The Erdös-Sós conjecture for geometric graphs
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Let $f(n, k)$ be the minimum number of edges that must be removed from some complete geometric graph $G$ on $n$ points, so that there exists a tree on $k$ vertices that is no longer a planar subgraph of $G$. In this paper we show that 

$$
\left(\frac{1}{3}\right) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2^{\frac{n(n-2)}{k-2}}.
$$

For the case when $k = n$, we show that $2 \leq f(n, n) \leq 3$. For the case when $k = n$ and $G$ is a geometric graph on a set of points in convex position, we completely solve the problem and prove that at least three edges must be removed.

**Keywords:** extremal graph theory, geometric graph, spanning tree

# 1 Introduction

One of the most notorious problems in extremal graph theory is the Erdős–Sós Conjecture, which states that every simple graph with average degree greater than $k - 2$ contains every tree on $k$ vertices as a subgraph. This conjecture was recently proved true for all sufficiently large $k$ (unpublished work of Ajtai, Komlós, Simonovits, and Szemerédi).

In this paper we investigate a variation of this conjecture in the setting of geometric graphs. Recall that a geometric graph $G$ consists of a set $S$ of points in the plane (these are the vertices of $G$), plus a set of straight line segments, each of which joins two points in $S$ (these are the edges of $G$). In particular, any set $S$ of points in the plane in general position (no three of its points are collinear) naturally induces a complete geometric graph. For brevity, we often refer to the edges of this graph simply as edges of $S$. If
A geometric graph is planar if no two of its edges cross each other. An embedding of an abstract graph \( H \) into a geometric graph \( G \) is an isomorphism from \( H \) to a planar geometric subgraph of \( G \). For \( r \geq 0 \), an \( r \)-edge is an edge of \( G \) such that in one of the two open semi-planes defined by the line containing it, there are exactly \( r \) points of \( G \). The convex hull of \( S \) is the intersection of all convex sets containing \( S \). We will frequently need to refer to the vertices and edges at the boundary of the convex hull of \( S \), which for brevity we will denote simply as convex hull vertices and convex hull edges of \( S \).

In this paper all point sets are in general position and \( G \) is a complete geometric graph on \( n \) points. It is well known that for every integer \( 1 \leq k \leq n \), \( G \) contains every tree on \( k \) vertices as a planar subgraph \([3]\). Even more, it is possible to embed any such tree into \( G \), when the image of a given vertex is prespecified \([5]\).

Let \( F \) be a subset of edges of \( G \), which we call forbidden edges. If \( T \) is a tree for which every embedding into \( G \) uses an edge of \( F \), then we say that \( F \) forbids \( T \). In this paper we study the question of what is the minimum size of \( F \) so that there is a tree on \( k \) vertices that is forbidden by \( F \). Let \( f(n,k) \) be the minimum of this number taken over all complete geometric graphs on \( n \) points. As \( f(2,2) = 1 \), \( f(3,3) = 2 \), \( f(4,3) = 3 \), \( f(4,4) = 2 \) and \( f(n,2) = \binom{n}{2} \), we assume through out the paper that \( n \geq 5 \) and \( k \geq 3 \).

We show the following bounds on \( f(n,k) \).

**Theorem 1.1**

\[
\left( \frac{1}{2} \right) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n,k) \leq 2 \frac{n(n-2)}{k-2}
\]

**Theorem 1.2**

\[2 \leq f(n,n) \leq 3\]

In the case when \( G \) is a convex complete geometric graph, we show that the minimum number of edges needed to forbid a tree on \( n \) vertices is three.

