

# The Chromatic Number of the Disjointness Graph of the Double Chain

Ruy Fabila-Monroy<sup>1\*</sup> Carlos Hidalgo-Toscano<sup>2†</sup> Jesús Leños<sup>3</sup> Mario Lomelí-Haro<sup>4‡</sup>

<sup>1</sup> Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, México.

<sup>2</sup> Centro de Investigación e Innovación en Tecnologías de la Información y Comunicación, México

<sup>3</sup> Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas, México

<sup>4</sup> Instituto de Física, Universidad Autónoma de San Luis Potosí, México

received 9<sup>th</sup> May 2018, revised 20<sup>th</sup> Feb. 2019, 3<sup>rd</sup> Sep. 2019, accepted 13<sup>th</sup> Feb. 2020.

Let  $P$  be a set of  $n \geq 4$  points in general position in the plane. Consider all the closed straight line segments with both endpoints in  $P$ . Suppose that these segments are colored with the rule that disjoint segments receive different colors. In this paper we show that if  $P$  is the point configuration known as the double chain, with  $k$  points in the upper convex chain and  $l \geq k$  points in the lower convex chain, then  $k + l - \lfloor \sqrt{2l + \frac{1}{4}} - \frac{1}{2} \rfloor$  colors are needed and that this number is sufficient.

**Keywords:** chromatic number, double chain, edge disjointness graph

## 1 Introduction

Throughout this paper,  $P$  is a set of  $n \geq 4$  points in general position in the plane. The *edge disjointness graph*,  $D(P)$ , of  $P$  is the graph whose vertices are all the closed straight line segments with endpoints in  $P$ ; two of which are adjacent in  $D(P)$  if and only if they are disjoint. The edge disjointness graph and other similar graphs were introduced by Araujo et al. (2005), as geometric analogs of the well known Kneser graphs. Let  $m$  and  $k$  be positive integers with  $k \leq m/2$ . We recall that the *Kneser graph*  $KG(m; k)$  is the graph whose vertices are all the  $k$ -subsets of  $\{1, 2, \dots, m\}$ ; two of which are adjacent if and only if they correspond to disjoint  $k$ -subsets.

The *chromatic number* of a graph  $G$  is the minimum number of colors needed to color its vertices so that adjacent vertices receive different colors; it is denoted by  $\chi(G)$ . Kneser (1956) posed the problem of finding the chromatic number of the Kneser graph. He conjectured that

$$\chi(KG(n; k)) = n - 2k + 2$$

\* was partially supported by CONACyT of Mexico grant 253261.

† was partially supported by CONACyT of Mexico grant 253261

‡ Gratefully acknowledges CONACyT for doctorate scholarship 163435.

for  $n \geq 2k - 1$ . The upper bound can be shown with simple combinatorial arguments. The lower bound was proved by Lovász (1978) using tools from algebraic topology (specifically the Borsuk-Ulam theorem). This is one of the earliest applications of Algebraic Topology to combinatorial problems. For a nice account of this connection see the book of Matoušek (2003).

Recently, Pach and Tomon (2019) have proved that if  $G$  is the disjointness graph of a family of grounded  $x$ -monotone curves such that  $\omega(G) = k$ , then  $\chi(G) \leq \binom{k+1}{2}$ , where  $\omega(G)$  denotes the clique number of  $G$ . We remark that the family of grounded  $x$ -monotone curves play the role of our closed straight line segments.

Clearly, the chromatic number is a well studied parameter of the Kneser graph and its relatives. A general upper bound of

$$\chi(D(P)) \leq \min \left\{ n - 2, n + \frac{1}{2} - \frac{\lfloor \log \log n \rfloor}{2} \right\}$$

was proved by Araujo et al. (2005). They obtained it as follows. Let  $C_n$  be a set of  $n$  points in convex position in the plane. Let

$$f(n) := \chi(D(C_n)).$$

They showed that  $f(n) \leq n - \frac{\lfloor \log_2 n \rfloor}{2}$ . Erdős and Szekeres (1935) proved that  $P$  has a subset of at least  $m = \lfloor \log_2(n)/2 \rfloor$  points in convex position. The segments with endpoints in this subset are colored using  $f(m)$  colors; the remaining segments are colored by deleting the remaining points one by one and in the process coloring all the segments with this point as an endpoint with the same new color.

