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We present simple graph-theoretic characterizations for the Cayley graphs of semigroups, monoids, right-cancellative monoids, left-cancellative monoids, and groups.

Keywords: Cayley graph, semigroup, monoid, group.

1 Introduction

Arthur Cayley was the first to define in 1854 [3] the notion of a group as well as the table of its operation known as the Cayley table. To describe the structure of a group $(G, \cdot)$, Cayley also introduced in 1878 [4] the concept of graph for G according to a generating subset S, namely the set of labeled oriented edges $g \xrightarrow{s} g{\cdot}s$ for every g of G and s of S. Such a graph, called Cayley graph, is directed and labeled in S (or an encoding of S by symbols called letters or colors). A characterization of unlabeled and undirected Cayley graphs was given by Sabidussi in 1958 [8] : an unlabeled and undirected graph is a Cayley graph if and only if we can find a group with a free and transitive action on the graph. Following a question asked by Hamkins in 2010 [6]: 'Which graphs are Cayley graphs?', we present simple graph-theoretic characterizations of Cayley graphs first for the monoids, then for the right-cancellative or/and left-cancellative monoids, and finally for the groups.

Let us present the main structural characterizations starting with the Cayley graphs of monoids. Among many properties of these graphs, we retain only basic ones. First and by definition, any Cayley graph is deterministic: there are no two edges of the same source and label. They are also source-complete: for any label of the graph and from any vertex, there is at least one edge. These first two properties are well known in automata theory. Every Cayley graph of a monoid is rooted: there is a path from the identity element to any vertex. The identity is also an out-simple vertex: it is not source of two edges with the same target. Finally, the identity is a propagating vertex meaning that if it is source of two paths labeled by u and v with the same target then any two paths labeled by u and v of the same source have the same target. These properties satisfied by the Cayley graphs of monoids are sufficient to characterize them: they are the deterministic and source-complete graphs with a propagating out-simple root (Theorem 4.4). In fact for any graph Γ and root r satisfying these properties, the vertex set of Γ is a monoid whose $s \cdot t$ is the target of the path from s labeled by u if u labels a path from r to t. It follows that Γ is a Cayley graph of this monoid generated by the set of successors of r. By embedding trivially every semigroup into a monoid, a characterization follows for the Cayley graphs of semigroups.

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We then characterize the Cayley graphs of right-cancellative monoids as being the Cayley graphs of monoids which are co-deterministic: there are no two edges of the same target and label (Theorem 6.2). We also characterize the Cayley graphs of left-cancellative monoids as being the Cayley graphs of monoids which are propagating: any vertex is propagating (Theorem 6.6). By extending to chains the vertex propagation, we get the Cayley graphs of groups: they are the deterministic and co-deterministic, chain-propagating connected simple graphs (Theorem 7.3).

2 Directed labeled graphs

We present some basic concepts on directed labeled graph, namely the determinism, the source-completeness, and the simplicity, which are basic notions in automata theory.

Let A be an arbitrary (finite or infinite) set. We denote by A^* the set of tuples (words) over A which is for the concatenation the free monoid generated by A. We denote by ε the 0-tuple *i.e.* the identity element called the *empty word*. A directed A-*graph* (V,G) is defined by a set V of *vertices* and a subset $G \subseteq V{\times}A{\times}V$ of *edges*. Any edge $(s,a,t) \in G$ is from the *source* s to the *target* t with *label* a, and is also written by the *transition* $s \xrightarrow{a}_G t$ or directly $s \xrightarrow{a} t$ if G is clear from the context. The sources and targets of edges form the set V_G of *non-isolated vertices* of G and A_G is the set of edge labels:

$$V_G \;=\; \{\; s \mid \exists\, a,t \;\; (s \xrightarrow{a} t \,\vee\, t \xrightarrow{a} s) \;\} \quad \text{and} \quad A_G \;=\; \{\; a \mid \exists\, s,t \;\; (s \xrightarrow{a} t) \;\}.$$

The set $V - V_G$ is the set of *isolated vertices*. From now on, we assume that any graph (V,G) is without isolated vertex (*i.e.* $V = V_G$), hence the graph can be identified with its edge set G. We also exclude the empty graph $\emptyset$: every graph is a non-empty set of labeled edges. Let $\longrightarrow_G$ be the *edge relation* defined by $s \longrightarrow_G t$ if $s \xrightarrow{a}_G t$ for some $a \in A_G$.

A *path* $(s_0,a_1,s_1,\ldots,a_n,s_n)$ of *length* $n \geq 0$ in a graph G is a sequence $s_0 \xrightarrow{a_1} s_1 \ldots \xrightarrow{a_n} s_n$ of n consecutive edges, and we write $s_0 \xrightarrow{a_1\ldots a_n} s_n$ for indicating the source s_0, the target s_n and the label word $a_1\ldots a_n \in A_G^*$ of the path. Let $\longrightarrow_G^*$ be the *path relation* defined by $s \longrightarrow_G^* t$ if $s \xrightarrow{u}_G t$ for some $u \in A_G^*$. We denote by $\longrightarrow_G(r) \;=\; \{\; s \mid r \longrightarrow_G s \;\}$ the set of *successors* of a vertex r. A graph G is *accessible* from $P \subseteq V_G$ if for any $s \in V_G$ there is $r \in P$ such that $r \longrightarrow_G^* s$. A vertex r is a *root* if the graph is accessible from $\{r\}$. A graph G is *strongly connected* if every vertex is a root:

$$s \longrightarrow_G^* t \quad \text{for all } s,t \in V_G\,.$$

A graph G is *deterministic* if there are no two edges of the same source and label *i.e.*

$$(r \xrightarrow{a}_G s \,\wedge\, r \xrightarrow{a}_G t) \implies s = t \;\text{ for any } r,s,t \in V_G \text{ and } a \in A_G$$

meaning that for any label $a \in A_G$, the relation $\xrightarrow{a}$ is one to one. This local condition is equivalent to the injectivity of $\xrightarrow{u}$ for any word $u \in A_G^*$.

A vertex s is a *source-complete vertex* if for any label a of the graph, there is at least one edge from s labeled by a *i.e.* $\forall\, a \in A_G \;\; \exists\, t\; (s \xrightarrow{a}_G t)$. A graph is a *source-complete graph* if all its vertices are source-complete. This condition is equivalent for any label word u to the existence of a path labeled by u from any vertex: $\forall\, u \in A_G^* \;\forall\, s \in V_G \;\; \exists\, t\; (s \xrightarrow{u}_G t)$. Thus a graph G is deterministic and source-complete if and only if for any label $a \in A_G$, $\xrightarrow{a}_G$ is a mapping from V_G into V_G or equivalently for any label word $u \in A_G^*$, $\xrightarrow{u}_G$ is a mapping.

A vertex s is an *out-simple vertex* if there are no two edges of source s with the same target:

$$(s \xrightarrow{a} t \,\wedge\, s \xrightarrow{b} t) \implies a = b \quad \text{for any } t \in V_G \text{ and } a,b \in A_G\,.$$

We also say that s is an *in-simple vertex* if there are no two edges of target s with the same source:

$$(t \xrightarrow{a} s \wedge t \xrightarrow{b} s) \implies a = b \quad \text{for any } t \in V_G \text{ and } a, b \in A_G\,.$$

A *simple vertex* is a vertex which is both in-simple and out-simple. A *simple graph* is a graph whose any vertex is simple (or in-simple, or out-simple).

3 Generalized Cayley graphs

We present graph-theoretic characterizations for the generalized Cayley graphs of magmas with a left identity, and with an identity.

Recall that a *magma* is a set M equipped with a binary operation $\cdot : M \times M \longrightarrow M$ that sends any two elements $p, q \in M$ to the element $p \cdot q$. Given a subset $Q \subseteq M$ and an injective mapping $[\![\ ]\!] : Q \longrightarrow A$, we define the *generalized Cayley graph* :

$$\mathcal{C}[\![M, Q]\!] \;=\; \{\, p \xrightarrow{[\![q]\!]} p \cdot q \mid p \in M \,\wedge\, q \in Q \,\}$$

of vertex set M and of label set $[\![Q]\!] = \{\, [\![q]\!] \mid q \in Q \,\}$.
We will be able to give a characterization of these graphs when the magma M has an identity (see Proposition 3.5). This characterization will then be refined when M is a monoid generated by Q, and then M is a group generated by Q.