An equivalent formulation of the problem studied in this paper is to ask how many edges must be removed from \( G \) so that it no longer contains every planar subtree on \( k \) vertices. A related problem is to ask how many edges must be removed from \( G \) so that it no longer contains any planar subtree on \( k \) vertices. For the case of \( k = n \), in \([6]\), it is proved that if any \( n-2 \) edges are removed from \( G \), it still contains a planar spanning subtree. Note that if the \( n-1 \) edges incident to any vertex of \( G \) are removed, then \( G \) no longer contains a spanning subtree. In general, for \( 2 \leq k \leq n-1 \), in \([1]\), it is proved that if any set of \( \left\lfloor \frac{n(n-k+1)}{2} \right\rfloor - 1 \) edges are removed from \( G \), it still contains a planar subtree on \( k \) vertices. In the same paper it is also shown that this bound is tight—a geometric graph on \( n \) vertices and a subset of \( \left\lfloor \frac{n(n-k+1)}{2} \right\rfloor \) of its edges are shown, so that when these edges are removed, every planar subtree has at most \( k-1 \) vertices. In \([4]\) the authors study the seemingly unrelated (non-geometric) problem of packing two trees into planar (abstract) graphs. That is, given two trees on \( n \) vertices, the authors consider the question of when it is possible to find a planar graph having both of them as spanning trees and in which the trees are edge disjoint. However, although theirs is a combinatorial question rather than

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\(^{10}\) A planar (abstract) graph is an abstract graph that can be embedded in the plane; the embedding may not be unique.
geometric, their Theorem 2.1 implies our Lemma 2.2. We provide a self contained proof of Lemma 2.2 for completeness.

A previous version of this paper appeared in the conference proceedings of EUROCG’12 [2].

2 Spanning Trees

In this section we consider the case when \( k = n \). Let \( T \) be a tree on \( n \) vertices. Consider the following algorithm to embed \( T \) into \( G \). Choose a vertex \( v \) of \( T \) and root \( T \) at \( v \). For every vertex of \( T \) choose an arbitrary order of its children. Suppose that the neighbors of \( v \) are \( u_1, \ldots, u_m \), and let \( n_1, \ldots, n_m \) be the number of nodes in their corresponding subtrees. Choose a convex hull point \( p \) of \( G \) and embed \( v \) into \( p \). Sort the remaining points of \( G \) counter-clockwise by angle around \( p \). A wedge is a region of the plane bounded by two infinite rays sharing a common apex. Choose \( m + 1 \) rays centered at \( p \) so that the wedge between two consecutive rays is convex and between the \( i \)-th ray and the \((i + 1)\)-th ray there are exactly \( n_i \) points of \( G \). Let \( S_i \) be this set of points. A convex hull vertex \( q \) of \( S_i \) is visible from \( p \) if the line segment with endpoints \( p \) and \( q \) intersects the convex hull of \( S \) only at \( q \). For each \( u_i \) choose a convex hull vertex of \( S_i \) visible from \( p \) and embed \( u_i \) into this point. Recursively embed the subtrees rooted at each \( u_i \) into \( S_i \). Note that this algorithm provides an embedding of \( T \) into \( G \). We will use this embedding frequently throughout the paper. See Figure 1.

For every integer \( n \geq 2 \) we define a tree \( T_n \) as follows: If \( n = 2 \), then \( T_n \) consists of only one edge; if \( n \) is odd, then \( T_n \) is constructed by subdividing once every edge of a star on \( \frac{n+1}{2} \) vertices; if \( n \) is even and greater than 2, then \( T_n \) is constructed by subdividing an edge of \( T_{n-1} \). See Figure 2.

We prove the lower bound of \( f(n, n) \geq 2 \) of Theorem 1.2.

**Theorem 2.1** If \( G \) has only one forbidden edge, then any tree on \( n \) vertices can be embedded into \( G \), without using the forbidden edge.

**Proof:** Let \( e \) be the forbidden edge of \( G \). Let \( T \) be a tree on \( n \) vertices. Choose a root for \( T \). Sort the children of each node of \( T \), by increasing size of their corresponding subtrees. Embed \( T \) into \( G \) with the embedding algorithm, choosing at all times the rightmost point (the first point when sorting clockwise around the root) as the root of the next subtree. Suppose that \( e \) is used in this embedding. Let \( e := pq \) so that \( u \) is embedded into \( p \) and \( v \) is embedded into \( q \) (note that \( u \) is the parent of \( v \) in \( T \)).
Suppose that the subtree rooted at $v$ has $m \geq 2$ nodes. In the algorithm, we embedded this subtree into a set of exactly $m$ points. We chose a convex hull point ($q$), of this set visible from $p$ to embed $v$. In this case we may choose another convex hull point visible from $p$ to embed $v$ and continue with the algorithm. Note that $pq$ is no longer used in the final embedding.

