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List Edge and List Total Colorings of Planar Graphs without non-induced 7-cycles

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Giving a planar graph $G$, let $\chi'_e(G)$ and $\chi''_v(G)$ denote the list edge chromatic number and list total chromatic number of $G$ respectively. It is proved that if $G$ is a planar graph without non-induced 7-cycles, then $\chi'_e(G) \leq \Delta(G) + 1$ and $\chi''_v(G) \leq \Delta(G) + 2$ where $\Delta(G) \geq 7$.

Keywords: List coloring; Planar graph; Choosability.

1 Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G$ be a graph and we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of $G$, respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (resp. face) $x$ in $G$. A vertex (resp. face) $x$ is called a $k$-vertex (resp. $k$-face), $k^+$-vertex (resp. $k^+$-face), or $k^-$-vertex, if $d(x) = k$, $d(x) \geq k$, or $d(x) \leq k$. We use $(d_1, d_2, \ldots, d_n)$ to denote a face $f$ if $d_1, d_2, \ldots, d_n$ are the degrees of vertices which are incident with the face $f$. If $u_1, u_2, \ldots, u_n$ are the vertices on the boundary walk of a face $f$, then we write $f = u_1u_2 \cdots u_n$. Let $\delta(f)$ denote the minimal degree of vertices which are incident with $f$. We use $f_i(v)$ to denote the number of $i$-faces which are incident with $v$ for each $v \in V(G)$. Let $n_i(f)$ denote the number of $i$-vertices which are incident with $f$ for each $f \in F(G)$. A cycle $C$ of length $k$ is called $k$-cycle, and if there is at least one edge $xy \in E(G) \setminus E(C)$ and $x, y \in V(C)$, the cycle $C$ is called non-induced $k$-cycle.

The mapping $L$ is said to be a total assignment for a graph $G$ if it assigns a list $L(x)$ of possible colors to each element $x \in V(G) \cup E(G)$. If $G$ has a proper total coloring $\phi(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that $G$ is total-$L$-colorable. Let $f : V(G) \cup E(G) \to N$ where $f$ is a function into the positive integers. We say that $G$ is total-$f$-choosability if it is total-$L$-colorable for every total assignment $L$.

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satisfying \(|L(x)| = f(x)| for all \(x \in V(G) \cup E(G)\). The list total coloring number \(\chi''_t(G)\) of \(G\) is the smallest integer \(k\) such that \(G\) is total-\(f\)-choosability when \(f(x) = k\) for each \(x \in V(G) \cup E(G)\). The list edge coloring number \(\chi'_e(G)\) is defined similarly in terms of coloring edges alone; and so is the concept of edge-\(f\)-choosability. On the list coloring number of a graph \(G\), there is a famous conjecture known as the List Coloring Conjecture.

**Conjecture 1** For a multigraph \(G\),

\[
(a) \chi'_t(G) = \chi'(G); \quad \quad (b) \chi''_t(G) = \chi''(G).
\]

Part \((a)\) of Conjecture 1 was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris [6, 11]. It is well known as the List Coloring Conjecture. Part \((b)\) was formulated by Borodin, Kostochka and Woodall [2]. Part \((a)\) has been proved for bipartite multigraphs [5]. Part \((a)\) and Part \((b)\) have been proved for outerplanar graphs [15], and graphs with \(\Delta \geq 12\) which can be embedded in a surface of nonnegative characteristic [2]. There are several related results for planar graphs, such as planar graphs without 4-cycles by Hou et al. [9], planar graphs without 4- and 5-cycles or planar graphs without intersecting 4-cycles by Liu et al. [13], planar graphs without triangles adjacent 4-cycles by Li et al. [14], planar graphs without intersecting triangles by Sheng et al. [18].

To confirm Conjecture 1 is a challenging work. From the Vizing Theorem and the Total Coloring Conjecture, the following weak conjecture is presented.

**Conjecture 2** For a multigraph \(G\),

\[
(a) \chi'_e(G) \leq \Delta(G) + 1; \quad \quad (b) \chi''_e(G) \leq \Delta(G) + 2.
\]

Part \((a)\) of Conjecture 2 has been proved for complete graphs of odd order [7]. Wang et al. confirmed part \((a)\) of Conjecture 2 for planar graphs without 6-cycles or without 5-cycles [17, 16]. Zhang et al. proved part \((a)\) of Conjecture 2 for planar graphs without triangles [19]. Hou et al. proved part \((a)\) of Conjecture 2 for planar graphs without adjacent triangles or 7-cycles [8]. Cai et al. confirmed part \((a)\) of Conjecture 2 for planar graphs without chordal 5-cycles [3]. Part \((b)\) of Conjecture 2 was proved by Hou et al. for planar graphs \(G\) with \(\Delta(G) \geq 9\) [10]. Dong et al. confirmed Conjecture 2 for planar graphs without 6-cycles with chord [4].

