$ ). We define Steiner triple system  $S$  on the set  $X = \{1, 2, \dots, v\} \times \{1, 2, 3\}$ . Let  $L = (\ell_{ij})$  be a Latin square on symbols  $V = \{1, 2, \dots, v\}$ . Let  $V_i = V \times \{i\}$  for  $i = 1, 2, 3$  and  $(V_1, \mathcal{B}_1)$ ,  $(V_2, \mathcal{B}_2)$  and  $(V_3, \mathcal{B}_3)$  be three Steiner triple systems  $S_1$ ,  $S_2$  and  $S_3$  respectively. It is easy to see that  $(X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B})$  is an  $STSS(3v)$ , where  $\mathcal{B} = \{(i, 1), (j, 2), (s, 3) \mid \ell_{ij} = s, 1 \leq i, j, s \leq v\}$ .

**Construction 2** ( $STSS(3v+1)$  from  $STSS(v+1)$ ). We define Steiner triple system  $S$  on the set  $X = (\{1, 2, \dots, v\} \times \{1, 2, 3\}) \cup \{\infty\}$ . Let  $L$  be a Latin square on symbols  $V = \{1, 2, \dots, v\}$ . Let  $V_i = V \times \{i\}$  for  $i = 1, 2, 3$  and  $(V_1 \cup \{\infty\}, \mathcal{B}_1)$ ,  $(V_2 \cup \{\infty\}, \mathcal{B}_2)$  and  $(V_3 \cup \{\infty\}, \mathcal{B}_3)$  be three  $STSS(v+1)$   $S_1$ ,  $S_2$  and  $S_3$  respectively. It is easy to see that  $(X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B})$  is an  $STSS(3v+1)$ , where  $\mathcal{B} = \{(i, 1), (j, 2), (s, 3) \mid \ell_{ij} = s, 1 \leq i, j, s \leq v\}$ .

**Construction 3** ( $STSS(3v+3)$  from  $STSS(v+3)$ ). We define Steiner triple system  $S$  on the set  $X = (\{1, 2, \dots, v\} \times \{1, 2, 3\}) \cup \{\infty_1, \infty_2, \infty_3\}$ . Let  $L = (\ell_{ij})$  be a Latin square on symbols  $V = \{1, 2, \dots, v\}$ . Let  $V_i = V \times \{i\}$  for  $i = 1, 2, 3$  and  $(V_1 \cup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B}_1)$ ,  $(V_2 \cup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B}_2)$  and  $(V_3 \cup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B}_3)$  be three  $STSS(v+3)$   $S_1$ ,  $S_2$  and  $S_3$  respectively, which each of them contains triple  $\{\infty_1, \infty_2, \infty_3\}$ . It is easy to see that  $(X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B})$  is an  $STSS(3v+3)$ , where  $\mathcal{B} = \{(i, 1), (j, 2), (s, 3) \mid \ell_{ij} = s, 1 \leq i, j, s \leq v\}$ .

Recall that a 1-factor of a graph  $G$  is a spanning subgraph of  $G$  that is regular of degree 1. A 1-factorization of a graph  $G$  is a set  $\mathcal{F} = \{F_1, \dots, F_k\}$  of edge-disjoint 1-factors of  $G$  whose edge-sets partition the edge-set of  $G$ . Colbourn and Dinitz (2006)

**Construction 4** ( $STSS(2v+1)$  from  $STSS(v)$ ). Let  $X = \{a_1, a_2, \dots, a_v\}$  and  $(X, \mathcal{B})$  be an  $STSS(v)$ . Put  $v+1 = 2n$  and let  $\mathcal{F} = \{F_i \mid i = 1, 2, \dots, 2n-1\}$  be a 1-factorization of  $K_{2n}$  with the vertex set  $V(K_{2n}) = Y$  where  $X \cap Y = \emptyset$ , ( $K_{2n}$  is complete graph with  $2n$  vertices). Put  $X^* = X \cup Y$  and  $\mathcal{B}^* = \mathcal{B} \cup \mathcal{C}$  where  $\mathcal{C} = \{\{a_i, x, y\} \mid [x, y] \in F_i, i = 1, 2, \dots, 2n-1\}$ . It is easy to see that  $(X^*, \mathcal{B}^*)$  is an  $STSS(2v+1)$ .

**Theorem 2** For all admissible  $r$ , if there exist three Latin squares on same  $2r+1$  symbols with the same constant secondary diagonal agreeing pairwise on exactly  $b+2r+1$  cells ( $2r+1$  of these are cells of the secondary diagonal), and if  $a_i \in J^3(2r+1)$  for  $i = 1, 2$  and  $a \in J_F^3(r)$ , then  $a_1 + a_2 + a + b \in J_F^3(3r+1)$ .

**Proof:** We use Construction 1 to construct a collection of three  $STSS(6r+3)$ s with flower intersection number  $a_1 + a_2 + a + b$ . For this purpose, start with three Latin squares  $L'$ ,  $L''$  and  $L'''$  of order  $2r+1$ , with the same constant secondary diagonal of 1's and agreeing pairwise on exactly  $b$  cells off the secondary diagonal. For  $i = 1, 2$ , let  $S'_i$ ,  $S''_i$  and  $S'''_i$  be a collection of three systems on  $\{1, 2, \dots, 2r+1\} \times \{i\}$  with exactly  $a_i$  triples in common. Let  $S'_3$ ,  $S''_3$  and  $S'''_3$  be a collection of three systems on  $\{1, 2, \dots, 2r+1\} \times \{3\}$  with the same  $r$  triples containing  $(1, 3)$  (a flower at element  $(1, 3)$ ), and  $a$  further triples in common. Then by using Construction 1 for  $v = 2r+1$ , we construct a collection of three Steiner triple systems  $S'$ ,  $S''$  and  $S'''$  of order  $6r+3$  with  $a_1 + a_2 + a + b + (3r+1)$  triples in common,  $3r+1$  of them are flower at element  $(1, 3)$ .  $\square$

**Remark 1** Theorem 2 is true also when Latin squares have same row or same column instead of same constant secondary diagonal. the proof is similar to the proof of Theorem 2.