Fact 3.1 *Any generalized Cayley graph of a magma is deterministic and source-complete.*

The deterministic source-complete graphs are the graphs such that for any label word u and for any vertex s, there is a unique path from s labeled by u. Among the deterministic and source-complete graphs, we want to determine those that are generalized Cayley graphs.
Given a vertex r of a graph G, a binary operation $\cdot$ on V_G is a r*-edge operation* of G if

$$r \xrightarrow{a} t \implies s \xrightarrow{a} s{\cdot}t \quad \text{for any } s, t \in V_G \text{ and } a \in A_G$$

Such an operation is illustrated below and will be refined later (see Figures 4.5, 7.4, 7.7).

$$s \xrightarrow{a} s \cdot t \qquad \textbf{for} \qquad r \xrightarrow{a} t \qquad \textbf{and} \qquad a \in A$$

Figure 3.2 *A r-edge operation on the vertex set of a graph.*

In particular if a graph G has a r-edge operation with r source-complete then G is source-complete. For any deterministic and source-complete graph, the existence of an edge-operation from an out-simple vertex allows to express it as a generalized Cayley graph.

Lemma 3.3 *Let r be an out-simple source-complete vertex of a deterministic graph G. Let $\cdot$ be a r-edge operation of G. So $G = \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!]$ with $[\![s]\!] = a$ for any $r \xrightarrow{a} s$.*

Proof.
We denote by $S_r = \longrightarrow_G(r) = \{\, s \mid \exists\, a \in A_G \ (r \xrightarrow{a} s) \,\}$ the set of successors of r.
As r is out-simple, $[\![\ ]\!]$ is a mapping from S_r into A_G. As G is deterministic, $[\![\ ]\!]$ is injective. As r is source-complete, $[\![\ ]\!]$ is surjective hence $[\![\ ]\!]$ is bijective.
Let us check that $G = \mathcal{C}[\![V_G, S_r]\!]$.
$\subseteq$: Let $s \xrightarrow{a}_G t$. As r is source-complete, there exists q such that $r \xrightarrow{a}_G q$. So $[\![q]\!] = a$.

As $\cdot$ is a r-edge operation, we have $s \xrightarrow[G]{a} s{\cdot}q$.
As G is deterministic, we get $t = s{\cdot}q$ hence $s \xrightarrow[\mathcal{C}[\![V_G\,,\,S_r]\!]]{a} s{\cdot}q \;=\; t$.
$\supseteq$: Let $s \xrightarrow[\mathcal{C}[\![V_G\,,\,S_r]\!]]{a} t$. There exists (a unique) $q \in S_r$ such that $[\![q]\!] = a$.
So $t = s{\cdot}q$ and $r \xrightarrow[G]{a} q$. As $\cdot$ is a r-edge operation, we have $s \xrightarrow[G]{a} s{\cdot}q = t$. $\square$

Recall that an element e of M is a *left identity* (resp. *right identity*) if $e{\cdot}p = p$ (resp. $p{\cdot}e = p$) for any $p \in M$. Note that any left identity of a magma is an out-simple vertex of its generalized Cayley graphs. This property with those of Fact 3.1 suffices to characterize the generalized Cayley graphs of magmas with a left identity.

Proposition 3.4 *A graph is a generalized Cayley graph of a magma with a left identity* if and only if *it is deterministic, source-complete with an out-simple vertex.*

Proof.
$\Longrightarrow$: Let $G = \mathcal{C}[\![M,Q]\!]$ for some magma M with a left identity e and some $Q \subseteq M$.
By Fact 3.1, it remains to check that e is out-simple.
Let $e \xrightarrow[G]{a} s$ and $e \xrightarrow[G]{b} s$ with $a, b \in A_G$.
As $A_G = [\![Q]\!]$, we have $a = [\![p]\!]$ and $b = [\![q]\!]$ for some $p, q \in Q$.
Thus $s = e{\cdot}p = p$ and $s = e{\cdot}q = q$ hence $p = q$ *i.e.* $a = b$.
$\Longleftarrow$: Let G be a deterministic and source-complete graph and let r be an out-simple vertex. We consider the magma $(V_G, \cdot)$ with $\cdot$ defined for any $s, t \in V_G$ by

$$s \cdot t \;=\; \begin{cases} x & \text{if } r \xrightarrow{a} t \text{ and } s \xrightarrow{a} x \text{ for some } a \in A_G \\ t & \text{if } s = r \\ r & \text{otherwise.} \end{cases}$$

As r is out-simple and G is deterministic, $\cdot$ is well-defined.
So r is a left identity and $\cdot$ is a r-edge operation.
As G is source-complete and by Lemma 3.3, G is a generalized Cayley graph of $(V_G, \cdot)$.
Note that we could also define more simply $\cdot$ for any $s, t \in V_G$ by

$$s \cdot t \;=\; \begin{cases} x & \text{if } r \xrightarrow{a} t \text{ and } s \xrightarrow{a} x \text{ for some } a \in A_G \\ t & \text{otherwise.} \end{cases}$$

However, this is only the first definition that can be generalized to the next proposition. $\square$

Let us introduce a graph notion. A vertex r of a graph G is a *loop-propagating vertex* if

$$r \xrightarrow{a} r \quad\Longrightarrow\quad s \xrightarrow{a} s \text{ for any } s \in V_G \text{ and } a \in A_G$$

meaning that if r has a loop labeled by a then any vertex has a loop labeled by a. In particular any vertex without loop is loop-propagating.
If a magma M has a left identity e and a right identity e' then $e = e \cdot e' = e'$ is the *identity* of M and we say that M is a *unital magma*. The identity e of a magma is a loop-propagating vertex for its generalized Cayley graphs. We can strengthen Proposition 3.4 to magmas with an identity.

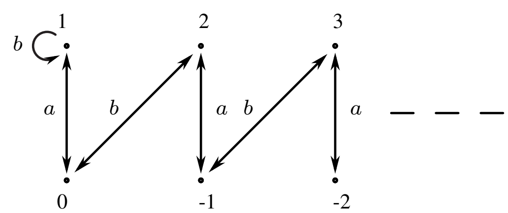
Proposition 3.5 *A graph is a generalized Cayley graph of a unital magma* if and only if *it is deterministic, source-complete with an out-simple and loop-propagating vertex.*

Proof.
$\Longrightarrow$: Let $G = \mathcal{C}[\![M,Q]\!]$ for some magma M with an identity e and some $Q \subseteq M$.
In the proof of Proposition 3.4, we have seen that e is out-simple. It remains to check that e is loop-propogating. Let $e \xrightarrow{a}_G e$ with $a \in A_G$ and let $s \in V_G = M$.
So $a = [\![q]\!]$ for some $q \in Q$ hence $e = e{\cdot}q = q$. Thus $s \xrightarrow{a}_G s{\cdot}q = s{\cdot}e = s$.
$\Longleftarrow$: We have just to refine the operation $\cdot$ of the previous proof as follows: for any $s, t \in V_G$

$$s \cdot t \ = \begin{cases} x & \text{if } r \xrightarrow{a} t \text{ and } s \xrightarrow{a} x \text{ for some } a \in A_G \\ t & \text{if } s = r \\ s & \text{if } t = r \\ r & \text{otherwise.} \end{cases}$$

As r is out-simple, loop-propagating and as G is deterministic, $\cdot$ is well-defined. $\square$

For instance consider the magma $(\mathbb{Z}, \cdot)$ where $m \cdot n = -m + n$ for any $m, n \in \mathbb{Z}$. There is no right identity and 0 is the unique left identity. The generalized Cayley graph $G = \mathcal{C}[\![\mathbb{Z}, \{1,2\}]\!]$ with $[\![1]\!] = a$ and $[\![2]\!] = b$ is the following simple graph:



Any vertex other than 1 is loop-propagating. By Proposition 3.5, G is a generalized Cayley graph of a unital magma. Precisely by applying the construction given in its proof for $r = 0$, we get

$$G \ = \ \mathcal{C}[\![\mathbb{Z}, \{1,2\}]\!] \ \text{ with } \ [\![1]\!] = a \ \text{ and } \ [\![2]\!] = b$$

for the magma $(\mathbb{Z}, \cdot)$ of identity 0 defined for any n by $0 \cdot n = n$ and for any $m \neq 0$ by

$$m \cdot n \ = \begin{cases} m & \text{if } n = 0 \\ -m + n & \text{if } n = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

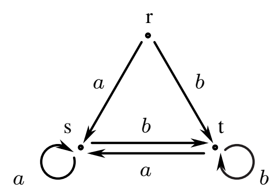
4 Cayley graphs of monoids

We present a graph theoretic characterization for the Cayley graphs of monoids. Such a characterization is based on a structural property describing a partial symmetry of these graphs: every vertex is the image of the unit element by an endomorphism. We express this general property by an elementary notion of propagation.