The exact value of  $f(n)$  has been computed. It is now known that

$$f(n) = n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor. \quad (1)$$

Indeed, Fabila-Monroy and Wood (2011) showed that the expression on the right hand side of Eq. (1) is a lower bound for  $f(n)$ ; and Jonsson (2011) established Eq. (1) by proving that such an expression is also an upper bound for  $f(n)$ . Repeating the above arguments, we have that

$$\chi(D(P)) \leq n - \left\lfloor \sqrt{\log n + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

As far as we know  $\{C_n\}_{n=1}^{\infty}$  is the only infinite family of point configurations<sup>(i)</sup> for which the exact value of the chromatic number of their disjointness graph has been computed. In this paper we compute the chromatic number of the disjointness graph of another infinite family of point configurations, called the double chain.

We now define this family. A  $k$ -cup is a set of  $k$  points in convex position in the plane such that its convex hull is bounded from above by an edge. Similarly, an  $l$ -cap is a set of  $l$  points in convex position whose convex hull is bounded from below by an edge.

**Definition 1** For  $k \leq l$ , a  $(k, l)$ -double-chain is the disjoint union of two point sets  $U$  and  $L$  such that:

- $U$  is a  $k$ -cup and  $L$  is an  $l$ -cap;

---

<sup>(i)</sup> with different order types, that is.

- every point of  $L$  is below every straight line determined by two points of  $U$ ; and
- every point of  $U$  is above every straight line determined by two points of  $L$ ;

In Figure 1 we illustrate a  $(5, 7)$ -double-chain and some of its edges. Note that Figure 1 suggests a natural way to construct a  $(k, l)$ -double-chain for any pair  $(k, l)$  of admissible integers. Moreover, it is a routine exercise to show that any two  $(k, l)$ -double-chains are the same (up to order type isomorphism). In view of this, we shall use  $C_{k,l}$  to denote any  $(k, l)$ -double-chain, and we often refer to it simply as the *double chain*. Each of the geometric properties of  $C_{k,l}$  in next remark follows easily from its definition, and they will be often used, without explicit mention, in our arguments.

**Remark 1** *Let  $U$  and  $L$  be the  $k$ -cup and the  $l$ -cap of  $C_{k,l}$ , respectively. Then the following holds:*

- *If  $U'$  and  $L'$  are proper subsets of  $U$  and  $L$ , respectively, then the set of points that results from  $C_{k,l}$  by deleting the points in  $U' \cup L'$  remains a double chain.*
- *Any straight line segment in the frontier of the convex hull of  $U$  (respectively,  $L$ ) does not cross any other straight line segment joining two points of  $L$  (respectively,  $U$ ) See Figure 1.*
- *Let  $g$  be a straight line segment with an endpoint in  $U$  and the other one in  $L$ , and let  $f$  be a straight line segment joining two points of  $X \in \{U, L\}$ . If  $g$  and  $f$  intersect each other, then they do at a common endpoint.*

The double chain was first introduced by Hurtado et al. (1999) as an example of a set of  $n$  points (in general position) whose flip graph of triangulations has diameter  $\Theta(n^2)$ . Since then the double chain has been used as an extremal example in various problems on point sets, see for example Aichholzer et al. (2007, 2015, 2008); Cibulka et al. (2009, 2013); Dumitrescu et al. (2013); García et al. (2000).

In this paper we show (Theorem 1) that for  $l \geq 3$

$$\chi(D(C_{k,l})) = k + f(l).$$

Note that for  $n$  even and  $k = l = n/2$ ,  $C_{\frac{n}{2}, \frac{n}{2}}$  is a set of  $n$  points for which

$$\chi(D(C_{\frac{n}{2}, \frac{n}{2}})) = n - \left\lfloor \sqrt{n + \frac{1}{4}} - \frac{1}{2} \right\rfloor \geq f(n) + c\sqrt{n},$$

for some positive constant  $c$ . So, to color the disjointness graph of  $C_{\frac{n}{2}, \frac{n}{2}}$ , more colors are needed than to color the disjointness graph of  $C_n$ . We conjecture that for every  $n \geq 3$ , and for every set  $P$  of  $n$  points

$$\chi(D(P)) \geq f(n).$$

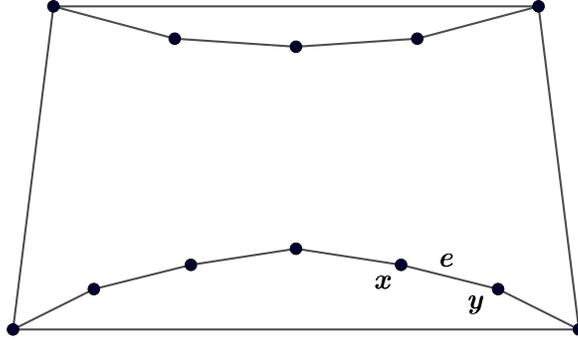
## 2 Preliminary Results and Definitions

Before proceeding we present some results and definitions. A *geometric graph* is a graph whose vertices are points in the plane, and whose edges are straight line segments joining these points. For exposition purposes, we abuse notation and use  $P$  to refer to the complete geometric graph with vertex set equal to  $P$ . Thus,  $\chi(D(P))$  is the minimum number of colors in an edge-coloring of  $P$  in which any two edges belonging to the same chromatic class cross or are incident.