**Theorem 3** For all admissible  $r$ , if  $b \in J^{3}(2r)$  and  $a_i \in J_F^3(r)$  for  $i = 1, 2, 3$ , then  $a_1 + a_2 + a_3 + b \in J_F^3(3r)$ .

**Proof:** We use Construction 2 to construct a collection of three  $STS(6r + 1)$ s with flower intersection number  $a_1 + a_2 + a_3 + b$ . For this purpose, start with three Latin squares  $L', L''$  and  $L'''$  of order  $2r$  with 3-way intersection number  $b$ . For  $i = 1, 2, 3$ , let  $S'_i, S''_i$  and  $S'''_i$  be a collection of three systems on  $(\{1, 2, \dots, 2r\} \times \{i\}) \cup \{\infty\}$  with the same  $r$  triples containing  $\infty$  (a flower at  $\infty$ ), and  $a_i$  further triples in common. Then by using Construction 2 for  $v = 2r$ , we construct a collection of three Steiner triple systems  $S', S''$  and  $S'''$  of order  $6r + 1$  with  $a_1 + a_2 + a_3 + b + 3r$  triples in common,  $3r$  of them are flower at  $\infty$ .  $\square$

**Theorem 4** For  $r \equiv 0, 2 \pmod{3}$ , if  $b \in J^{3}(2r)$  and  $a_i \in J_F^3(r+1)$  for  $i = 1, 2, 3$ , then  $a_1 + a_2 + a_3 + b \in J_F^3(3r + 1)$ .

**Proof:** We use Construction 3 to construct a collection of three  $STS(6r + 3)$ s with flower intersection number  $a_1 + a_2 + a_3 + b$ . For this purpose, start with three Latin squares  $L', L''$  and  $L'''$  of order  $2r$  with 3-way intersection number  $b$ . For  $i = 1, 2, 3$ , let  $S'_i, S''_i$  and  $S'''_i$  be a collection of three systems on  $(\{1, 2, \dots, 2r\} \times \{i\}) \cup \{\infty_1, \infty_2, \infty_3\}$  which contain triple  $\{\infty_1, \infty_2, \infty_3\}$  and with the same  $r + 1$  triples containing  $\infty_1$  (a flower at  $\infty_1$ ), and  $a_i$  further triples in common. Then by using Construction 3 for  $v = 2r$ , we construct a collection of three Steiner triple systems  $S', S''$  and  $S'''$  of order  $6r + 3$  with  $a_1 + a_2 + a_3 + b + (3r + 1)$  triples in common,  $3r + 1$  of them are flower at  $\infty_1$ .  $\square$

**Theorem 5** For all admissible  $r$ , if  $k \in J_F^3(r)$  then  $k + (s - 1)(r + 1) \in J_F^3(2r + 1)$  for every  $s = 1, 2, \dots, 2r - 2, 2r + 1$ .

**Proof:** Let  $v = 2r + 1$  and  $X, Y, X^*$  and  $\mathcal{F}$  be as in Construction 4. Let  $(X, \mathcal{B}_i)$  for  $i = 1, 2, 3$  be a collection of three  $STS(v)$ s, with 3-way intersection number  $k + r$  which  $r$  of them being the triples of a common flower at point  $a_1$ . For  $s = 1, 2, \dots, v - 3, v$ , let  $\alpha_1$  and  $\alpha_2$  be two permutations of  $X$  fixing exactly  $s$  elements  $\{a_1, a_2, \dots, a_s\}$  and for  $s + 1 \leq i \leq v$ ,  $\alpha_1(a_i) \neq \alpha_2(a_i)$ . Now let  $\mathcal{C}$  be as in Construction 4, i.e.

$$\begin{aligned} \mathcal{C} &= \{\{a_i, x, y\} \mid [x, y] \in F_i, i = 1, 2, \dots, v\}, \text{ and put} \\ \alpha_1(\mathcal{C}) &= \{\{\alpha_1(a_i), x, y\} \mid [x, y] \in F_i, i = 1, 2, \dots, v\}, \text{ and} \\ \alpha_2(\mathcal{C}) &= \{\{\alpha_2(a_i), x, y\} \mid [x, y] \in F_i, i = 1, 2, \dots, v\}. \end{aligned}$$

Each 1-factor of  $\mathcal{F}$  contains  $(v + 1)/2 = r + 1$  edges, so  $\mathcal{C}, \alpha_1(\mathcal{C})$  and  $\alpha_2(\mathcal{C})$  pairwise have exactly  $(s - 1)(r + 1) + (r + 1)$  triples in common, which  $(r + 1)$  of them being the triples which contain  $a_1$ . So three  $STS(2v + 1)$ s,  $(X^*, \mathcal{B}_1 \cup \mathcal{C})$ ,  $(X^*, \mathcal{B}_2 \cup \alpha_1(\mathcal{C}))$  and  $(X^*, \mathcal{B}_3 \cup \alpha_2(\mathcal{C}))$  have 3-way intersection number  $k + (s - 1)(r + 1) + 2r + 1$  which  $2r + 1$  of them being the triples of a common flower at point  $a_1$ . So  $k + (s - 1)(r + 1) \in J_F^3(2r + 1)$ .  $\square$

**Theorem D** Colbourn and Dinitz (2006)(Section III-3-4), Todorov (2012) For  $n \geq 5$  and  $n \neq 6$ , there exist four mutually orthogonal Latin squares of order  $n$  except possibly for  $n \in \{10, 18, 22\}$ .