We write $s \xrightarrow{u,v}_G t$ if $s \xrightarrow{u}_G t$ and $s \xrightarrow{v}_G t$. We write $s \xrightarrow{u,v}_G$ when $s \xrightarrow{u,v}_G t$ for some t.
A vertex r of a graph G is a *propagating vertex* if

$$r \xrightarrow{u,v} \implies s \xrightarrow{u,v} \text{ for any } s \in V_G \text{ and } u,v \in A_G^*$$

meaning that if there are two paths from r of the same target labeled by u and v then from any vertex, there are two paths of the same target labeled by u and v.
In particular for $u = v \in A$, a graph is source-complete if it has a source-complete propagating vertex. Furthermore for $u = \varepsilon$ and $v \in A$, a propagating vertex is loop-propagating.
For the following deterministic and source-complete graph:



the vertex r is propagating but the vertices s, t are not propagating: we have $s \xrightarrow{\varepsilon,a}$ (resp. $t \xrightarrow{\varepsilon,b}$) which is not the case for the other vertices.
Let us express the existence of a propagating vertex by a partial symmetry of the graph when it is deterministic. Given a vertex r of a graph G, we denote by $G_{\downarrow r}$ the restriction of G to the vertices accessible from r:

$$G_{\downarrow r} \;=\; \{\, (s,a,t) \in G \mid r \longrightarrow_G^* s \,\}.$$

A *morphism* from a graph G into a graph H is a mapping h from V_G into V_H such that

$$s \xrightarrow{a}_G t \implies h(s) \xrightarrow{a}_H h(t).$$

A propagating vertex r of a deterministic graph G is a vertex from which there is for any vertex s a morphism from $G_{\downarrow r}$ into $G_{\downarrow s}$ linking r to s.

Lemma 4.1 *For any deterministic graph G and vertices r, s, we have*

$$r \xrightarrow{u,v}_G \implies s \xrightarrow{u,v}_G \text{ for any } u,v \in A_G^*$$

if and only if *there is a morphism h from $G_{\downarrow r}$ to $G_{\downarrow s}$ such that $h(r) = s$.*

Proof.
$\Longleftarrow$: Immediate for any graph.
$\Longrightarrow$: As G is deterministic, it allows to define the mapping $h : V_{G_{\downarrow r}} \longrightarrow V_{G_{\downarrow s}}$ by

$$h(p) \;=\; q \text{ if } r \xrightarrow{u}_G p \text{ and } s \xrightarrow{u}_G q \text{ for some } u \in A^*.$$

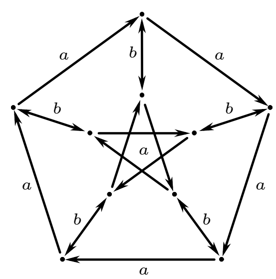
Thus $h(r) = s$. It remains to check that h is a morphism.
Let $p \xrightarrow{a}_{G_{\downarrow r}} q$. There exists $u \in A_G^*$ such that $r \xrightarrow{u}_G p$.
As $r \xrightarrow{ua}_G$, we have $s \xrightarrow{ua}_G$ *i.e.* $s \xrightarrow{u}_G p' \xrightarrow{a}_G q'$ for some vertices p', q'.
As G is deterministic, $h(p) = p'$ and $h(q) = q'$ hence $h(p) \xrightarrow{a}_{G_{\downarrow s}} h(q)$. $\Box$

Recall that a magma $(M,\cdot)$ is a *semigroup* if $\cdot$ is associative: $(p\cdot q)\cdot r \;=\; p\cdot(q\cdot r)$ for any $p,q,r \in M$. A *monoid* M is a semigroup with an identity 1. In that case, 1 is a propagating vertex of its generalized Cayley graphs.

Fact 4.2 *For any generalized Cayley graph of a monoid,* 1 *is a propagating vertex.*

Proof.
Let $1 \overset{u,v}{\longrightarrow}_G s$ with $u, v \in A_G^*$. We have $u = [\![p_1]\!] \ldots [\![p_m]\!]$ and $v = [\![q_1]\!] \ldots [\![q_n]\!]$ for some $m, n \geq 0$ and $p_1, \ldots, p_m, q_1, \ldots, q_n \in Q$.
Thus $(\ldots(1{\cdot}p_1)\ldots){\cdot}p_m = s = (\ldots(1{\cdot}q_1)\ldots){\cdot}q_n$. As $\cdot$ is associative, $p_1{\cdot}\ldots{\cdot}p_m = q_1{\cdot}\ldots{\cdot}q_n$.
Let t be any vertex of G. So $t \overset{u}{\longrightarrow}_G (\ldots(t{\cdot}p_1)\ldots){\cdot}p_m = t{\cdot}(p_1{\cdot}\ldots{\cdot}p_m)$ and
$t \overset{v}{\longrightarrow}_G t{\cdot}(q_1{\cdot}\ldots{\cdot}q_n) = t{\cdot}(p_1{\cdot}\ldots{\cdot}p_m)$. Hence $t \overset{u,v}{\longrightarrow}_G$. □

For instance the following oriented and labeled Petersen graph:



is not a generalized Cayley graph of a monoid since it has no propagating vertex: for s an inner vertex and t an outer vertex, we have $s \overset{\varepsilon,ababa}{\longrightarrow}$ and $t \overset{\varepsilon,baaba}{\longrightarrow}$ while $t \overset{ababa}{\not\longrightarrow} t$ and $s \overset{baaba}{\not\longrightarrow} s$.

Recall that the *submonoid generated* by a subset Q of a monoid M is the least submonoid $Q^* = \{\, q_1{\cdot}\ldots{\cdot}q_n \mid n \geq 0 \wedge q_1, \ldots, q_n \in Q \,\}$ containing Q. A *Cayley graph of a monoid* M is a generalized Cayley graph $\mathcal{C}[\![M, Q]\!]$ for M generated by Q *i.e.* $M = Q^*$.

Fact 4.3 *For any Cayley graph of a monoid,* 1 *is a root.*

Proof.
Let $G = \mathcal{C}[\![M, Q]\!]$ for some monoid M generated by Q.
For any $m \in M$, we have $m = q_1{\cdot}\ldots{\cdot}q_n$ for some $n \geq 0$ and $q_1, \ldots, q_n \in Q$, thus $1 \overset{v}{\longrightarrow}_G m$ for the label word $v = [\![q_1]\!] \ldots [\![q_n]\!]$. □

Thus a generalized Cayley graph of a monoid is a Cayley graph if and only if it is rooted by 1. We can give a simple graph-theoretic characterization for the Cayley graphs of monoids.

Theorem 4.4 *A graph is a Cayley graph of a monoid* if and only if
it is deterministic, source-complete with a propagating out-simple root.

Proof.
$\Longrightarrow$: Let $G = \mathcal{C}[\![M, Q]\!]$ for some monoid M and some $Q \subseteq M$ such that $Q^* = M$.
By Fact 3.1, G is deterministic and source-complete. By (the proof of) Proposition 3.4, the vertex 1 is out-simple. By Facts 4.2 and 4.3, the vertex 1 is a propagating root.

$\Longleftarrow$: Let G be a deterministic and source-complete graph and r be a propagating out-simple root. We extend a r-edge operation into a *r-path operation* as being a binary operation $\cdot$ on V_G such that

$$r \xrightarrow{u} t \implies s \xrightarrow{u} s \cdot t \ \text{ for any } \ s, t \in V_G \ \text{ and } \ u \in A_G^*$$

which is illustrated below.

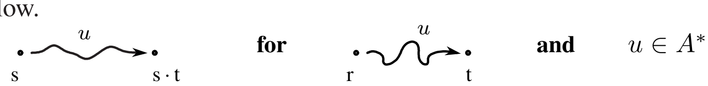


Figure 4.5 *A r-path operation on the vertex set of a graph.*

Let us check the existence and the unicity of a r-path operation.

Let $s, t \in V_G$. Let us check that $s \cdot t$ is well-defined.
As r is a root, there exists $u \in A_G^*$ such that $r \xrightarrow{u}_G t$.
As G is source-complete, there exists x such that $s \xrightarrow{u}_G x$.
Let $r \xrightarrow{v}_G t$ for some $v \in A_G^*$. So $r \xrightarrow{u,v}_G$. As r is propagating, $s \xrightarrow{u,v}_G$.
As G is deterministic, we get $s \xrightarrow{v}_G x$. Thus $s \cdot t = x$ is well-defined.