In this paper, we shall show the following result.

**Theorem** Let \(G\) be a planar graph without non-induced 7-cycles, if \(\Delta(G) \geq 7\), then \(\chi'_t(G) \leq \Delta(G) + 1\) and \(\chi''_t(G) \leq \Delta(G) + 2\).

## 2 Planar graphs without non-induced 7-cycles

First let us introduce some important lemmas.

**Lemma 3** Let \(G\) be a planar graph without non-induced 7-cycles. Then there is an edge \(uv \in E(G)\) such that \(\min\{d(u), d(v)\} \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil\) and \(d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}\).

**Proof:** Suppose to the contrary that \(G\) is a minimal counterexample to Lemma 3 in terms of the number of vertices and edges. Then we have \(\delta(G) \geq 3\).
By Euler’s formula $|V| - |E| + |F| = 2$ and $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E|$, we have

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -6(|V| - |E| + |F|) = -12.$$

Define an initial charge function $w$ on $V(G) \cup F(G)$ by setting $w(v) = 2d(v) - 6$ if $v \in V(G)$ and $w(f) = d(f) - 6$ if $f \in F(G)$, so that $\sum_{x \in V(G) \cup F(G)} w(x) = -12$. Now redistribute the charge according to the following discharging rules.

For convenience, let $\bar{w}(v)$ denote the total charge transferred from a vertex $v$ to all its incident 4- and 5-faces where $d(v) = 5$.

D1 Let $f$ be a 3-face incident with a vertex $v$. Then $v$ gives $f$ charge $\frac{4 - \bar{w}(v)}{f_3(v)}$ if $d(v) = 5$, $\frac{3}{2}$ if $d(v) \geq 6$.

D2 Let $f$ be a 4-face incident with a vertex $v$. Then $v$ gives $f$ charge $\frac{1}{2}$ if $d(v) = 4$, 5 and 6, 1 if $d(v) \geq 7$.

D3 Let $f$ be a 5-face incident with a vertex $v$. Then $v$ gives $f$ charge $\frac{1}{5}$ if $d(v) = 4$, 5 and 6, $\frac{1}{3}$ if $d(v) \geq 7$.

Let the new charge of each element $x$ be $w'(x)$ for each $x \in V(G) \cup F(G)$.

In the following, let us check the new charge $w'(x)$ of each element $x \in V(G) \cup F(G)$.

Suppose $d(v) = 3$. Then $w'(v) = w(v) = 0$.

Suppose $d(v) = 4$. Then $w(v) = 2f_4(v) \leq 4$. If $2 \leq f_4(v) \leq 4$, then $f_5(v)$ is $0$ for $G$ contains no non-induced 7-cycles. We have $w'(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by D2. Otherwise, i.e. $f_4(v) \leq 1$, then $f_5(v)$ is $4$. Thus we have $w'(v) > 2 - \frac{1}{2} \times 4 = \frac{5}{2}$ if $D1$ and D3.

Suppose $d(v) = 5$. Then $w(v) = 4$, $f_3(v) \leq 5$. If $1 \leq f_3(v)$, then $w'(v) \geq 4 - \frac{4 - \bar{w}(v)}{f_3(v)}f_3(v) = \bar{w}(v) = 0$ by D1. Otherwise, i.e. $f_3(v) = 0$, then $f_4(v) + f_5(v) \leq 5$. It is clear that $w'(v) > 4 - \frac{1}{2} \times 5 = \frac{3}{2}$ if $D2$ and D3.

Suppose $d(v) = 6$. Then $w(v) = 6$, $f_3(v) \leq 4$ for $G$ contains no non-induced 7-cycles. If $f_3(v) = 4$, then $f_4(v) = 0$ and $f_5(v) = 0$ for $G$ contains no non-induced 7-cycles. We have $w'(v) \geq 6 - \frac{3}{2} \times 4 = 0$ by D1. If $f_5(v) \leq 3$, then it is clear that $w'(v) > 6 - \frac{3}{2} \times 3 - \frac{1}{2} \times 3 = 0$ by D1, D2 and D3.

Suppose $d(v) = 7$. Then $w(v) = 8$, $f_3(v) \leq 5$ for $G$ contains no non-induced 7-cycles.