**Theorem 6** Let  $t \geq 5$  and  $t \notin \{6, 10, 18, 22\}$ , there exists a pairwise balanced design (PBD) of order  $6t$ , with six blocks of size  $t$  and  $t^2$  blocks of size 6. (or a  $B(\{6, t\}, 1, 6t)$ ).

**Proof:** Let  $V = \{1, 2, \dots, t\}$ , we define a pairwise balanced design  $B(\{6, t\}, 1, 6t)$  on set  $V \times \{1, 2, \dots, 6\}$ . By Theorem D there exist four mutually orthogonal Latin squares of order  $t$ , where  $t \geq 5$  and  $t \notin \{6, 10, 18, 22\}$ . Let  $L_1, L_2, L_3$  and  $L_4$  be four mutually orthogonal Latin squares of order  $t$ , where  $L_n = (\ell_{ij}^{\{n\}})$ ,  $i \in V \times \{5\}, j \in V \times \{6\}$  and  $\ell_{ij}^{\{n\}} \in V \times \{n\}$  for  $n \in \{1, 2, 3, 4\}$ . It is enough to consider six blocks  $\{V \times \{m\}\}$  for  $m \in \{1, 2, \dots, 6\}$  and  $t^2$  blocks  $\{\{a_1, a_2, a_3, a_4, i, j\} \mid a_n \in V \times \{n\}, 1 \leq n \leq 4, i \in V \times \{5\}, j \in V \times \{6\}\}$ , where  $\ell_{ij}^{\{n\}} = a_n$  for  $1 \leq n \leq 4$ .  $\square$

The following theorem is similar to Theorem 4.1.3 of Lindner and Rodger (2008).

**Theorem 7** If there exists a  $B(\{k_1, k_2, \dots, k_x\}, 1, r)$  for  $r \equiv 0, 1 \pmod{3}$ , and if there exists an  $STS(2k_i + 1)$  for  $1 \leq i \leq x$ , then there exists an  $STS(2r + 1)$ .

**Proof:** Let  $(X, \mathcal{B})$  be a  $B(\{k_1, k_2, \dots, k_x\}, 1, r)$  with  $X = \{1, 2, \dots, r\}$ . Define an  $STS(2r + 1)$   $(Y, \mathcal{A})$  with  $Y = \{\infty\} \cup (\{1, 2, \dots, r\} \times \{1, 2\})$  as follows.

- (1) For  $1 \leq i \leq r$ ,  $\{\infty, (i, 1), (i, 2)\} \in \mathcal{A}$ ,
- (2) For each block  $B \in \mathcal{B}$ , let  $(Y(B), \mathcal{A}(B))$  be an  $STS(2|B| + 1)$ , where  $Y(B) = \{\infty\} \cup (B \times \{1, 2\})$  and where the symbols have been named so that  $\{\infty, (i, 1), (i, 2)\} \in \mathcal{A}(B)$  for all  $i \in B$ , and let  $\mathcal{A}(B) \setminus \{\{\infty, (i, 1), (i, 2)\} \mid i \in B\} \subseteq \mathcal{A}$ .

It is easy to see that  $(Y, \mathcal{A})$  is an  $STS(2r + 1)$ .  $\square$

**Theorem 8** Let  $k \equiv 0, 1 \pmod{3}$  for each  $k \in K$  and  $r \equiv 0, 1 \pmod{3}$ . If there exists a  $B(K, 1, r)$   $(X, \mathcal{B})$  such that  $k_B \in J_F^3(|B|)$  for each  $B \in \mathcal{B}$ , then  $\sum_{B \in \mathcal{B}} k_B \in J_F^3(r)$ .

**Proof:** Since  $r \equiv 0, 1 \pmod{3}$  and for each  $k \in K$ ,  $k \equiv 0, 1 \pmod{3}$ , we can use the construction which has been explained in Theorem 7 to form a collection of three  $STS(2r + 1)$ s  $(Y, \mathcal{A}_1)$ ,  $(Y, \mathcal{A}_2)$  and  $(Y, \mathcal{A}_3)$  as follows. For each  $B \in \mathcal{B}$ , since  $k_B \in J_F^3(|B|)$ , then we may form a collection of three  $STS(2|B| + 1)$ s  $(Y(B), \mathcal{A}_1(B))$ ,  $(Y(B), \mathcal{A}_2(B))$  and  $(Y(B), \mathcal{A}_3(B))$  such that their 3-way flower intersection number is  $k_B$  and with flower at  $\infty$ . For  $i = 1, 2, 3$ , we define

$$\mathcal{A}_i = \{\{\infty, x_1, x_2\} \mid x \in B\} \cup \{\cup_{B \in \mathcal{B}} \{\mathcal{A}_i(B) \setminus \{\{\infty, x_1, x_2\} \mid x \in B\}\}\}.$$

So  $(Y, \mathcal{A}_1)$ ,  $(Y, \mathcal{A}_2)$  and  $(Y, \mathcal{A}_3)$  are three  $STS(2r + 1)$ s with 3-way flower intersection number  $\sum_{B \in \mathcal{B}} k_B$  and the flower is on  $\infty$ . So  $\sum_{B \in \mathcal{B}} k_B \in J_F^3(r)$ .  $\square$

In the following, some auxiliary lemmas are expressed.