The path operation $\cdot$ is a r-edge operation.
By Lemma 3.3, $G = \mathcal{C}[\![V_G \,,\, \longrightarrow_G(r)]\!]$ where $[\![s]\!] = a$ for any $r \xrightarrow{a} s$.
It remains to check that $(V_G, \cdot)$ is a monoid generated by $\longrightarrow_G(r)$.

i) Let us check that r is a left identity.
Let $s \in V_G$. As r is a root, there exists $u \in A_G^*$ such that $r \xrightarrow{u}_G s$.
By definition of $\cdot$, we have $r \xrightarrow{u}_G r \cdot s$. As G is deterministic, we get $r \cdot s = s$.

ii) Let us check that r is also a right identity.
Let $s \in V_G$. As $r \xrightarrow{\varepsilon}_G r$ and by definition of $\cdot$, we have $s \xrightarrow{\varepsilon}_G s \cdot r$ *i.e.* $s \cdot r = s$.

iii) Let us check that $\cdot$ is associative. Let $x, y, z \in V_G$.
As r is a root, there exists $v, w \in A_G^*$ such that $r \xrightarrow{v} y$ and $r \xrightarrow{w} z$.
Thus $x \xrightarrow{v} x \cdot y \xrightarrow{w} (x \cdot y) \cdot z$ and $y \xrightarrow{w} y \cdot z$. So $r \xrightarrow{vw} y \cdot z$ hence $x \xrightarrow{vw} x \cdot (y \cdot z)$.
As G is deterministic, we get $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

iv) Let us check that $\longrightarrow_G(r)$ generates V_G. Let $s \in V_G$.
There exists a path $r = s_0 \xrightarrow{a_1} s_1 \ldots s_{n-1} \xrightarrow{a_n} s_n = s$.
As r is source-complete, there exists $r_1, \ldots, r_n$ such that $r \xrightarrow{a_1} r_1, \ldots, r \xrightarrow{a_n} r_n$.
By induction on $1 \le i \le n$, we have $s_i = s_{i-1} \cdot r_i$.
Therefore $s = (\ldots(r \cdot r_1) \cdot \ldots) \cdot r_n = r_1 \cdot \ldots \cdot r_n \in (\longrightarrow_G(r))^*$. $\square$

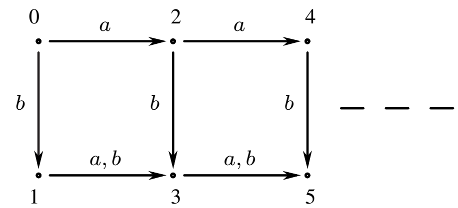
By Theorem 4.4, the graph of the first figure of this section is a Cayley graph of a monoid: it is equal to $\mathcal{C}[\![\{r, s, t\} \,,\, \{s, t\}]\!]$ with $[\![s]\!] = a$, $[\![t]\!] = b$ for the monoid $(\{r, s, t\}, \cdot)$ where $\cdot$ is the r-path operation defined by the following Cayley table:

$\cdot$	r	s	t
r	r	s	t
s	s	s	t
t	t	s	t

Another example is given by the following infinite graph:

$$G = \{\, n \xrightarrow{a} n+2 \mid n \in \mathbb{N} \,\} \cup \{\, 2n \xrightarrow{b} 2n+1 \mid n \in \mathbb{N} \,\} \cup \{\, 2n+1 \xrightarrow{a,b} 2n+3 \mid n \in \mathbb{N} \,\}$$

represented as follows:



Such a graph is deterministic, source-complete, and its root 0 is out-simple and propagating. By Theorem 4.4, G is a Cayley graph of a monoid. Precisely $G = \mathcal{C}[\![\mathbb{N},\{1,2\}]\!]$ where $[\![1]\!] = b$ and $[\![2]\!] = a$ for the 0-path operation $\cdot$ defined for any $p,q \in \mathbb{N}$ by

$$p \cdot q \;=\; \begin{cases} p+q & \text{if } p \text{ or } q \text{ is even,} \\ p+q+1 & \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

which is indeed associative.

5 Cayley graphs of semigroups

We exploit the previous characterization for the Cayley graphs of monoids to obtain a characterization for the Cayley graphs of semigroups.

Recall that a *Cayley graph of a semigroup* M is a generalized Cayley graph $\mathcal{C}[\![M,Q]\!]$ such that $M = Q^+$ whose $Q^+ = \{\, q_1 \cdot \ldots \cdot q_n \mid n > 0 \wedge q_1,\ldots,q_n \in Q \,\}$ is the *subsemigroup generated* by Q. Let us extend Theorem 4.4 into a characterization of these graphs. Every semigroup trivially embeds into a monoid by adding an element. On the graph side, we just need to express properties that make it possible to embed into a graph satisfying the conditions of Theorem 4.4 just by adding a single vertex.

Theorem 5.1 *A graph G is a Cayley graph of a semigroup* if and only if
it is deterministic and there is an injection i from A_G into V_G such that
G is accessible from $i(A_G)$
$i(a) \xrightarrow{u}\xleftarrow{v} i(b) \;\Longrightarrow\; s \xrightarrow{au,bv}$ *for any $s \in V_G$, $a,b \in A_G$ and $u,v \in A_G^*$.*

Proof.
$\Longrightarrow$: Let $G = \mathcal{C}[\![M,Q]\!]$ for some semigroup $(M,\cdot)$ generated by Q.
By Fact 3.1, G is deterministic.
If M is not a monoid *i.e.* it has no identity, we turn M into a monoid $M' = M \cup \{1\}$ by just adding an identity 1 *i.e.* $p{\cdot}1 = 1{\cdot}p = p$ for any $p \in M'$.
Let $M' = M$ when M is a monoid.
In both cases, M' is a monoid of identity 1. Let $G' = \mathcal{C}[\![M',Q]\!] = G \cup \{\, 1 \xrightarrow{[\![q]\!]} q \mid q \in Q \,\}$.
Let $i : A_G \longrightarrow Q$ defined by $i([\![q]\!]) = q$ for any $q \in Q$.
As M is generated by Q, the graph G is accessible from $Q = i(A_G)$.
Let $i([\![p]\!]) \xrightarrow{u}_G {}_G\!\xleftarrow{v} i([\![q]\!])$ for some $p,q \in Q$ and $u,v \in A_G^*$.
So $p \xrightarrow{u}_G {}_G\!\xleftarrow{v} q$ hence $1 \xrightarrow{[\![p]\!]u,[\![q]\!]v}_{G'}$. By Fact 4.2, 1 is a propagating vertex of G'.
Let $s \in V_G$. Therefore $s \xrightarrow{[\![p]\!]u,[\![q]\!]v}_{G'}$ hence $s \xrightarrow{[\![p]\!]u,[\![q]\!]v}_{G}$.

$\Longleftarrow$: Let G be a deterministic graph.
Let an injection $i : A_G \longrightarrow V_G$ such that G is accessible from $Q = i(A_G)$ and such that

$$s \xrightarrow{au,bv}_G \text{ when } i(a) \xrightarrow{u}_G {}_G\!\xleftarrow{v} i(b) \text{ for any } s \in V_G\,,\ a,b \in A_G \text{ and } u,v \in A_G^*\,. \qquad (1)$$

Note that by (1), we get $s \xrightarrow{a}_G$ for any $s \in V_G$ and $a \in A_G$, hence G is source-complete.
We take a new vertex r and we define the graph

$$\widehat{G} \;=\; G \cup \{\; r \xrightarrow{a} i(a) \mid a \in A_G \;\}.$$

So $\widehat{G}$ remains deterministic and source-complete, and r is out-simple.
As G is accessible from Q, r is a root of $\widehat{G}$. By (1), r is a propagating vertex of $\widehat{G}$.
By the proof of Theorem 4.4, $V_{\widehat{G}} \;=\; V_G \cup \{r\}$ is a monoid generated by Q for the r-path operation $\cdot$ of identity r with

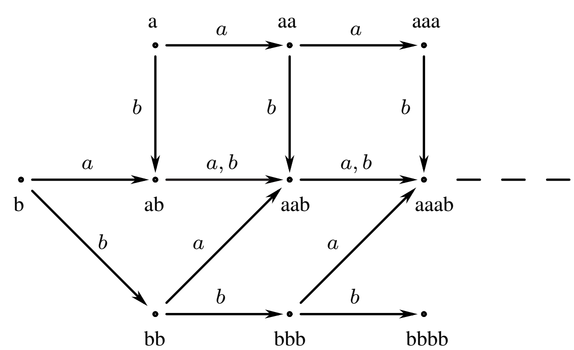
$$\widehat{G} \;=\; \mathcal{C}[\![V_{\widehat{G}}, Q]\!] \;=\; \mathcal{C}[\![V_G, Q]\!] \cup \{\; r \xrightarrow{[\![q]\!]} q \mid q \in Q \;\} \text{ where } i([\![q]\!]) \;=\; q \text{ for every } q \in Q.$$

As r is not the target of an edge of $\widehat{G}$ and by definition, $\cdot$ remains an internal operation on V_G *i.e.* $p \cdot q \neq r$ for any $p,q \in V_G$. Thus $\cdot$ remains associative on V_G.
Finally $G \;=\; \mathcal{C}[\![V_G\,,\,Q]\!]$ and $(V_G\,,\,\cdot)$ is a semigroup. □