Let  $c$  be a proper vertex coloring<sup>(ii)</sup> of  $D(P)$  and let  $S$  be a chromatic class of  $D(P)$  in this coloring.

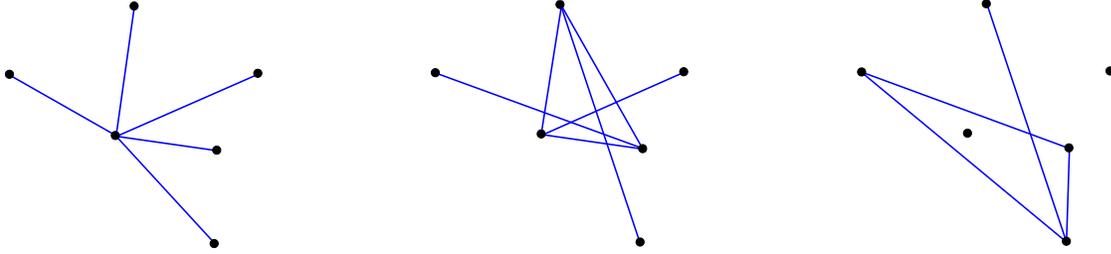
---

<sup>(ii)</sup> a coloring in which pairs of adjacent vertices receive different colors.



**Fig. 1:** This is a drawing of  $C_{5,7}$  and some of its edges. The edge  $e = xy$  is in the convex hull of  $L$  and is not crossed by any of remaining edges of  $C_{5,7}$ . Thus any edge receiving the same color as  $e$  in any proper coloring of  $D(C_{5,7})$  must be incident with exactly one of  $x$  or  $y$ .

We say that  $S$  is a *star* if all of its edges share a common vertex, which we call an *apex*. If  $S$  is not a star then it is a *thrackle*. See Figure 2.



**Fig. 2:** A star and two distinct thrackles of the same set of 6 points.

**Proposition 1** Let  $c$  be an optimal coloring of  $D(P)$  and let  $S_1, \dots, S_r$  be different stars of  $c$  with apices  $v_1, \dots, v_r$ , respectively. Then

$$\chi(D(P \setminus \{v_1, \dots, v_r\})) = \chi(D(P)) - r.$$

**Proof:** Suppose that there exists a coloring  $\chi(D(P \setminus \{v_1, \dots, v_r\}))$  with less than  $\chi(D(P)) - r$  colors. Extend this coloring to a coloring of  $D(P)$  by using a new different color for each  $S_i$ . This produces a coloring of  $D(P)$  with less than  $\chi(D(P))$  colors.  $\square$

Let

$$g(n) := \max \left\{ i : i \in \mathbb{Z}^+, \binom{i}{2} \leq n \right\}. \quad (2)$$

Jonsson (2011) observed, in the remark following Theorem 1.1, that

$$f(n) = n - g(n) + 1.$$

This implies the following result.

**Proposition 2**

$$f(n+1) = \begin{cases} f(n) & \text{if } n = \binom{i}{2} - 1 \text{ for some positive integer } i \text{ and} \\ f(n) + 1 & \text{otherwise.} \end{cases}$$

Therefore,  $f(n+k) - f(n) \leq k$ , for every nonnegative integer  $k$ .

**Proposition 3** *In every optimal coloring of  $D(C_n)$  there is at most one chromatic class consisting of a single edge of  $P$ .*

**Proof:** Suppose for a contradiction that for some  $n$  there exists an optimal coloring  $c$  of  $D(C_n)$  with two chromatic classes,  $S_1$  and  $S_2$ , consisting of a single edge. Furthermore, suppose that  $n$  is the minimum such integer. The minimality of  $n$  and Proposition 1 imply that  $S_1$  and  $S_2$  are the only stars of  $c$ .