Suppose that $v$ is a leaf, and that $v$ has a sibling $v'$ whose subtree has at least two nodes. Then we may interchange $v$ and $v'$ in the order of the children of $u$, so that $e$ is no longer used in the embedding, or if it is, then $v'$ is embedded into $q$, but then we proceed as above.

Suppose that $v$ is a leaf, has at least one sibling and all its siblings are leaves. The subtree rooted at $u$ is a star. We choose a point distinct from $p$ and $q$ in the point set where this subtree is embedded, and embed $u$ into this point. Afterward we join it to the remaining points. This produces an embedding that avoids $e$.

Assume then, that $v$ is a leaf and that it has no siblings. We distinguish the following cases:

1. **$u$ has no siblings.** In this case, the subtree rooted at the parent of $u$ is a path of length two. If $u$ has no grandparent then $n = 3$ and $T$ can be trivially embedded into $G$ without using $e$. Suppose $u$ has a grandparent. In this case there are only four vertices to consider: $v$, $u$, the parent of $u$ and the grandparent of $u$. We keep the current location of the grandparent of $u$, and change the points into which the remaining vertices are embedded. This can always be done so that $e$ is not used in the embedding. All possible cases are shown in Figure [5]

2. **$u$ has a sibling $u'$ whose subtree is not an edge.** We may change the order of the siblings of $u$, with respect to their parent, so that the subtree rooted at $u'$ will be embedded into the point set containing $p$ and $q$. In the initial order—increasing by size of their corresponding subtrees—$u'$ is after $u$. We may assume that in the new ordering, the order of the siblings of $u$ before it, stays the same. Therefore $p$ is the rightmost point of the set into which the subtree rooted at $u'$ will be embedded. Embed $u'$ into $p$. Either we find an embedding not using $e$, or this embedding falls into one of the cases considered before.

3. **$u$ has at least one sibling, and the subtree at every sibling of $u$ is an edge.**

Suppose that $u$ has no grandparent; then $T$ is equal to $T_n$ and $n$ is odd. Let $w$ be the parent of $u$. Embed $w$ into $p$. Let $p_1, \ldots, p_{n-1}$ be the points of $G$ different from $p$ sorted counter-clockwise
Fig. 3: The embedding of a path of length three. The grandparent of $u$ is highlighted and the forbidden edge is dashed.

by angle around $p$; choose $p_1$ so that the angle between two consecutive points is less than $\pi$. Let $u_1, \ldots, u_{(n-1)/2}$ be the neighbors of $w$. Embed each $u_i$ into $p_{2i-1}$ and its child into $p_{2i}$. If $q$ equals $p_{2j-1}$ for some $j$ then embed $u_j$ into $p_{2j}$ and its child into $p_{2j-1}$. This embedding avoids $e$.

Suppose that $w$ is the grandparent of $u$ and let $p'$ be the point into which $w$ is embedded. Let $S$ be the point set into which the subtree rooted at the parent of $u$ is embedded. Note that $S$ has an odd number of points. We replace the embedding as follows. Sort $S$ counter-clockwise by angle around $p'$. Call a point even if it has an even number of points before it in this ordering. Call a point odd if it has an odd number of points before it in this ordering. If $e$ is incident to an odd point, then we embed the parent of $u$ into this point. The remaining subtree rooted at $u$ can be embedded without using $e$. If the endpoints of $e$ are both even, between them there is an odd point. We embed the parent of $u$ into this point. The remaining vertices can be embedded without using $e$ (see Figure 4).