For instance, the two semilines $G_a \;=\; \{\; n \xrightarrow{a} n+1 \mid n \in \mathbb{N} - \{0\} \;\} \cup \{\; n \xrightarrow{a} n-1 \mid n \in \mathbb{Z} - \mathbb{N} \;\}$ is not a Cayley graph of a semigroup since it is not connected and has a unique label. On the other hand, $G_a \cup G_b$ is a Cayley graph of a semigroup. Precisely $G_a \cup G_b \;=\; \mathcal{C}[\![\mathbb{Z} - \{0\}, \{1,-1\}]\!]$ where $[\![1]\!] \;=\; a$ and $[\![-1]\!] \;=\; b$ for the associative operation $\cdot$ defined by

$$m \cdot n \;=\; m + \mathrm{sign}(m)\,|n| \text{ for any } m,n \in \mathbb{Z} - \{0\}$$

where $\mathrm{sign}(m) = 1$ for any $m > 0$ and $\mathrm{sign}(m) = -1$ for any $m < 0$.
Another example is given by the following graph G:



whose its vertex set V is the regular set $a^*a \,\cup\, a^*b \,\cup\, b^*bb$.
By Theorem 5.1, G is the Cayley graph $\mathcal{C}(V, \{a,b\})$ for the associative binary operation $\cdot$ on V defined for any $m,n > 0$, $i,j \geq 0$, $p,q > 1$ by the following Cayley table:

$\cdot$	a^n	$a^j b$	b^q
a^m	a^{m+n}	$a^{j+m}b$	$a^{m+q-1}b$
$a^i b$	$a^{i+n}b$	$a^{i+j+1}b$	$a^{i+q}b$
b^p	$a^{n+p-1}b$	$a^{j+p}b$	b^{p+q}

6 Cayley graphs of right-cancellative and left-cancellative monoids

We strengthen the characterization for the Cayley graphs of monoids (Theorem 4.4) to the Cayley graphs of right-cancellative monoids and of left-cancellative monoids.

Recall that a magma $(M,\cdot)$ is *right-cancellative* if $p \cdot r = q \cdot r \implies p = q$ for any $p,q,r \in M$. Let us give a basic property of their generalized Cayley graphs.
A graph G is *co-deterministic* if there are no two edges of the same target and label *i.e.*

$$(s \xrightarrow{a}_G r \wedge t \xrightarrow{a}_G r) \implies s = t \text{ for any } r,s,t \in V_G \text{ and } a \in A_G\,.$$

Fact 6.1 *Any generalized Cayley graph of a right-cancellative magma is co-deterministic.*

By adding the condition of co-determinism in Theorem 4.4, we get a characterization for the Cayley graphs of right-cancellative monoids.

Theorem 6.2 *A graph is a Cayley graph of a right-cancellative monoid* if and only if *it is deterministic, co-deterministic, source-complete with a propagating out-simple root.*

Proof.
$\Longrightarrow$: By Theorem 4.4 and Fact 6.1.
$\Longleftarrow$: Let G be a deterministic and co-deterministic source-complete graph with a propagating out-simple root r. It remains to check that the r-path operation $\cdot$ defined in the proof of Theorem 4.4 is right-cancellative. Let $s,s',t \in V_G$ such that $s{\cdot}t = s'{\cdot}t$.
There exists $u \in A_G^*$ such that $r \xrightarrow{u} t$. So $s \xrightarrow{u} s{\cdot}t$ and $s' \xrightarrow{u} s'{\cdot}t = s{\cdot}t$.
As G is co-deterministic, we get $s = s'$. □

For instance, let us consider the following source-complete, deterministic and co-deterministic graph G:

$$\cdots \quad -3 \xleftarrow{a,b} -2 \xleftarrow{a,b} -1 \xleftarrow{b} 0 \xrightarrow{a} 1 \xrightarrow{a,b} 2 \xrightarrow{a,b} 3 \quad \cdots$$

which is a completion of $G_a \cup G_b$ from the first example after Theorem 5.1. The vertex 0 is the unique out-simple vertex, is the unique root and is the unique propagating vertex. By Theorem 6.2, G is a Cayley graph of a right-cancellative monoid. Precisely $G = \mathcal{C}[\![\mathbb{Z},\{1,-1\}]\!]$ where $[\![1]\!] = a$ and $[\![-1]\!] = b$ for the right-cancellative monoid $(\mathbb{Z},\cdot)$ defined for any $m,n \in \mathbb{Z}$ by

$$m \cdot n \;=\; \begin{cases} n & \text{if } m = 0 \\ m + \mathrm{sign}(m)\,|n| & \text{if } m \neq 0 \end{cases}$$

which is the natural completion of the semigroup that we have associated with $G_a \cup G_b$.

We will now consider the Cayley graphs of left-cancellative monoids. Recall that a magma $(M,\cdot)$ is *left-cancellative* if $r \cdot p = r \cdot q \implies p = q$ for any $p,q,r \in M$.

Fact 6.3 *Any generalized Cayley graph of a left-cancellative magma is simple.*

Let us strengthen Theorem 4.4 in order to characterize the Cayley graphs of left-cancellative monoids. We say that a graph is a *propagating graph* when all its vertices are propagating.

The previous graph is not propagating since only its root is a propagating vertex. Here is a propagating graph:

$$\bullet \xrightarrow{a,b} \bullet \xrightarrow{a,b} \bullet \quad - \; - \; -$$

Note that every propagating graph with an out-simple vertex is simple.
Let us express differently the propagating graphs when they are deterministic.
If a morphism h from a graph G into a graph H is bijective and h^{-1} is a morphism, h is called an *isomorphism* from G to H. We say vertices s,t of a graph G are *accessible-isomorphic* if $t = h(s)$ for some isomorphism h from $G_{\downarrow s}$ to $G_{\downarrow t}$. Let us apply Lemma 4.1.

Corollary 6.4 *A deterministic graph is propagating* if and only if *all its vertices are accessible-isomorphic.*

Proof.
For the necessary condition, Lemma 4.1 provides two morphisms between $G_{\downarrow s}$ and $G_{\downarrow t}$ which are mutual inverses since G is deterministic. $\square$

When a monoid is left-cancellative, its generalized Cayley graphs are propagating.

Fact 6.5 *Any generalized Cayley graph of a left-cancellative monoid is propagating.*

Proof.
Let $r \overset{u,v}{\longrightarrow}_G s$ with $u,v \in A_G^*$. We have $u = [\![p_1]\!] \ldots [\![p_m]\!]$ and $v = [\![q_1]\!] \ldots [\![q_n]\!]$ for some $m,n \geq 0$ and $p_1,\ldots,p_m,q_1,\ldots,q_n \in Q$. Thus $(\ldots(r{\cdot}p_1)\ldots){\cdot}p_m \ = \ s \ = \ (\ldots(r{\cdot}q_1)\ldots){\cdot}q_n$.
As $\cdot$ is associative and left-cancellative with an identity, we get $p_1{\cdot}\ldots{\cdot}p_m \ = \ q_1{\cdot}\ldots{\cdot}q_n$.
Let t be any vertex of G. So $t \overset{u,v}{\longrightarrow}_G t{\cdot}(p_1{\cdot}\ldots{\cdot}p_m) \ = \ t{\cdot}(q_1{\cdot}\ldots{\cdot}q_n)$. $\square$

We get a graph-theoretic characterization for the Cayley graphs of left-cancellative monoids: they are the Cayley graphs of monoids which are propagating.

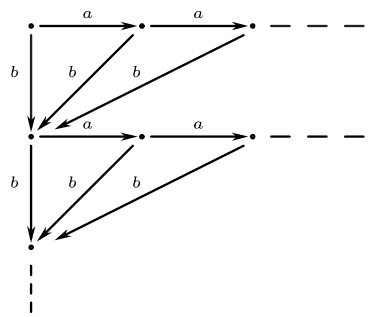
Theorem 6.6 *A graph is a Cayley graph of a left-cancellative monoid* if and only if *it is rooted, simple, deterministic and propagating.*

Proof.
$\Longrightarrow$: By Facts 3.1, 4.3, 6.3, 6.5.
$\Longleftarrow$: Let G be a simple, deterministic and propagating graph with a root r.
It remains to check that the r-path operation $\cdot$ defined in the proof of Theorem 4.4 is left-cancellative.
Assume that $s \cdot t \ = \ s \cdot t'$.
As r is a root, there exists $u,v \in A_G^*$ such that $r \overset{u}{\longrightarrow}_G t$ and $r \overset{v}{\longrightarrow}_G t'$.
By definition of $\cdot$ we get $s \overset{u}{\longrightarrow}_G s \cdot t$ and $s \overset{v}{\longrightarrow}_G s \cdot t' \ = \ s \cdot t$. Thus $s \overset{u,v}{\longrightarrow}_G$.
As G is propagating, $r \overset{u,v}{\longrightarrow}_G$. As G is deterministic, it follows that $t = t'$. $\square$

For instance, let us consider the following graph:

$$G \ = \ \{\, b^m a^n \overset{a}{\longrightarrow} b^m a^{n+1} \mid m,n \geq 0 \,\} \cup \{\, b^m a^n \overset{b}{\longrightarrow} b^{m+1} \mid m,n \geq 0 \,\}$$

represented below.