Let  $T_1, \dots, T_k$  be the chromatic classes of  $c$  different from  $S_1$  and  $S_2$ . Note that these are thrackles. Fabila-Monroy and Wood (2011) showed that  $T_1 \cup \dots \cup T_k$  consists of at most  $kn - \binom{k}{2}$  edges of  $C_n$ . Therefore,  $\binom{n}{2} \leq kn - \binom{k}{2} + 2$ . This implies that  $(n-k)^2 \leq n+k+4$ . Since  $k = f(n) - 2 = n - g(n) - 1$ , we have that  $(g(n) + 1)^2 \leq 2n - (g(n) + 1) + 4$ . Rearranging terms in the previous inequality we arrive at  $\binom{g(n)+1}{2} \leq n - g(n) + 1$ . By the definition of  $g(n)$ ,  $\binom{g(n)+1}{2} > n$ . Therefore,  $g(n) < 1$  – a contradiction.  $\square$

### 3 The Chromatic Number of $D(C_{k,l})$

It is relatively easy to find an optimal coloring of  $D(C_{k,l})$ .

**Lemma 1** *For all positive integers  $k \leq l$ ,*

$$\chi(D(C_{k,l})) \leq k + f(l).$$

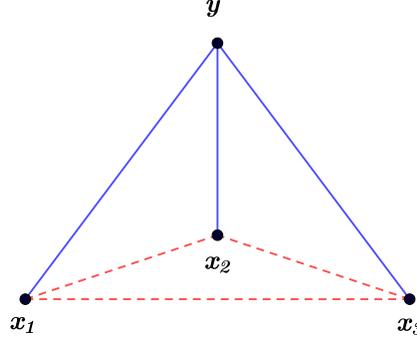
**Proof:** Color the edges of  $L$  of  $C_{k,l}$  with  $f(l)$  colors. For each of the  $k$  vertices in  $U$ , color the edges incident to them, that have not been colored yet, with a new color. This yields a proper coloring of  $D(C_{k,l})$  with  $k + f(l)$  colors.  $\square$

The following lemma is needed to prove the lower bound on  $\chi(D(C_{k,l}))$ .

**Lemma 2** *If  $l \geq 3$ , then  $\chi(D(C_{1,l})) \geq 1 + f(l)$ .*

**Proof:**

From Eq. (1) we know that  $f(3) = 1$ . Now we shall show that  $\chi(D(C_{1,3})) = 1 + f(3) = 2$ . The proper coloring of  $D(C_{1,3})$  given in Figure 3 shows that  $\chi(D(C_{1,3})) \leq 2$ . On the other hand, since the straight line segments  $yx_2$  and  $x_1x_3$  are disjoint, then they cannot receive the same color in any proper coloring of  $D(C_{1,3})$ . This implies that  $\chi(D(C_{1,3})) \geq 2$ , as required.



**Fig. 3:** A proper coloring of  $D(C_{1,3})$ .

Assume that  $l \geq 4$  and that the result holds for smaller values of  $l$ . Let  $c$  be an optimal coloring of  $D(C_{1,l})$ . We may assume that  $c$ , when restricted to  $L$  uses  $f(l)$  colors, as otherwise we are done.

Suppose that  $c$  has a star with apex  $v$ . Then by Proposition 1, the graph  $D(C_{1,l} \setminus \{v\})$  can be properly colored with one color less. If  $v$  is the single point in  $U$ , then  $c$  uses at least  $f(l) + 1$  colors. If  $v$  is in  $L$ , then by induction,  $c$  uses at least  $\chi(D(C_{1,l-1})) + 1 = f(l-1) + 2$  colors. By Proposition 2 this is at least  $f(l) + 1$ . Then we can assume that all chromatic classes of  $c$  are thrackles.

We claim that if all the edges incident to the single vertex  $u$  in  $U$  are in the same chromatic class  $H$ , then  $H$  is a star with apex  $u$ . Indeed, let  $h_1, h_2, \dots, h_l$  be the edges incident with  $u$ , and let  $w_1, w_2, \dots, w_l \in L$  be their respective endpoints. Then  $\{h_1, h_2, \dots, h_l\} \subseteq H$  and  $L = \{w_1, w_2, \dots, w_l\}$ . Now, suppose by way of contradiction that there is an edge  $w_i w_j$  belonging to  $H$ . Since  $l \geq 3$ , then there exists a point  $w_k \in L \setminus \{w_i, w_j\}$ . The existence of such  $w_k$  and the fact that  $L$  is an  $l$ -cap imply that  $h_k$  is disjoint from  $w_i w_j$ . But this contradicts that  $h_k$  and  $w_i w_j$  are in the chromatic class  $H$ . Thus we may assume that there are two edges incident to  $u$  with different color.