**Lemma 2** For all admissible  $r \geq 4$ ,  $0 \in J_F^3(r)$ .

**Proof:** Let  $(X, \mathcal{B})$  be an  $STS(2r + 1)$  and write  $\mathcal{B} = \mathcal{F}_x \cup \mathcal{C}$ , where  $\mathcal{F}_x$  is the flower at the point  $x$ . Let  $Y = X - \{x\}$  and  $\mathcal{G} = \{\{a, b\} : \{a, b, x\} \in \mathcal{F}_x\}$ . Then  $(Y, \mathcal{G}, \mathcal{C})$  is a 3-GDD of type  $2^r$ . Since the maximum number of disjoint 3-GDDs of type  $2^r$  is  $2(r - 2)$  and there exists a large set of 3-GDD of type  $2^r$  for  $r \equiv 0, 1 \pmod{3}$  Cao et al. (2001), the result follows.  $\square$

**Lemma 3** For all admissible  $r$ ,  $\frac{2r(r-1)}{3} \in J_F^3(r)$ .

**Proof:** It is enough to consider the same  $STS(2r + 1)$  three times.  $\square$

The following lemma is similar to Lemma 1 of Hoffman and Lindner (1987).

**Lemma 4** Let  $v = 2r + 1 \equiv 1, 3 \pmod{6}$ , for  $r \geq 6$  and  $k \geq \frac{2r(r-3)}{3}$ , if  $k + r \in J^3(v)$ , then  $k \in J_F^3(r)$ .

**Proof:** It is enough to assume a point  $x$  is contained in a triple of the first system that is not a triple of the other systems, then obviously it must be contained in at least two such triples. A simple calculation now shows that there must be at least one point  $x$  for which the triples through  $x$  are the same in all systems.  $\square$

**Corollary 1** For all admissible  $r \geq 9$ , let  $k \in I_F^3(r)$  with  $k \geq \frac{2r(r-3)}{3}$ , then  $k \in J_F^3(r)$ .

**Proof:** From Milici and Quattrocchi (1987),  $J^3(v) = I^3(v)$  for every  $v \geq 19$ . Lemma 4 completes the proof.  $\square$

### 3 Small Cases

In this section we discuss some small admissible values of  $r$ , needed for general constructions.

For a  $v$ -set  $X$ , let  $(X, \mathcal{A})$ ,  $(X, \mathcal{B})$  and  $(X, \mathcal{C})$  be three Steiner triple systems of order  $v$ . For convenience, we introduce a notation in this section.  $|\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}|_F = k$ , means that three  $STS(v)$ s,  $(X, \mathcal{A})$ ,  $(X, \mathcal{B})$  and  $(X, \mathcal{C})$ , have 3-way flower intersection number  $k$ .

The following theorem is obvious.

**Theorem 9**  $J_F^3(1) = \{0\}$ .

**Theorem 10**  $J_F^3(3) = \{4\}$ .

**Proof:** By Theorem A,  $J^3(7) = \{1, 7\}$ , so the only 3-way flower intersection number of  $STS(7)$  can be 4. Using Lemma 3 completes the proof.  $\square$

**Theorem 11**  $J_F^3(4) = \{0, 8\}$ .

**Proof:** By Theorem A,  $J^3(9) = \{0, 1, 3, 4, 12\}$ , so the only 3-way flower intersection numbers of  $STS(9)$  can be  $\{0, 8\}$ . Lemmas 2 and 3 completes the proof.  $\square$

**Theorem 12**  $[0, 5] \cup \{7, 20\} \subseteq J_F^3(6) \subseteq [0, 7] \cup \{20\}$ .

**Proof:** By Theorem A,  $J^3(13) = I^3(13) \setminus \{14, 15, 16, 17, 18, 20\}$ , so  $J_F^3(6) \subseteq [0, 7] \cup \{20\}$ . By Lemmas 2 and 3,  $\{0, 20\} \subseteq J_F^3(6)$ . In Milici and Quattrocchi (1987), the  $STS(13)$ s which are given in Tables 1 – 7, 1 – 8 and 1 – 9 have 3-way flower intersection numbers 2, 3 and 4 respectively with flower at 1. Also the  $STS(13)$ s which are given in Table 1 – 11 have 3-way flower intersection number 7 with flower at 13. Let  $X = \{1, 2, \dots, 13\}$  and  $(X, \mathcal{A})$  and  $(X, \mathcal{B})$  be the following  $STS(13)$ s.

	1	2	11	1	3	13	1	4	10	1	5	7	1	6	12	1	8	9	2	3	6	2	4	9	2	5	12
$\mathcal{A}$	2	7	13	2	8	10	3	4	7	3	5	10	3	8	12	3	9	11	4	5	8	4	6	11	4	12	13
	5	6	9	5	11	13	6	7	10	6	8	13	7	8	11	7	9	12	9	10	13	10	11	12			

	1	2	11	1	3	13	1	4	10	1	5	7	1	6	12	1	8	9	2	3	6	2	4	13	2	5	12
$\mathcal{B}$	2	7	9	2	8	10	3	4	7	3	5	10	3	8	12	3	9	11	4	5	8	4	6	11	4	9	12
	5	6	9	5	11	13	6	7	10	6	8	13	7	8	11	7	12	13	9	10	13	10	11	12			

Consider the following permutations on  $X$ :

$$\pi_1 = (3\ 4\ 7)(13\ 10\ 5), \quad \pi_2 = (13\ 8)(12\ 4)(2\ 11)(3\ 9)(6\ 10),$$

$$\pi'_1 = (7\ 4\ 3)(5\ 10\ 13), \quad \pi'_2 = (12\ 11\ 9)(2\ 8\ 6)(4\ 10)(5\ 7).$$

It is checked by computer programming that  $|\mathcal{A} \cap \pi_1(\mathcal{A}) \cap \pi'_1(\mathcal{A})|_F = 1$  and  $|\mathcal{B} \cap \pi_2(\mathcal{A}) \cap \pi'_2(\mathcal{A})|_F = 5$ .