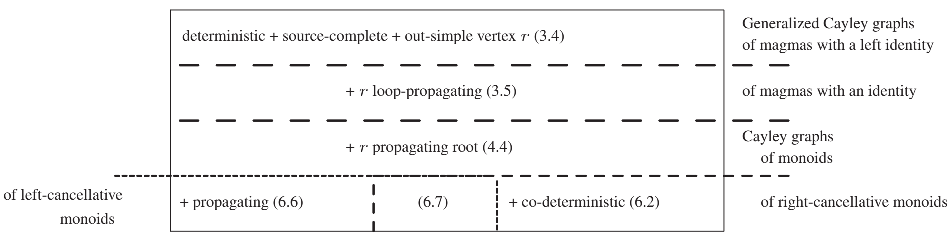
It is a *skeleton* of the graph of ω^2 where a is the successor and b goes to the next limit ordinal: $(V_G, \longrightarrow_G^*)$ is isomorphic to $(\omega^2, \leq)$. This graph is not co-deterministic hence by Fact 6.1 is not a generalized Cayley graph of a right-cancellative monoid. By Theorem 6.6, $G = \mathcal{C}(V_G, \{a,b\})$ where $(V_G, \cdot)$ is the left-cancellative monoid defined for any $m,n,p,q \geq 0$ by

$$b^m a^n \cdot b^p a^q = \begin{cases} b^m a^{n+q} & \text{if } p = 0 \\ b^{m+p} a^q & \text{if } p \neq 0. \end{cases}$$

As the converse of Theorems 6.2 and 6.6 define the same monoid, it follows a characterization for the Cayley graphs of monoids which are *cancellative i.e.* both left-cancellative and right-cancellative.

Corollary 6.7 *A graph is a Cayley graph of a cancellative monoid* if and only if *it is rooted, simple, deterministic, co-deterministic, propagating.*

Let us summarize the previous characterizations.



It remains to extend the path propagation to chains in Corollary 6.7 to get a characterization for the Cayley graphs of groups.

7 Cayley graphs of groups

We present a graph-theoretic characterization for the Cayley graphs of groups in a strong form [1] (see Theorem 7.3.b) and in a weak form [2] (see Theorem 7.3.c). It follows a characterization for the generalized Cayley graphs of groups (see Theorem 7.6).

We start by introducing some basic graph notions. We need to circulate in a graph G in the direct and inverse direction of the arrows. Let $\overline{} : A_G \longrightarrow A - A_G$ be an injective mapping whose $\overline{A_G} = \{ \overline{a} \mid a \in A_G \}$ is a disjoint copy of A_G. This allows to define the *inverse* of G as the graph $\overline{G} = \{ t \xrightarrow{\overline{a}} s \mid s \xrightarrow{a}_G t \}$. So $V_{\overline{G}} = V_G$ and $A_{\overline{G}} = \overline{A_G}$.
A path of $G \cup \overline{G}$ *i.e.* $s \xrightarrow{u}_{G \cup \overline{G}} t$ with $u \in (A_G \cup \overline{A_G})^*$ is a *chain* of G also denoted by $s \xrightarrow{u}_G t$

where $s \xrightarrow{\overline{a}}_G t$ means that $t \xrightarrow{a}_G s$ for any $a \in A_G$. So the chain $r \xrightarrow{a} s \xrightarrow{\overline{b}} t$ corresponds to $r \xrightarrow{a} s \xleftarrow{b} t$. We write $\longleftrightarrow_G$ for the *adjacency relation* $\longrightarrow_{G \cup \overline{G}}$ *i.e.* $s \longleftrightarrow_G t$ for $s \longrightarrow_G t$ or $t \longrightarrow_G s$. We denote by $\longleftrightarrow_G^*$ for the *chain relation* $\longrightarrow_{G \cup \overline{G}}^*$. Recall that a graph G is *connected* if $G \cup \overline{G}$ is strongly connected *i.e.* $s \longleftrightarrow_G^* t$ for all $s,t \in V_G$. So G is co-deterministic if $\overline{G}$ is deterministic. We say that G is *target-complete* if $\overline{G}$ is source-complete *i.e.* for any label a and any vertex s, there is at least one edge labeled by a of target s.
A *chain-propagating vertex* r of a graph G is a propagating vertex of $G \cup \overline{G}$ *i.e.*

$$r \xrightarrow{u,v}_G \quad \Longrightarrow \quad s \xrightarrow{u,v}_G \text{ for any } s \in V_G \text{ and } u,v \in (A_G \cup \overline{A_G})^*$$

meaning that if there are two chains from r of the same target labeled by u and v then from any vertex, there are two chains of the same target labeled by u and v.
We say that a graph is a *chain-propagating graph* when all its vertices are chain-propagating.
Let us recall the vertex-transitivity of a graph. An *automorphism* of G is an isomorphism from G to G. Two vertices s,t of a graph G are *isomorphic* and we write $s \simeq_G t$ if $t = h(s)$ for some automorphism h of G. A graph G is *vertex-transitive* if its automorphism group acts transitively upon its vertices *i.e.* if all its vertices are isomorphic: $s \simeq_G t$ for every $s,t \in V_G$. This implies that the graph is chain-propagating. The converse is true when the graph is both deterministic and co-deterministic.

Lemma 7.1 *A deterministic and co-deterministic graph is vertex-transitive* if and only if *it is chain-propagating.*

Proof.
$\Longrightarrow$: Let G be a vertex-transitive graph and $s \xrightarrow{u,v}_G$ for some $u,v \in (A_G \cup A_{\overline{G}})^*$.
For any $t \in V_G$, we have $s \simeq_G t$ hence $t \xrightarrow{u,v}_G$ thus s is chain-propagating.
$\Longleftarrow$: Let G be a deterministic and co-deterministic graph which is chain-propagating.
In particular G is source and target-complete since these properties are only on the labels of the graph.
Let $s,t \in V_G$. Let us check that $s \simeq_G t$.
As s,t are chain-propagating, we have $s \xrightarrow{u,v}_G \iff t \xrightarrow{u,v}_G$ for any $u,v \in (A_G \cup A_{\overline{G}})^*$.
So we can define the mapping $f_{s,t} : V_G \longrightarrow V_G$ for any $r \in V_G$ by

$$f_{s,t}(r) = \begin{cases} r' & \text{if } s \xrightarrow{u} r \text{ and } t \xrightarrow{u} r' \text{ for some } u \in (A_G \cup A_{\overline{G}})^* \\ r' & \text{otherwise and if } s \xrightarrow{u} r' \text{ and } t \xrightarrow{u} r \text{ for some } u \in (A_G \cup A_{\overline{G}})^* \\ r & \text{otherwise.} \end{cases}$$

Let us check that $f_{s,t}$ is a bijective morphism of inverse $f_{t,s}$. Thus and as $f_{s,t}(s) = t$, we get $s \simeq_G t$.
Let $r \xrightarrow{a}_G r'$ with $a \in A_G \cup \{\varepsilon\}$. We have to show that $f_{t,s}(f_{s,t}(r)) = r$ and $f_{s,t}(r) \xrightarrow{a}_G f_{s,t}(r')$.
We distinguish the three complementary cases below.
Case 1: $r \longleftrightarrow_G^* s$. So $s \xrightarrow{u}_G r$ for some $u \in (A_G \cup A_{\overline{G}})^*$.
Thus $t \xrightarrow{u}_G f_{s,t}(r)$ and $t \xrightarrow{ua}_G f_{s,t}(r')$. As G is deterministic, $f_{s,t}(r) \xrightarrow{a}_G f_{s,t}(r')$.
Moreover $s \xrightarrow{u}_G f_{t,s}(f_{s,t}(r))$. As G is deterministic and co-deterministic, $f_{t,s}(f_{s,t}(r)) = r$.
Case 2: $\neg(r \longleftrightarrow_G^* s)$ and $r \longleftrightarrow_G^* t$. So $\neg(r' \longleftrightarrow_G^* s)$ and $t \xrightarrow{u}_G r$ for some $u \in (A_G \cup A_{\overline{G}})^*$.
Thus $s \xrightarrow{u}_G f_{s,t}(r)$ and $s \xrightarrow{ua}_G f_{s,t}(r')$. As G is deterministic, $f_{s,t}(r) \xrightarrow{a}_G f_{s,t}(r')$.
Moreover $t \xrightarrow{u}_G f_{t,s}(f_{s,t}(r))$. As G is deterministic and co-deterministic, $f_{t,s}(f_{s,t}(r)) = r$.
Case 3: $\neg(r \longleftrightarrow_G^* s)$ and $\neg(r \longleftrightarrow_G^* t)$. So $f_{s,t}(r) = r$ and $f_{s,t}(r') = r'$ and $f_{t,s}(r) = r$. $\square$

A *group* $(M, \cdot)$ is a monoid whose each element p has an *inverse* p^{-1}: $p \cdot p^{-1} = p^{-1} \cdot p = 1$.