Let  $e_1$  and  $e_2$  be two edges incident to  $u$  of different colors. Suppose that  $e_1$  is colored *red* and  $e_2$  is colored *blue*. Let  $v_1$  and  $v_2$  be their respective endpoints in  $L$ . Since the *red* and *blue* edges are not stars, there exist edges  $f_1$  and  $f_2$ , both with endpoints in  $L$ , of colors *red* and *blue*, respectively. Note also that all the *red* edges of  $L$  must be incident to  $v_1$  and that all the *blue* edges of  $L$  must be incident to  $v_2$ . Since the *red* and the *blue* edges are not stars, then there exist other edges incident to  $u$  of colors *red* and *blue*. Let  $g_1$  and  $g_2$  be such edges, and suppose that  $g_1$  is *red* and that  $g_2$  is *blue*.

We claim that  $f_1$  and  $f_2$  are the only *red* and *blue* edges in  $L$ . Seeking a contradiction, suppose that there exists a *red* edge  $f'_1 \neq f_1$  with endpoints in  $L$ . From previous paragraph we know that both  $f_1$  and  $f'_1$  are incident with  $v_1$ . Let  $v$  and  $v'$  be the other endpoints of  $f_1$  and  $f'_1$ , respectively. Then  $v \neq v'$ , and as a consequence, there is a  $w \in \{v, v'\}$  such that  $w$  is not in  $g_1$ . This implies that the element of  $\{f_1, f'_1\}$  that is incident with  $w$  is disjoint from  $g_1$ . This last statement contradicts the assumption that  $f_1, f'_1$ , and  $g_1$  are all *red*. A totally analogous argument shows that  $f_2$  is the only *blue* edge in  $L$ . Therefore,  $c$

when restricted to  $L$  is an optimal coloring of  $C_l$  in which two chromatic classes consist of a single edge. The last conclusion contradicts Proposition 3, yielding that the restriction of  $c$  to  $L$  is not optimal. This contradicts our earlier supposition that  $c$ , when restricted to  $L$  uses  $f(l)$  colors.  $\square$

**Lemma 3** *If  $l \geq 3$ , then  $\chi(D(C_{k,l})) \geq k + f(l)$ .*

**Proof:** Suppose for a contradiction that there exist  $k$  and  $l$  such that there exists an optimal coloring  $c$  of  $D(C_{k,l})$  with less than  $k + f(l)$  colors. Furthermore suppose that  $k$  and  $l$  are such that  $k + l$  is minimum. It can be checked by hand that the theorem holds for  $k \leq l \leq 3$ , and by Lemma 2 it holds for  $k = 1$ . Therefore,  $k \geq 2$  and  $l \geq 4$ .

Suppose that  $c$  has a star with apex  $v$ . By Proposition 1,  $D(C_{k,l} \setminus \{v\})$  can be colored with less than  $k + f(l) - 1$  colors. If  $v$  is in  $U$  then we have  $C_{k,l} \setminus \{v\} = C_{k-1,l}$  and  $D(C_{k-1,l})$  can be colored with less than  $(k - 1) + f(l)$  colors; this contradicts the minimality of  $k + l$ . If  $k = l$ , we can assume without loss of generality that  $v$  is in  $U$ . Thus, we assume that  $v$  is in  $L$  and that  $k < l$ . Then  $C_{k,l} \setminus \{v\} = C_{k,l-1}$  and, by Proposition 1,  $D(C_{k,l-1})$  can be colored with less than  $k + f(l) - 1$  colors. By Proposition 2, we know that  $k + f(l) - 1 \leq k + f(l - 1)$ ; this contradicts the minimality of  $k + l$ . Thus we can assume that all the chromatic classes of  $c$  are thrackles.

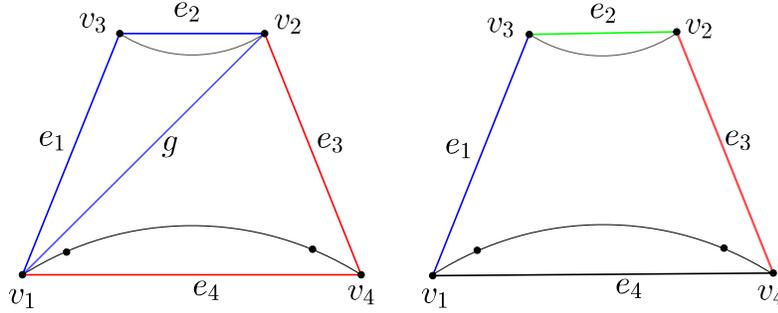
Note that there are exactly four edges  $e_1, e_2, e_3$  and  $e_4$  in the convex hull of  $C_{k,l}$ , and let  $v_1, v_2, v_3$  and  $v_4$  be the set of endpoints of  $e_1, e_2, e_3$  and  $e_4$ . Since each  $e_i$  does not cross any other edge, then every edge of the same color as  $e_i$  must be incident to one of the endpoints of  $e_i$ . Let  $\gamma$  be the number of different colors received by these four edges in  $c$ . Note that  $\gamma = 2, 3$  or  $4$ .