□

The following construction is similar to construction used in Lemma 4 of Hoffman and Lindner (1987). Let us call the following Latin square,  $L$ , of order 8 on symbols  $0, 1, \dots, 7$ , *special* of order 8, where  $A$  is a Latin square of order 4 on symbols  $4, 5, 6$  and  $7$ .

$$L = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 2 \\ \hline 2 & 3 & 0 & 1 \\ \hline 3 & 2 & 1 & 0 \\ \hline & & & A \\ \hline & & 0 & 1 \\ & & 1 & 0 \\ & & 2 & 3 \\ & & 3 & 2 \\ \hline & A^T & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \\ & & 2 & 3 \\ & & 3 & 2 \\ \hline \end{array}$$

Denote by  $K$  the set of integers  $k$  for which there exists a collection of three special Latin squares of order 8 which pairwise agree in exactly  $k$  of the 24 cells above the  $2 \times 2$  diagonal blocks.

**Lemma 5**  $\{8, 9, 12\} \subseteq K$ .

**Proof:** Let  $k \in K$ , write  $k = a + b$  where  $a \in J^{t^3}(4)$  and  $b = 8$  (the number of cells which contain elements 2 and 3). By Theorem C,  $\{0, 1, 4\} \subseteq J^{t^3}(4)$ , so  $\{8, 9, 12\} \subseteq K$ . □

**Lemma 6**  $\{12, 13, 16\} \subseteq J_F^3(7)$ .

**Proof:** Let  $K$  be the set of integers  $k$  for which there exists a collection of three special Latin squares of order 8,  $L_{ij}^{(n)}$  for  $n = 1, 2, 3$ , which pairwise agree in exactly  $k$  of the 24 cells above the  $2 \times 2$  diagonal blocks. We construct a collection of three  $STS(15)$ s with 3-way flower intersection number  $h$  for  $h \in \{12, 13, 16\}$ . Write  $h = \ell + k$  where  $\ell \in J_F^3(3)$  and  $k \in K$ . Let  $X_1 = \{\infty_i : 1 \leq i \leq 7\}$ ,  $X_2 = \{1, 2, \dots, 8\}$  and  $(X_1, \mathcal{B}_n)$  for  $n = 1, 2, 3$  be three  $STS(7)$ s with 3-way intersection number  $\ell + 3$  where three of these will constitute the flower at  $\infty_1$ . It is not hard to check that  $(X_1 \cup X_2, \mathcal{B}_n \cup \mathcal{C}_n)$  for  $n = 1, 2, 3$  are three  $STS(15)$ s, where  $\mathcal{C}_n = \{\{\infty_s, i, j\} | \ell_{ij}^{(n)} = s, 1 \leq i < j \leq 8, 1 \leq s \leq 7\}$  which have 3-way intersection number  $\ell + k + 7$ , seven of these will constitute the flower at  $\infty_1$ . So  $h = \ell + k \in J_F^3(7)$ . By Theorem 10,  $\ell = 4$  and by Lemma 5,  $\{8, 9, 12\} \subseteq K$ . So  $\{12, 13, 16\} \subseteq J_F^3(7)$ . □

**Theorem 13**  $[0, 8] \cup [10, 13] \cup \{16, 22, 28\} \subseteq J_F^3(7) \subseteq [0, 16] \cup \{22, 28\}$ .

**Proof:** By Theorem A,  $J^3(15) = I^3(15) \setminus \{24, 25, 26, 27\}$ , so  $J_F^3(7) \subseteq [0, 16] \cup \{22, 28\}$ . By Lemmas 2, 3, 4 and 6,  $\{0, 12, 13, 16, 22, 28\} \subseteq J_F^3(7)$ . In Lemma 2 of Amjadi and Soltankhah (2017), the three  $KTS(15)$ s which have intersection numbers 15 and 17 actually have 3-way flower intersection numbers



8 and 10 respectively with a common flower at element 1. Also in Lemma 3 – 4 of Milici and Quattrocchi (1987), the three  $STS(15)$ s which have intersection numbers 12 and 18 actually have 3-way flower intersection numbers 5 and 11 respectively with a common flower at element 4 (Tables 2 – 1, 2 – 4 and 2 – 5). Now let  $X = \{1, 2, \dots, 15\}$  and  $(X, \mathcal{A}), (X, \mathcal{B}), (X, \mathcal{C}), (X, \mathcal{D})$  and  $(X, \mathcal{E})$  be the following  $STS(15)$ s.