Fact 7.2 *Any generalized Cayley graph of a group is co-deterministic, simple and vertex-transitive.*

Proof.
Let G be a generalized Cayley graph of a group. By Facts 6.1 and 6.3, G is co-deterministic and simple. It remains to chack that G is vertex-transitive. Let $r \xrightarrow{u,v} s$ for $u, v \in (A_G \cup A_{\overline{G}})^*$. Let $t \in V_G$.
By Lemma 7.1, it remains to check that $t \xrightarrow{u,v}$. As

$$x \xrightarrow{\overline{[\![q]\!]}} y \iff y \xrightarrow{[\![q]\!]} x \iff x = y \cdot q \iff y = x \cdot q^{-1}$$

we extend $[\![\ ]\!]$ on $Q^{-1} - Q$ by $[\![q^{-1}]\!] = \overline{[\![q]\!]}$ for any $q \in Q$ such that $q^{-1} \notin Q$.
Thus $u = [\![p_1]\!] \dots [\![p_m]\!]$ and $v = [\![q_1]\!] \dots [\![q_n]\!]$ for some $m, n \geq 0$ and $p_1, \dots, p_m, q_1, \dots, q_n$ in $Q \cup Q^{-1}$. So $r \cdot p_1 \cdot \ldots \cdot p_m = s = r \cdot q_1 \cdot \ldots \cdot q_n$ *i.e.* $p_1 \cdot \ldots \cdot p_m = q_1 \cdot \ldots \cdot q_n$.
Hence $t \xrightarrow{u,v} t \cdot p_1 \cdot \ldots \cdot p_m = t \cdot q_1 \cdot \ldots \cdot q_n$. $\square$

A *Cayley graph of a group* M has been defined in [4] and is a generalized Cayley graph $\mathcal{C}[\![M, Q]\!]$ such that M is generated by Q *i.e.* M is equal to the least subgroup $(Q \cup Q^{-1})^*$ containing Q where $Q^{-1} = \{ q^{-1} \mid q \in Q \}$ is the set of inverses of the elements in Q.
We can give a simple characterization for the Cayley graphs of groups.

Theorem 7.3 *For any graph G, the following properties are equivalent:*
a) *G is a Cayley graph of a group,*
b) *G is connected, deterministic, co-deterministic, simple, vertex-transitive,*
c) *G is connected, deterministic, co-deterministic, with a chain-propagating simple vertex which is source and target-complete.*

Proof.
(a) $\Longrightarrow$ (b): Let $G = \mathcal{C}[\![M, Q]\!]$ for some group $(M, \cdot)$ generated by Q.
By Facts 3.1 and 7.2, it remains to check that G is connected. Let $s \in V_G = M$.
So $s = q_1^{\epsilon_1} \cdot \ldots \cdot q_n^{\epsilon_n}$ for some $n \geq 0, q_1, \dots, q_n \in Q$ and $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$.
Thus $1 \xrightarrow{a_1 \dots a_n} s$ where for any $1 \leq i \leq n$, $a_i = [\![q_i]\!]$ if $\epsilon_i = 1$ and $a_i = \overline{[\![q_i]\!]}$ if $\epsilon_i = -1$.

(b) $\Longrightarrow$ (c): any vertex-transitive graph is source and target-complete, and by Lemma 7.1.

(c) $\Longrightarrow$ (a): Let r be a chain-propagating vertex of G which is also simple, source and target-complete.
So G is source-complete and target-complete.
We extend a r-path operation into a *r-chain operation* as being a binary operation $\cdot$ on V_G such that

$$r \xrightarrow{u} t \implies s \xrightarrow{u} s \cdot t \quad \text{for any } s, t \in V_G \text{ and } u \in (A_G \cup \overline{A_G})^*$$

which is illustrated below.

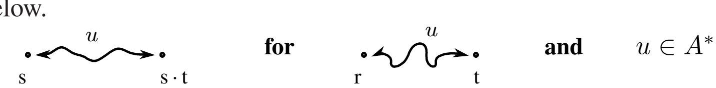


Figure 7.4 *A r-chain operation on the vertex set of a graph.*

Let us check the existence and the unicity of a r-chain operation.
Let $s, t \in V_G$. Let us check that $s \cdot t$ is well-defined.
As G is connected, there exists $u \in (A_G \cup A_{\overline{G}})^*$ such that $r \xrightarrow{u}_G t$.

As r is chain-propagating, there exists x such that $s \xrightarrow{u}_G x$.
Let $r \xrightarrow{v}_G t$ for some $v \in (A_G \cup A_{\overline{G}})^*$. So $r \xrightarrow{u,v}_G$.
As r is chain-propagating, we have $s \xrightarrow{u,v}_G$.
As G is deterministic and co-deterministic, we get $s \xrightarrow{v}_G x$.
Thus $s \cdot t \,=\, x$ is well-defined.

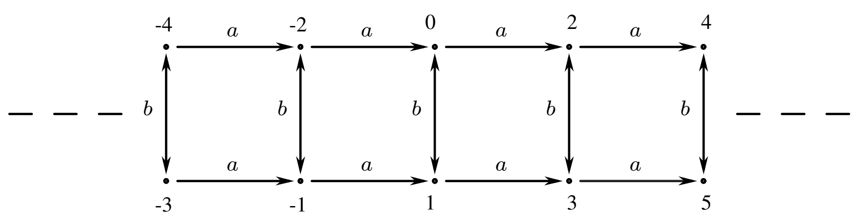
The chain operation $\cdot$ is a r-edge operation.
By Lemma 3.3, $G \,=\, \mathcal{C}[\![V_G\,,\, \longrightarrow_G(r)]\!]$ where $[\![s]\!] = a$ for any $r \xrightarrow{a} s$.
It remains to check that $(V_G, \cdot)$ is a group.
In that case and having $\mathcal{C}[\![V_G\,,\, \longrightarrow_G(r)]\!]$ connected, it is generated by $\longrightarrow_G(r)$.

i) Let us check that r is an identity. Let $s \in V_G$.
As $r \xrightarrow{\varepsilon}_G r$ and by definition of $\cdot$, we have $s \xrightarrow{\varepsilon}_G s \cdot r$ *i.e.* $s \cdot r = s$.
Furthermore and as G is connected, there exists $u \in (A_G \cup A_{\overline{G}})^*$ such that $r \xrightarrow{u}_G s$.
By definition of $\cdot$, we have $r \xrightarrow{u}_G r \cdot s$. As G is deterministic and co-deterministic, $r \cdot s = s$.

ii) Let us check that $\cdot$ is associative. Let $x, y, z \in V_G$.
As G is connected, there exists $v, w \in (A_G \cup A_{\overline{G}})^*$ such that $r \xrightarrow{v} y$ and $r \xrightarrow{w} z$.
Thus $x \xrightarrow{v} x \cdot y \xrightarrow{w} (x \cdot y) \cdot z$ and $y \xrightarrow{w} y \cdot z$. So $r \xrightarrow{vw} y \cdot z$ hence $x \xrightarrow{vw} x \cdot (y \cdot z)$.
As G is deterministic and co-deterministic, we get $(x \cdot y) \cdot z \,=\, x \cdot (y \cdot z)$.

iii) Let $s \in V_G$. Let us check that s has an inverse.
As G is connected, there exists $u \in (A_G \cup A_{\overline{G}})^*$ such that $r \xrightarrow{u} s$.
As G is source and target-complete, there exists $\overline{s} \in V_G$ such that $\overline{s} \xrightarrow{u} r$.
We have $\overline{s} \xrightarrow{u} \overline{s} \cdot s$. As G is deterministic and co-deterministic, $\overline{s} \cdot s \,=\, r$.
For $u \,=\, a_1 \ldots a_n$ with $n \geq 0$ and $a_1, \ldots, a_n \in A_G \cup A_{\overline{G}}$, let $u' \,=\, \overline{a_n} \ldots \overline{a_1}$ with $\overline{\overline{a}} \,=\, a$ for any $a \in A_G$. Thus $s \xrightarrow{u'} r$ and $r \xrightarrow{u'} \overline{s}$. So $s \xrightarrow{u'} s \cdot \overline{s}$.
As G is deterministic and co-deterministic, $s \cdot \overline{s} \,=\, r$. $\square$

Let us consider the following graph:



This graph satisfies the properties of Theorem 7.3 (b) or (c) hence is the Cayley graph $\mathcal{C}[\![\mathbb{Z}, \{1,2\}]\!]$ with $[\![1]\!] = b$ and $[\![2]\!] = a$, where $\cdot$ is the 0-chain operation defined for any $p, q \in \mathbb{Z}$ by

$$p \cdot q \;=\; \begin{cases} p+q & \text{if } p \text{ or } q \text{ is even,} \\ p+q-2 & \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

and makes that $(\mathbb{Z}, \cdot)$ is a group.