Suppose $f_5(v) = 5$. Then $f_4(v) = 0$ and $f_5(v) = 0$ for $G$ contains no non-induced 7-cycles. We can get $w'(v) \geq 8 - \frac{3}{2} \times 5 = \frac{1}{2}$ if $0$ by D1.

Suppose $f_3(v) = 4$. Then $f_3(v) \leq 2$. If $f_4(v) = 2$, then $f_5(v) = 0$ for $G$ contains no non-induced 7-cycles. We have $w'(v) \geq 8 - \frac{3}{2} \times 4 - 1 \times 2 = 0$ by D1 and D2. If $f_5(v) \geq 1$, then $f_5(v) \leq 1$ for $G$ contains no non-induced 7-cycles. We have $w'(v) \geq 8 - \frac{3}{2} \times 4 - 1 - \frac{1}{4} = \frac{5}{2} > 0$ by D1, D2 and D3.

Suppose $f_3(v) = 3$. Then $f_4(v) \leq 2$ and $f_5(v) \leq 2$ for $G$ contains no non-induced 7-cycles. It is clear that $w'(v) > 8 - \frac{3}{2} \times 3 - 1 \times 2 - \frac{1}{2} \times 2 = \frac{5}{2} > 0$ by D1, D2 and D3.

Suppose $f_4(v) = 2$. Then it is clear that $w'(v) > 8 - \frac{3}{2} \times 2 - 1 \times 5 = 0$ by D1, D2 and D3.

Suppose $d(v) = 8$. Then $w(v) = 10$, $f_3(v)$ is $6$ for $G$ contains no non-induced 7-cycles. If $f_5(v) = 6$, then $f_4(v) = 0$ and $f_5(v) = 0$ for $G$ contains no non-induced 7-cycles. We have $w'(v) \geq 10 - \frac{3}{2} \times 6 = 1 > 0$ by D1. If $f_3(v) = 5$, then $f_4(v) \leq 1$ and $f_5(v) \leq 1$ for $G$ contains no non-induced 7-cycles. We can get $w'(v) \geq 10 - \frac{3}{2} \times 5 - 1 - \frac{1}{4} = \frac{7}{2} > 0$ by D1, D2 and D3. If $f_5(v) \leq 4$, then it is clear that $w'(v) \geq 10 - \frac{3}{2} \times 4 - 1 \times 4 = 0$ by D1, D2 and D3.
Lemma 4 Let $G$ be a planar graph without non-induced 7-cycles. Then $\chi'_1(G) \leq k + 1$ and $\chi''_1(G) \leq k + 2$ where $k = \max\{\Delta(G), 7\}$.

**Proof:** Suppose to the contrary that $G'$ and $G''$ are minimal counterexamples to the conclusions for $\chi'_1$ and $\chi''_1$ respectively. Let $L'$ and $L''$ be list assignments such that $|L'(v)| = k + 1$ for each $v \in E(G)$, $G'$ is not edge-$L'$-colorable, and $|L''(x)| = k + 2$ for each $x \in V(G) \cup E(G)$, $G''$ is not total-$L''$-colorable. By Lemma 3 $G'$ and $G''$ contain an edge $vw \in E(G)$ such that $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta(G) + 1}{2} \rfloor$ and $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\} = k + 2$.
Let $\bar{G}' = G' - uv$. Then $\bar{G}'$ is edge-$L'$-colorable by assumption. For $d(u) + d(v) \leq k + 2$, there are at most $k$ edges which are adjacent to $uv$ in $\bar{G}'$. Thus there is at least one color in $L'(uv)$ which we can use to color $uv$. Then $G'$ is edge-$L'$-colorable, a contradiction.

Let $\bar{G}'' = G'' - uv$. Then $\bar{G}''$ is total-$L''$-colorable by assumption. For $d(u) + d(v) \leq k + 2$, there are at most $k$ edges which are adjacent to $uv$ in $\bar{G}''$. Thus there is at least one color in $L''(uv)$ which we can use to color $uv$. Then $G''$ is total-$L''$-colorable, a contradiction. From the above discussion, we have $\chi'_l(G) \leq k + 1$ and $\chi''_l(G) \leq k + 2$ where $k = \max \{\Delta(G), 7\}$. □

By Lemma 4, it is easy to obtain the main theorem.

**Theorem** Let $G$ be a planar graph without non-induced 7-cycles, if $\Delta(G) \geq 7$, then $\chi'_l(G) \leq \Delta(G) + 1$ and $\chi''_l(G) \leq \Delta(G) + 2$.

**References**