	1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	4 10 14	2 9 11	2 8 10	2 13 15	2 12 14	2 4 6	2 5 7
$\mathcal{A}$	5 8 13	3 8 12	3 9 14	3 4 7	3 5 6	3 11 15	3 10 13
	6 9 15	6 13 14	4 11 13	5 11 14	4 8 15	5 9 10	4 9 12
	7 11 12	7 10 15	5 12 15	6 10 12	7 9 13	7 8 14	6 8 11
	1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	4 10 15	2 9 11	2 8 10	2 13 15	2 12 14	2 4 6	2 5 7
$\mathcal{B}$	5 9 13	3 8 11	3 9 10	3 4 7	3 5 6	3 13 14	3 12 15
	6 9 15	6 8 13	4 11 13	5 11 14	4 8 12	5 10 12	4 9 14
	7 11 15	7 10 13	5 8 15	6 10 14	7 9 12	7 8 14	6 11 12
	1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	4 8 15	2 9 11	2 8 10	2 13 15	2 12 14	2 4 6	2 5 7
$\mathcal{C}$	5 9 13	3 8 12	3 9 14	3 4 7	3 5 6	3 11 13	3 10 15
	6 9 12	6 8 14	4 13 14	5 10 14	4 11 12	5 12 15	4 9 10
	7 11 14	7 10 12	5 8 11	6 10 13	7 9 15	7 8 13	6 11 15
	1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	4 8 13	2 9 11	2 8 10	2 13 15	2 12 14	2 4 6	2 5 7
$\mathcal{D}$	5 9 14	3 8 11	3 9 12	3 4 7	3 5 6	3 13 14	3 10 15
	6 9 15	6 8 14	4 11 14	5 11 13	4 12 15	5 10 12	4 9 10
	7 11 15	7 10 14	5 8 15	6 10 13	7 9 13	7 8 12	6 11 12
	1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	4 10 14	2 9 11	2 8 10	2 13 15	2 12 14	2 4 6	2 5 7
$\mathcal{E}$	5 8 13	3 8 11	3 9 10	3 4 7	3 5 6	3 12 15	3 13 14
	6 9 15	6 11 13	4 9 13	5 11 14	4 8 12	5 9 12	4 9 13
	7 11 12	7 10 13	5 10 15	6 10 12	7 9 14	7 8 15	6 8 14

Consider the following permutations on  $X$ :

$$\begin{aligned} \pi_1 &= (5\ 13\ 3\ 14\ 4\ 12\ 2\ 15)(6\ 10)(7\ 11), & \pi'_1 &= (2\ 10)(3\ 11)(4\ 8\ 14)(5\ 9\ 15)(6\ 12)(7\ 13), \\ \pi_2 &= (4\ 10\ 14\ 5\ 11\ 15), & \pi'_2 &= (15\ 11\ 5\ 14\ 10\ 4), \\ \pi_3 &= (2\ 10\ 15\ 13\ 9\ 7\ 3\ 11\ 14\ 12\ 8\ 6), & \pi'_3 &= (2\ 6)(3\ 7)(4\ 8)(5\ 9)(10\ 15\ 13\ 11\ 14\ 12), \\ \pi_4 &= (2\ 4\ 6)(3\ 5\ 7), & \pi'_4 &= (6\ 4\ 2)(7\ 5\ 3), \\ \pi_5 &= (2\ 12\ 4\ 14\ 6)(3\ 13\ 5\ 15\ 7), & \pi'_5 &= (15\ 6\ 2)(3\ 14\ 7)(4\ 13)(5\ 12)(8\ 10)(9\ 11), \\ \pi_6 &= (15\ 13\ 3)(2\ 14\ 12)(4\ 10\ 6)(5\ 11\ 7), & \pi'_6 &= (15\ 3\ 13)(2\ 12\ 14)(4\ 6\ 10)(7\ 11\ 5). \end{aligned}$$

It is checked by computer programming that

$$\begin{aligned} |\mathcal{B} \cap \pi_1(\mathcal{D}) \cap \pi'_1(\mathcal{D})|_F &= 1, & |\mathcal{B} \cap \pi_2(\mathcal{B}) \cap \pi'_2(\mathcal{B})|_F &= 2, & |\mathcal{B} \cap \pi_3(\mathcal{D}) \cap \pi'_3(\mathcal{D})|_F &= 3, \\ |\mathcal{E} \cap \pi_4(\mathcal{E}) \cap \pi'_4(\mathcal{E})|_F &= 4, & |\mathcal{B} \cap \pi_5(\mathcal{C}) \cap \pi'_5(\mathcal{C})|_F &= 6, & |\mathcal{A} \cap \pi_6(\mathcal{A}) \cap \pi'_6(\mathcal{A})|_F &= 7. \end{aligned}$$

with flower at 1. □

**Theorem 14**  $I^3(9) \setminus \{4, 6, 7, 9, 11, 34, 35\} \subseteq J_F^3(9)$ .

**Proof:** By Theorem 3 for  $r = 3$ , and Theorems C and 10,  $[12, 31] \cup \{39, 48\} \subseteq J_F^3(9)$ . By Theorem 5 for  $r = 4$ , and Theorem 11,  $\{5, 8, 10, 33\} \subseteq J_F^3(9)$ . By Lemma 2,  $0 \in J_F^3(9)$  and by Corollary 1,  $\{36, 37, 38, 40, 42\} \subseteq J_F^3(9)$ . In Milici and Quattrocchi (1987), the  $STS(19)$ s which are given in Table

3 – 2 have 3-way flower intersection number 32 with flower at 19. Let  $X = \{1, 2, \dots, 19\}$  and  $(X, \mathcal{A})$  be the following  $STS(19)$ .

1 2 3	2 9 14	3 8 12	4 8 19	7 8 10	1 18 19	4 13 16	6 14 19	9 10 18	10 17 19
1 4 5	2 5 15	3 9 19	5 6 10	7 9 13	2 11 13	4 15 17	7 15 19	9 11 15	11 14 17
1 6 7	2 6 18	3 5 17	5 7 14	1 10 11	2 12 19	5 11 12	7 17 18	9 12 17	11 16 19
1 8 9	2 7 16	3 6 15	5 8 18	1 12 13	3 10 13	5 13 19	8 13 14	10 12 15	12 14 18
4 6 9	2 8 17	3 7 11	5 9 16	1 14 15	3 16 18	6 12 16	8 15 16	10 14 16	13 15 18
2 4 10	3 4 14	4 7 12	6 8 11	1 16 17	4 11 18	6 13 17			

Consider the following permutations on  $X$ :