The upper bound of $f(n,n) \leq 3$ of Theorem 1.2 follows directly from Lemma 3.2. Now we prove in Lemma 2.2 and Theorem 2.3 that if $G$ is a convex geometric graph, at least three edges are needed to forbid some tree on $n$ vertices.

**Lemma 2.2** Let $T$ be a tree on $n$ vertices. If $G$ is a convex geometric graph, then $T$ can be embedded into $G$ using less than $\frac{n}{2}$ convex hull edges of $G$.

**Proof:** If $T$ is a star, then any embedding of $T$ into $G$ uses only two convex hull edges. If $T$ is a path then it can be embedded into $G$ using at most two convex hull edges. Therefore, we may assume that $T$ is neither a star nor a path.

Since $T$ is not a path, it has a vertex of degree at least three. Choose this vertex as the root. Since $T$ is not a star, the root has a child whose subtree has at least two nodes. Order the children of $T$ so that this node is first. Embed $T$ into $G$ with the embedding algorithm.

Let $u$ and $v$ be vertices of $T$, so that $u$ is the parent of $v$. Suppose that the subtree rooted at $v$ has at least two nodes. Then in the embedding algorithm we have at least two choices to embed $v$ once the ordering
of the children of $u$ has been chosen. At least one of the choices is such that $uv$ is not embedded into a convex hull edge. Therefore, we may assume that the embedding is such that each convex hull edge used, is incident to a leaf.

Note that every vertex of $T$, distinct from the root, is incident to at most one convex hull edge in the embedding. Since the first child of the root is not a leaf, no convex hull edge is used to embed this child. Only in the embedding of the last child of the root a convex hull may have been used. Therefore every vertex of $T$ is incident to at most one convex hull edge. Thus the set of convex hull edges used in the embedding is a matching. Therefore at most $\frac{n}{2}$ convex hull edges are used in the embedding.

Suppose that exactly $\frac{n}{2}$ convex hull edges are used. One of these edges must be incident to the root. Since the root was chosen of degree at least three it has a child which is not a leaf nor the first child; we place this vertex last in the ordering of the children of the root. The leaf adjacent to the root can no longer be a convex hull edge and the embedding uses less than $\frac{n}{2}$ convex hull edges.

**Theorem 2.3** If $G$ is a convex geometric graph and has at most two forbidden edges, then any tree on $n$ vertices can be embedded into $G$, without using a forbidden edge.

**Proof:** Let $f_0$ be an embedding given by Lemma 2.2 of $T$ into $G$. For $0 \leq i \leq n$, let $f_i$ be the embedding produced by rotating $f_0$, $i$ places to the right. Assume that in each of these rotations at least one forbidden edge is used, as otherwise we are done. Let $e_1, \ldots, e_m$ be the edges of $T$ that are mapped to a forbidden edge in some rotation. Assume that the two forbidden edges are an $l$-edge and an $r$-edge respectively.

Suppose that $l \neq r$. Then, each edge of $T$ can be embedded into a forbidden edge at most once in all of the $n$ rotations. Thus $m \geq n$. This is a contradiction, since $T$ has $n - 1$ edges.

Suppose that $l = r$. Then, each of the $e_i$ is mapped twice to a forbidden edge. Thus $m \geq n/2$. By Lemma 2.2, $f_0$ uses less than $n/2$ convex hull edges. Therefore, $l = r > 0$. But a set of $n/2$ or more $r$-edges, with $r > 0$, must contain a pair of edges that cross. And we are done, since $f_0$ is an embedding.
3 Bounds on $f(n, k)$

In this section we prove Theorem 1.1. First we show the upper bound.

**Lemma 3.1** If $T_n$ is embedded into $G_n$ then every edge incident to a leaf of $T_n$ must be embedded into a convex hull edge.