Suppose that  $\gamma = 2$ . Without loss of generality assume that  $e_1$  and  $e_2$  are *blue*;  $e_3$  and  $e_4$  are *red*;  $v_3$  is the common endpoint of  $e_1$  and  $e_2$ ; and that  $v_4$  is the common endpoint of  $e_3$  and  $e_4$ . See Figure 4 (left). We claim that at least one of these two chromatic classes is a star. Suppose that the *blue* chromatic class is not a star. Then there is a *blue* edge  $g$  which is not incident to  $v_3$ . As neither  $e_1$  nor  $e_2$  is crossed by any other edge, then such a  $g$  must be  $v_1v_2$ . Since  $g$  is *blue* and it is the the only edge that intersects both  $e_3$  and  $e_4$  but not at  $v_4$ , then the *red* chromatic class is a star with apex  $v_4$ , a contradiction to the assumption that all the chromatic classes of  $c$  are thrackles.

Suppose that  $\gamma = 4$ . Then there are no edges with the same color as any of the  $e_i$  in  $C_{k,l} \setminus \{v_1, v_2, v_3, v_4\}$ . Therefore,  $c$  when restricted to the subgraph  $D(C_{k,l} \setminus \{v_1, v_2, v_3, v_4\})$  uses less than  $k + f(l) - 4$  colors. See Figure 4 (right). Note that  $C_{k,l} \setminus \{v_1, v_2, v_3, v_4\} = C_{k-2,l-2}$ ; by Proposition 2,  $k + f(l) - 4$  is at most  $(k - 2) + f(l - 2)$ ; this contradicts the minimality of  $k + l$ .

Finally, suppose that  $\gamma = 3$ . Then exactly two of the  $e_i$  are of the same color; moreover these edges share an endpoint. Without loss of generality assume that: these edges are  $e_1$  and  $e_2$ ; their common endpoint is  $v_3$ ; and that the other endpoints of  $e_1$  and  $e_2$  are  $v_1$  and  $v_2$ , respectively. Assume that  $e_1$  and  $e_2$  are colored *blue*. Since all the chromatic classes in  $c$  are thrackles then the edge  $v_1v_2$  must also be colored *blue*. Let  $S := U$  if  $v_3$  is in  $U$  and let  $S := L$  if  $v_3$  is in  $L$ . Without loss of generality assume that  $v_1$  is not in  $S$ . Note that any other *blue* edge must be incident to  $v_3$  and its other endpoint is not in  $S$ . Now we recolor *blue* all the edges incident with  $v_3$  and having the other endpoint not in  $S$ . See Figure 5.

First let us assume that  $|S| \geq 3$ . We only show the case in which  $S = U$ . The proof for the case  $S = L$  is totally analogous. Then  $S \setminus \{v_2, v_3\}$  is not empty. Let  $w$  be the vertex in  $S \setminus \{v_2, v_3\}$  which is the closest to  $v_3$ . See Figure 5 (left). From the definition of  $w$  we have that the edge  $v_3w$  does not cross any other edge, and in particular  $v_3w$  cannot be *blue*. Suppose that  $v_3w$  is *red*. If  $v_1w$  is also colored *red*,



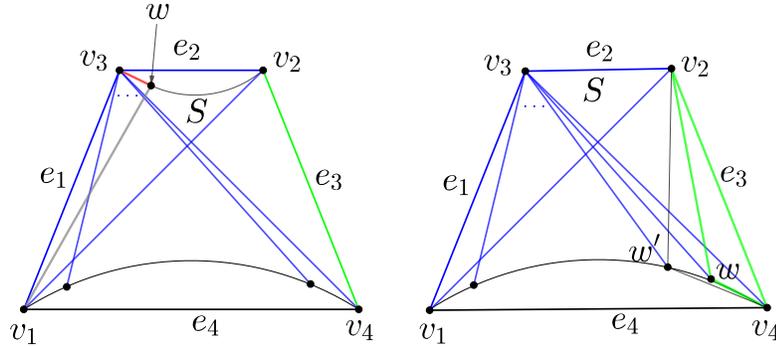
**Fig. 4:** The cases  $\gamma = 2$  (left) and  $\gamma = 4$  (right) in the proof of Lemma 3.

then the *red* chromatic class is a star, a contradiction. Thus  $v_1w$  is not red. Since  $v_1w$  cannot be colored *blue*, we assume that it is colored *gray*. See Figure 5 (left). Since  $v_1w$  is crossed only by *blue* edges, then any other *gray* edge must be incident to  $v_1$  or  $w$ . Also note that every *red* edge must be incident to  $v_3$  or  $w$ . These observations together imply that  $c$  when restricted to  $C_{k,l} \setminus \{v_1, w, v_3\}$  is a coloring of  $D(C_{k,l} \setminus \{v_1, w, v_3\})$  with less than  $k + f(l) - 3$  colors. Then  $C_{k,l} \setminus \{v_1, w, v_3\} = C_{k-2, l-1}$ . By Proposition 2,  $k + f(l) - 3 \leq (k - 2) + f(l - 1)$ ; this contradicts the minimality of  $k + l$ .