- $\pi_1 = (4\ 16)(5\ 17)(10\ 8\ 12\ 6\ 14\ 2\ 18)(11\ 9\ 13\ 7\ 15\ 3\ 19)$ ,
- $\pi'_1 = (2\ 10)(3\ 11)(12\ 14)(13\ 15)(17\ 7\ 9\ 5\ 18\ 16\ 6\ 8\ 4\ 19)$ ,
- $\pi_2 = (6\ 14)(7\ 15)(8\ 12\ 4\ 16\ 2\ 18)(9\ 13\ 5\ 17\ 3\ 19)$ ,
- $\pi'_2 = (14\ 6\ 4\ 2\ 10\ 16\ 8\ 19\ 15\ 7\ 5\ 3\ 11\ 17\ 9\ 18)$ ,
- $\pi_3 = (2\ 8\ 10\ 18\ 4\ 12\ 6)(3\ 9\ 11\ 19\ 5\ 13\ 7)$ ,
- $\pi'_3 = (2\ 6\ 4\ 14\ 9\ 13\ 11)(3\ 7\ 5\ 15\ 8\ 12\ 10)(16\ 19\ 17\ 18)$ .

It is checked by computer programming that  $|\mathcal{A} \cap \pi_i(\mathcal{A}) \cap \pi'_i(\mathcal{A})|_F = i$  for  $i = 1, 2, 3$  with flower at 1.  $\square$

**Theorem 15**  $J_F^3(10) = I_F^3(10)$ .

**Proof:** By Corollary 1,  $[47, 52] \cup \{54, 60\} \subseteq J_F^3(10)$ . By Theorem 4 for  $r = 3$ , and Theorems C and 11,  $[0, 44] \subseteq J_F^3(10)$  and in Theorem 2 for  $r = 3$ , with the aim of Theorems A, 1 and 10, we can assume that  $b = 33$ ,  $a_1 = 1$ ,  $a_2 = 7$  and  $a = 4$ , so  $45 \in J_F^3(10)$ . In Lemma 2.8 of Milici and Quattrocchi (1987), the three systems with 3-way intersection number 56, actually have 3-way flower intersection number 46, so  $46 \in J_F^3(10)$ .  $\square$

Let us call the following Latin square,  $L$ , of order 16 on symbols  $0, 1, \dots, 15$ , *special* of order 16, where  $A$  and  $C$  are Latin squares of order 4 on symbol set  $\{4, 5, 6, 7\}$  and  $B$  is a Latin square of order 8 on symbol set  $\{8, 9, \dots, 15\}$ .

$$L = \begin{array}{|c|c|c|c|c|c|} \hline \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 0 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 0 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 0 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 0 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 0 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & 2 \\ \hline \end{array} \\ \hline \end{array}$$

Denote by  $M$  the set of integers  $m$  for which there exists a collection of three special Latin squares of order 16 which pairwise agree in exactly  $m$  of the 112 cells above the  $2 \times 2$  diagonal blocks.

**Lemma 7**  $M = [16, 97] \cup \{100, 103, 112\}$ .

**Proof:** Let  $m \in M$ , write  $m = a + b + c + d$  where  $a, c \in J^3(4)$ ,  $b \in J^3(8)$  and  $d = 16$  (the number of cells which contain elements 2 and 3). By Theorem C  $J^3(4) = \{0, 1, 4, 16\}$  and  $J^3(8) = [0, 49] \cup \{52, 55, 64\}$ , so  $m \in [16, 97] \cup \{100, 103, 112\}$ .  $\square$

**Theorem 16**  $J_F^3(15) = I_F^3(15)$ .

**Proof:** Let  $M$  be the set of integers  $m$  for which there exists a collection of three special Latin squares of order 16,  $L_{ij}^{(n)}$  for  $n = 1, 2, 3$ , which pairwise agree in exactly  $m$  of the 112 cells above the  $2 \times 2$  diagonal blocks. We construct a collection of three  $STS(31)$ s with 3-way flower intersection number  $h$  for  $h \in [16, 125] \cup \{128, 131, 134, 140\}$ . Write  $h = \ell + m$  where  $\ell \in J_F^3(7)$  and  $m \in M$ . Let  $X_1 = \{\infty_i : 1 \leq i \leq 15\}$ ,  $X_2 = \{1, 2, \dots, 16\}$  and  $(X_1, \mathcal{B}_n)$  for  $n = 1, 2, 3$  be three  $STS(15)$ s with 3-way intersection number  $\ell + 7$  where seven of these will constitute the flower at  $\infty_1$ . It is not hard to check that  $(X_1 \cup X_2, \mathcal{B}_n \cup \mathcal{C}_n)$  for  $n = 1, 2, 3$  are three  $STS(31)$ s, where  $\mathcal{C}_n = \{\{\infty_s, i, j\} | \ell_{ij}^{(n)} = s, 1 \leq i < j \leq 16, 1 \leq s \leq 15\}$  which have 3-way intersection number  $\ell + m + 15$ , fifteen of these will constitute the flower at  $\infty_1$ . So  $h = \ell + m \in J_F^3(15)$ . By Theorem 13,  $\ell \in [0, 8] \cup [10, 13] \cup \{16, 22, 28\}$  and by Lemma 7,  $m \in [16, 97] \cup \{100, 103, 112\}$ . So  $[16, 125] \cup \{128, 131, 134, 140\} \subseteq J_F^3(15)$ . By Theorem 5 for  $r = 7$ ,  $[0, 15] \subseteq J_F^3(15)$ . Existence of the remaining flower intersection numbers is guaranteed by Corollary 1.  $\square$