We can now obtain a characterization for the generalized Cayley graphs of groups. First, let us start with a basic fact on their *connected components* which are their maximal connected subsets.

Fact 7.5 *The connected components of the generalized Cayley graph of groups are the Cayley graphs of groups.*

Proof.
$\Longrightarrow$: By Facts 3.1, 7.2 and Theorem 7.3.
$\Longleftarrow$: Any Cayley graph of a group is a connected generalized Cayley graph. □

To characterize the generalized Cayley graphs of groups, we will give before the conclusion two examples showing that we cannot get rid of the axiom of choice. The axiom of choice is equivalent in ZF set theory to the property that any non-empty set has a group structure [5].

Theorem 7.6 *In ZFC set theory, a graph is a generalized Cayley graph of a group* if and only if *it is simple, deterministic, co-deterministic, vertex-transitive.*

Proof.
$\Longrightarrow$: By Facts 3.1 and 7.2.
$\Longleftarrow$: Let G be a simple, deterministic, co-deterministic, vertex-transitive graph.
Let $\mathrm{Comp}(G)$ be the set of connected components of G. Let us recall that a *representative set* (or a *transversal*) P of $\mathrm{Comp}(G)$ is a subset of V_G that has exactly one vertex in each connected component: $|P \cap V_C| = 1$ for any $C \in \mathrm{Comp}(G)$. Its *canonical mapping* $\pi_P : V_G \longrightarrow P$ is the mapping associating to each vertex s the vertex of P in the same connected component: $s \longleftrightarrow^*_G \pi_P(s)$ for any $s \in V_G$. So $\pi_P(p) = p$ for any $p \in P$.
Using ZFC set theory, there exists a representive set P and a binary operation $\circ$ such that $(P, \circ)$ is a group. Let r be its identity and $C \in \mathrm{Comp}(G)$ having r as vertex.
By (the proof of) Theorem 7.3, $(V_C, {}_r\cdot)$ is a group for the r-chain-operation ${}_r\cdot$ and of identity r. We take the group product $(P{\times}V_C, \cdot)$ with

$$(p, x) \cdot (q, y) \;=\; (p \circ q, x \,{}_r\!\cdot y) \text{ for any } p, q \in P \text{ and } x, y \in V_C\,.$$

This group is of identity (r, r).
As G is vertex-transitive, deterministic and co-deterministic, we can define the mapping

$$f : P{\times}V_C \longrightarrow V_G \text{ such that } r \xrightarrow{u}_G x \;\Longrightarrow\; p \xrightarrow{u}_G f(p, x) \text{ for (any) } u \in (A_G \cup \overline{A_G})^*.$$

Thus f is a bijection hence $(V_G, \cdot)$ is a group where

$$f(p, x) \cdot f(q, y) \;=\; f(p \circ q, x \,{}_r\!\cdot y) \text{ for any } p, q \in P \text{ and } x, y \in V_C\,.$$

This group $(V_G, \cdot)$ is of identity $f(r, r) = r$.
Let us show that this operation $\cdot$ is a r-chain operation hence by Lemma 3.3,

$$G \;=\; \mathcal{C}[\![V_G\,, \longrightarrow_G(r)]\!] \text{ with } [\![s]\!] = a \text{ for any } r \xrightarrow{a} s.$$

We say that a binary operation ${}_P\cdot$ on V_G is a P*-chain operation* if for any $s, t \in V_G$,

$$(\pi_P(s) \xrightarrow{u}_G s \;\wedge\; \pi_P(t) \xrightarrow{v}_G t) \;\Longrightarrow\; \pi_P(s) \circ \pi_P(t) \xrightarrow{uv}_G s \,{}_P\!\cdot t \text{ for any } u, v \in (A_G \cup \overline{A_G})^*$$

which is illustrated below. Any P-chain operation is a r-chain operation.

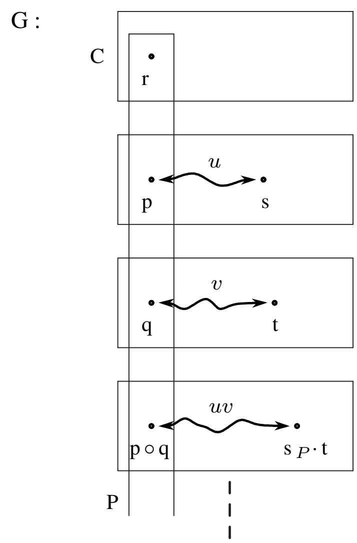


Figure 7.7 *A P-chain operation on the vertex set of a graph.*

As G is vertex-transitive, deterministic and co-deterministic, ${}_P\cdot$ exists and is unique.
Let us check that $\cdot$ and ${}_P\cdot$ are equal.
Let $p,q \in P$ and $x,y \in V_C$. We have to check that $f(p,x)\ {}_P\cdot\ f(q,y) \ = \ f(p,x)\cdot f(q,y)$.
Let $u,v \in (A_G \cup \overline{A_G})^*$ such that $r \xrightarrow{u}_G x$ and $r \xrightarrow{v}_G y$.
By definition of ${}_r\cdot$, we have $x \xrightarrow{v}_G x\ {}_r\cdot\ y$ hence $r \xrightarrow{uv}_G x\ {}_r\cdot\ y$.
By definition of f, we get $p\circ q \xrightarrow{uv}_G f(p\circ q, x\ {}_r\cdot\ y)$, $p \xrightarrow{u}_G f(p,x)$ and $q \xrightarrow{v}_G f(q,y)$.
By definition of ${}_P\cdot$, we have $p\circ q \xrightarrow{uv}_G f(p,x)\ {}_P\cdot\ f(q,y)$.
As G is deterministic, we get $f(p,x)\ {}_P\cdot\ f(q,y) \ = \ f(p\circ q, x\ {}_r\cdot\ y) \ = \ f(p,x)\cdot f(q,y)$. □

For instance let us consider the following graph:

$$\begin{aligned} G \quad &= \quad \{\ (m,i,p) \xrightarrow{a} (m+1,i,p) \mid m,p \in \mathbb{Z} \wedge i \in \{0,1\}\ \} \\ &\cup \quad \{\ (m,i,p) \xrightarrow{b} (m,1-i,p) \mid m,p \in \mathbb{Z} \wedge i \in \{0,1\}\ \} \end{aligned}$$

of representation the countable repetition of the preceding one.
By Theorem 7.6, $G \ = \ \mathcal{C}[\![\mathbb{Z}{\times}\{0,1\}{\times}\mathbb{Z}, \{(1,0,0),(0,1,0)\}]\!]$ with $[\![(1,0,0)]\!] = a$, $[\![(0,1,0)]\!] = b$ and for the group operation $(m,i,p)\cdot(n,j,q) \ = \ (m+n, i+j\,(\mathrm{mod} 2), p+q)$.
Note that the graph $G \ = \ \{\ s \xrightarrow{q} s+q \mid s \in \mathbb{R} \wedge q \in \mathbb{Q}\ \}$ is the generalized Cayley graph $\mathcal{C}(\mathbb{R},\mathbb{Q})$ for the group $(\mathbb{R},+)$. Its unlabeled edge relation $\longrightarrow_G$ is the *Vitali equivalence* (1905) for which we need the axiom of choice to select an element in each class (and the selected elements form a set which is not Lebesgue measurable).
Similarly the graph $\{\ P \xrightarrow{Q} P \vartriangle Q \mid P,Q \subseteq \mathbb{N} \wedge Q$ finite $\}$ is the generalized Cayley graph $\mathcal{C}(2^{\mathbf{N}}, 2^{\mathbf{N}}_f)$ for the group $2^{\mathbf{N}}$ of the subsets of $\mathbb{N}$ with the symmetric difference operation $\vartriangle$, and for the set $2^{\mathbf{N}}_f$ of finite subsets of $\mathbb{N}$. Its unlabeled edge relation is the *Borel equivalence* E_0 and we also need the axiom of choice to have a selector for $2^{\mathbf{N}}/E_0$ usually denoted by $\mathrm{P}(\omega)/\mathrm{Fin}$.

Conclusion

We obtained simple graph-theoretic characterizations for the Cayley graphs of semigroups, monoids and groups. These characterizations have been extended to semilattices, commutative monoids and abelian groups [2]. Simple characterizations have also been given for the generalized Cayley graphs of left-quasigroups and quasigroups [1]. The characterizations obtained are relevant to decide whether a graph is a Cayley graph of a semigroup, a monoid, a group. This was done [1] for the family of *end-regular graphs* [7] which includes all the finite graphs.

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