**Proof:** Let $e := uv$ be an edge of $T_n$ incident to leaf. Suppose that $u$ is the leaf vertex. Then $v$ is of degree two. Suppose that $e$ is not embedded into a convex hull edge of $G$. Then $e$ divides $S \setminus \{u, v\}$ into two non-empty subsets $S_1$ and $S_2$, so that $S_1$ lies on the opposite side of $S_2$ with respect to $e$. Assume that the parent of $v$ is embedded into $S_1$. Then no vertex of $T_n$ can be embedded into $S_2$ without crossing $e$. Therefore $e$ must be a convex hull edge of $G$. \qed

**Lemma 3.2** If $G$ is a convex geometric graph, then forbidding three consecutive convex hull edges of $G$ forbids the embedding of $T_n$.

**Proof:** Recall that $T_n$ comes from subdividing a star, let $v$ be the non leaf vertex of this star. Let $p_1p_2, p_2p_3, p_3p_4$ be the forbidden edges, in clockwise order around the convex hull of $G$. Note that by Lemma 3.1 in any embedding of $T_n$ into $G$, an edge incident to a leaf of $T_n$, must be embedded into a convex hull edge. Neither the leaves of $T_n$ nor its neighbors can be embedded into $p_2$ or $p_3$, without using a forbidden edge. Thus, $v$ must be embedded into $p_2$ or $p_3$. Without loss of generality assume that $v$ is embedded into $p_2$. But then, the embedding must use $p_2p_3$ or $p_3p_4$. \qed

**Lemma 3.3** If $G$ is a convex geometric graph, then forbidding the convex hull edges incident to any three vertices $p_1, p_2$ and $p_3$ of $G$, forbids the embedding of $T_n$.

**Proof:** Note that by Lemma 3.1 neither a leaf of $T_n$, nor its neighbor can be embedded into $p_1, p_2$ or $p_3$, without using a forbidden edge. But at most two points do not fall into this category. \qed

**Lemma 3.4**

$$f(n, k) \leq \frac{2n(n-2)}{k-2}$$

**Proof:** Let $G$ be a complete convex geometric graph. We forbid every $r$-edge of $G$ for $r = 0, \ldots, \left\lfloor \frac{n-2}{k-2} \right\rfloor - 2$. Note that, in total we are forbidding at most $n \left( \left\lfloor \frac{n-2}{k-2} \right\rfloor - 2 \right) + 1 \leq 2^n \left( \frac{n-2}{k-2} \right)$ edges. As every subset of points of $G$ is in convex position, it suffices to show that every induced subgraph $H$ of $G$ on $k$ vertices is in one of the two configurations of Lemmas 3.2 and 3.3.

Assume then, that $H$ does not contain three consecutive forbidden edges in its convex hull nor three vertices, each with its two convex hull edges forbidden. $H$ has at most two (non-adjacent) pairs of consecutive forbidden edges in its convex hull. Therefore every forbidden edge of $H$ in its convex hull—with the exception of at most two—must be preceded by an $\ell$-edge (of $G$), with $\ell > \left\lfloor \frac{n-2}{k-2} \right\rfloor$. The number of these $\ell$-edges contained in $H$ is at least $\frac{k-2}{2} \left\lfloor \frac{n-2}{k-2} \right\rfloor - 2 \geq n - k$ points of $G$. This is a contradiction, since together with the $k$ points of $H$ this is strictly more than $n$. \qed


Now, we show the lower bound of Theorem 1.1.

Lemma 3.5

\[ f(n, k) \geq \left( \frac{1}{2} \right) \frac{n^2}{k - 1} - \frac{n}{2} \]

Proof: Let \( F \) be a set of edges whose removal from \( G \) forbids some \( k \)-tree. Let \( H := G \setminus F \). Note that \( H \) contains no complete \( K_k \) as a subgraph, otherwise any \( k \)-tree can be embedded into this subgraph. By Turán’s Theorem [7], \( H \) cannot contain more than \( \left( \frac{k - 2}{k - 1} \right) \frac{n^2}{2} \) edges. Thus \( F \) must have size at least

\[ \left( \frac{n}{2} \right) - \left( \frac{k - 2}{k - 1} \right) \frac{n^2}{2} = \left( \frac{1}{2} \right) \frac{n^2}{k - 1} - \frac{n}{2} \]

\[ \square \]

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References


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