**Theorem 17**  $I_F^3(24) \setminus [1, 15] \subseteq J_F^3(24)$ .

**Proof:** By Lemma 2,  $0 \in J_F^3(24)$  and by Corollary 1,  $[336, 360] \cup \{362, 368\} \subseteq J_F^3(24)$ . There exists a  $\{4\}$ -GDD of type  $3^4 6^2$  Kreher and Stinson (1997). All input designs required in Theorem 8 to achieve remaining intersection numbers, is guaranteed by Theorems 10, 11 and 12.  $\square$

**Theorem 18**  $J_F^3(60) = I_F^3(60)$ .

**Proof:** By Corollary 1,  $[2280, 2352] \cup \{2354, 2360\} \subseteq J_F^3(60)$ . There exists a  $\{4\}$ -GDD of type  $6^{10}$  Wei and Ge (2014). All input designs required in Theorem 8 to achieve remaining intersection numbers, is guaranteed by Theorems 11 and 12.  $\square$

**Theorem 19**  $J_F^3(132) = I_F^3(132)$ .

**Proof:** By Corollary 1,  $[11352, 11520] \cup \{11522, 11528\} \subseteq J_F^3(132)$ . There exists a  $\{4\}$ -GDD of type  $9^{12} 24^1$  Wei and Ge (2013). All input designs required in Theorem 8 to achieve remaining intersection numbers, is guaranteed by Theorems 11, 14 and 17.  $\square$

## 4 Main results

Now, we are in position to present the main theorem.

**Theorem 20 (Main Theorem)**

Let  $S^3[m]$  denote the set of non-negative integers less than or equal to  $m$ , with the exception of  $m - 1$ ,  $m - 2$ ,  $m - 3$ ,  $m - 4$ ,  $m - 5$  and  $m - 7$  and let  $I_F^3(n) = S^3[\frac{2n(n-1)}{3}]$ . For  $n \equiv 0, 1 \pmod{3}$ ,  $n \geq 10$  but  $n \neq 24$ ,  $J_F^3(n) = I_F^3(n)$ .  $J_F^3(3) = \{4\}$ ,  $J_F^3(4) = \{0, 8\}$ ,  $[0, 5] \cup \{7, 20\} \subseteq J_F^3(6) \subseteq [0, 7] \cup \{20\}$ ,  $[0, 8] \cup [10, 13] \cup \{16, 22, 28\} \subseteq J_F^3(7) \subseteq [0, 16] \cup \{22, 28\}$ ,  $I_F^3(9) \setminus \{4, 6, 7, 9, 11, 34, 35\} \subseteq J_F^3(9)$  and  $I_F^3(24) \setminus [1, 15] \subseteq J_F^3(24)$ .

**Proof:** The proof is based on recursive constructions where expressed before. For any admissible  $n$ , consider the following five cases. It is worth mentioning that in all cases, to construct  $STS(2n + 1)$ , recursive constructions may use small  $STS$ s of order  $2m + 1$  where  $n$  and  $m$  are not congruent modulo 9.

1.  $n \equiv 1, 4 \pmod{9}$

$J_F^3(1)$ ,  $J_F^3(4)$  and  $J_F^3(10)$  have been obtained in Theorems 9, 11 and 15, so let  $n \geq 13$ . Let  $n = 3r + 1$ , clearly  $r$  is admissible, since  $n \equiv 1, 4 \pmod{9}$ . All required objects in Theorem 2, is guaranteed by Theorems A and 1 and the 3-way flower intersection numbers of  $STS(2r + 1)$ .

2.  $n \equiv 0, 3 \pmod{9}$

$J_F^3(3)$  and  $J_F^3(9)$  have been obtained in Theorems 10 and 14, so let  $n \geq 12$ . Let  $n = 3r$ , clearly  $r$  is admissible, since  $n \equiv 0, 3 \pmod{9}$ . All required objects in Theorem 3, is guaranteed by Theorem C and the 3-way flower intersection numbers of  $STS(2r + 1)$ .

3.  $n \equiv 7 \pmod{9}$

$J_F^3(7)$  has been obtained in Theorem 13, so let  $n \geq 16$ . Let  $n = 3r + 1$ , clearly  $r \equiv 0, 2 \pmod{3}$  (it means  $r + 1$  is admissible), since  $n \equiv 7 \pmod{9}$ . All required objects in Theorem 4, is guaranteed by Theorem C and the 3-way flower intersection numbers of  $STS(2r + 3)$ .

4.  $n \equiv 15 \pmod{18}$  (It means  $n = 9k + 6$ , where  $k$  is odd).

$J_F^3(15)$  has been obtained in Theorem 16, so let  $n \geq 33$ . Let  $n = 2r + 1$ , clearly  $r$  is admissible, since  $n \equiv 15 \pmod{18}$ . All required objects in Theorem 5, is guaranteed by the 3-way flower intersection numbers of  $STS(2r + 1)$ .

5.  $n \equiv 6 \pmod{18}$  (It means for  $n = 9k + 6$ , where  $k$  is even).

$J_F^3(6)$ ,  $J_F^3(24)$ ,  $J_F^3(60)$  and  $J_F^3(132)$  have been obtained in Theorems 12, 17, 18 and 19, so let  $n \geq 42$  and  $n \notin \{60, 132\}$ . Let  $n = 6t$ , clearly  $t \equiv 1 \pmod{3}$ ,  $t \geq 7$  and  $t \notin \{10, 22\}$ . By Theorem 6, there exists a  $B(\{6, t\}, 1, 6t)$  and since  $t$  and 6 are admissible, by Theorem 7, there exists an  $STS(12t + 1)$ . All required objects in Theorem 8, is guaranteed by Theorem 12 and the 3-way flower intersection numbers of  $STS(2t + 1)$ .

□

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