Now suppose that  $|S| = 2$ . Then  $S = U = \{v_2, v_3\}$ . By symmetry, we may assume that  $e_1, e_2, e_3$  and  $e_4$  are placed as in Figure 5 (right), and that  $e_3 = v_2v_4$  is *green*. Let  $w$  be the vertex in  $L \setminus \{v_1, v_4\}$  which is closest to  $v_4$ . Then  $wv_4$  does not cross any other edge, and any edge crossing  $wv_2$  is *blue*. Also note that  $wv_2$  cannot be *blue*. If  $wv_2$  and  $wv_4$  receive the same color, different from *green*, then the chromatic class containing them must be a star. Similarly, if  $wv_2, wv_4$  and  $v_2v_4$  receive distinct colors, then we can proceed as in previous paragraph and deduce that  $C_{1, l-2} = C_{2, l} \setminus \{v_2, w, v_4\}$  is a counterexample that contradicts the minimality of  $k + l$ .

Thus we may assume that at least one of  $wv_2$  or  $wv_4$  is *green*. We claim that both are *green*. Because  $v_2v_4$  is not crossed by any edge, then any other *green* edge must be adjacent to exactly one of  $v_2$  or  $v_4$ . This and the fact that the *green* chromatic class is not a star, imply that for each  $v \in \{v_2, v_4\}$  there exists at least one *green* edge distinct of  $v_2v_4$  which is incident with  $v$ . Let  $v_2x$  and  $v_4y$  be any couple of such *green* edges. Clearly,  $x, y \in L \setminus \{v_4\}$ . Since the *green* edges incident with  $v_2$  are crossed only by *blue* edges, then we must have that  $x = y$ . This and the fact that at least one of  $wv_2$  or  $wv_4$  is *green* imply that  $w = x = y$ . This implies that the *green* chromatic class consists precisely of  $wv_2, wv_4$  and  $v_2v_4$ .

Let  $w'$  be the vertex in  $L \setminus \{v_1, w, v_4\}$  which is the closest to  $w$ . See Figure 5 (right). Note that  $ww'$  does not cross any other edge, and that any edge crossing  $w'v_2$  is *blue*. Also note that none of  $w'v_2$  and  $w'v_4$  can be *blue* or *green*. Again, if  $w'v_2$  and  $w'v_4$  receive the same color, then the chromatic class containing them must be a star. Thus we assume that  $w'v_2$  and  $w'v_4$  have distinct colors. This implies that the color of at least one of  $w'v_2$  or  $w'v_4$  is different from the color of  $ww'$ . Let  $v \in \{v_2, v_4\}$  such that  $c(ww') \neq c(w'v)$ . Since none of  $ww'$  and  $w'v$  can be *green*, then the colors of  $ww', wv$ , and  $w'v$



**Fig. 5:** Here we illustrate the only two (up to symmetry) possibilities for the case  $\gamma = 3$ . On the left we have the case in which  $|S| \geq 3$  and  $S = U$ . On the right we have the case in which  $|S| = 2$  and hence  $S = U$ .

are distinct. From this and the fact that any edge crossing  $w'v$  is *blue* or incident with  $w$  it follows that  $C_{2,l} \setminus \{v, w, w'\}$  is a counterexample that contradicts the minimality of  $k + l$ . The result follows.  $\square$

Summarizing, we have the following result.

**Theorem 1** For  $l \geq 3$ ,  $\chi(D(C_{k,l})) = k + f(l)$ .

## Acknowledgements

We thank two anonymous referees for their valuable comments and improvements to the presentation.

## References

- O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber. On the number of plane geometric graphs. *Graphs Combin.*, 23(suppl. 1):67–84, 2007. ISSN 0911-0119. doi: 10.1007/s00373-007-0704-5. URL <https://doi.org/10.1007/s00373-007-0704-5>.
- O. Aichholzer, D. Orden, F. Santos, and B. Speckmann. On the number of pseudo-triangulations of certain point sets. *J. Combin. Theory Ser. A*, 115(2):254–278, 2008. ISSN 0097-3165. doi: 10.1016/j.jcta.2007.06.002. URL <https://doi.org/10.1016/j.jcta.2007.06.002>.
- O. Aichholzer, W. Mulzer, and A. Pilz. Flip distance between triangulations of a simple polygon is NP-complete. *Discrete Comput. Geom.*, 54(2):368–389, 2015. ISSN 0179-5376. doi: 10.1007/s00454-015-9709-7. URL <https://doi.org/10.1007/s00454-015-9709-7>.
- G. Araujo, A. Dumitrescu, F. Hurtado, M. Noy, and J. Urrutia. On the chromatic number of some geometric type Kneser graphs. *Comput. Geom.*, 32(1):59–69, 2005. ISSN 0925-7721. doi: 10.1016/j.comgeo.2004.10.003. URL <https://doi.org/10.1016/j.comgeo.2004.10.003>.

- J. Cibulka, J. Kynčl, V. Mészáros, R. Stolař, and P. Valtr. Hamiltonian alternating paths on bicolored double-chains. In *Graph drawing*, volume 5417 of *Lecture Notes in Comput. Sci.*, pages 181–192. Springer, Berlin, 2009. doi: 10.1007/978-3-642-00219-9\_18. URL [https://doi.org/10.1007/978-3-642-00219-9\\_18](https://doi.org/10.1007/978-3-642-00219-9_18).
- J. Cibulka, J. Kynčl, V. Mészáros, R. Stolař, and P. Valtr. Universal sets for straight-line embeddings of bicolored graphs. In *Thirty essays on geometric graph theory*, pages 101–119. Springer, New York, 2013. doi: 10.1007/978-1-4614-0110-0\_8. URL [https://doi.org/10.1007/978-1-4614-0110-0\\_8](https://doi.org/10.1007/978-1-4614-0110-0_8).
- A. Dumitrescu, A. Schulz, A. Sheffer, and C. D. Tóth. Bounds on the maximum multiplicity of some common geometric graphs. *SIAM J. Discrete Math.*, 27(2):802–826, 2013. ISSN 0895-4801. doi: 10.1137/110849407. URL <https://doi.org/10.1137/110849407>.
- P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935. ISSN 0010-437X. URL [http://www.numdam.org/item?id=CM\\_1935\\_\\_2\\_\\_463\\_0](http://www.numdam.org/item?id=CM_1935__2__463_0).
- R. Fabila-Monroy and D. R. Wood. The chromatic number of the convex segment disjointness graph. In *Computational geometry*, volume 7579 of *Lecture Notes in Comput. Sci.*, pages 79–84. Springer, Cham, 2011. doi: 10.1007/978-3-642-34191-5\_7. URL [https://doi.org/10.1007/978-3-642-34191-5\\_7](https://doi.org/10.1007/978-3-642-34191-5_7).
- A. García, M. Noy, and J. Tejel. Lower bounds on the number of crossing-free subgraphs of  $K_N$ . *Comput. Geom.*, 16(4):211–221, 2000. ISSN 0925-7721. doi: 10.1016/S0925-7721(00)00010-9. URL [https://doi.org/10.1016/S0925-7721\(00\)00010-9](https://doi.org/10.1016/S0925-7721(00)00010-9).
- F. Hurtado, M. Noy, and J. Urrutia. Flipping edges in triangulations. *Discrete Comput. Geom.*, 22(3):333–346, 1999. ISSN 0179-5376. doi: 10.1007/PL00009464. URL <https://doi.org/10.1007/PL00009464>.
- J. Jonsson. The exact chromatic number of the convex segment disjointness graph. 2011. URL <https://arxiv.org/abs/1804.01057>.
- M. Kneser. Aufgabe 360. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 58:27, 1956. URL [http://www.digizeitschriften.de/dms/img/?PID=PPN37721857X\\_0058%7Clog20](http://www.digizeitschriften.de/dms/img/?PID=PPN37721857X_0058%7Clog20).
- L. Lovász. Kneser’s conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978. ISSN 0097-3165. doi: 10.1016/0097-3165(78)90022-5. URL [https://doi.org/10.1016/0097-3165\(78\)90022-5](https://doi.org/10.1016/0097-3165(78)90022-5).
- J. Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003. ISBN 3-540-00362-2. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.
- J. Pach and I. Tomon. On the chromatic number of disjointness graphs of curves. In *35th International Symposium on Computational Geometry*, volume 129 of *LIPICs. Leibniz Int. Proc. Inform.*, pages Art. No. 54, 17. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019. doi: 10.1016/j.jctb.2020.02.003. URL <https://doi.org/10.1016/j.jctb.2020.02.003>.