Non-representable hyperbolic matroids (Extended abstract)

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Abstract. The generalized Lax conjecture asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices. Hyperbolic polynomials give rise to a class of (hyperbolic) matroids which properly contains the class of matroids representable over the complex numbers. This connection was used by the first author to construct counterexamples to algebraic (stronger) versions of the generalized Lax conjecture by considering a non-representable hyperbolic matroid. The Vámos matroid and a generalization of it are to this day the only known instances of non-representable hyperbolic matroids. We prove that the Non-Pappus and Non-Desargues matroids are non-representable hyperbolic matroids by exploiting a connection, due to Jordan, between Euclidean Jordan algebras and projective geometries. We further identify a large class of hyperbolic matroids that are parametrized by uniform hypergraphs and prove that many of them are non-representable. Finally we explore consequences to algebraic versions of the generalized Lax conjecture.

Résumé. La conjecture généralisée de Lax affirme que chaque cône hyperbolique est une partie linéaire du cône des matrices définies positives. Les polynômes hyperboliques engendrent une classe de matroïdes qui contient l'ensemble des matroïdes qui pensent être représenter sur les nombres complexes. Cette observation a été utilisé par le premier auteur pour construire un contre-exemple à des versions algébriques de la conjecture généralisée de Lax qui dépend d'un matroïde hyperbolique que l'on ne peut pas représenter sur les nombres complexes. Le matroïde de Vamos et une généralisation sont les seuls exemples de matroïdes que l'on connait aujourd'hui et qui ne peuvent pas être représenter sur les nombres complexes. On montre que les Non-Pappus et Non-Desarques matroïdes ne peuvent pas être représenté non plus en utilisant un lien, attribué à Jordan, entre l'Algébre de Jordan Euclidienne et la géométrie projective. On identifie aussi une classe de matroïdes hyperboliques qui sont paramétrées par des hypergraphes dont la plupart ne sont pas représentables. Finalement, on exploite des conséquences des versions algébriques de la conjecture de Lax généralisée.

Keywords. Matroid, hyperbolic polynomial, Generalized Lax conjecture, Jordan algebra

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1 Introduction

Although hyperbolic polynomials have their origin in PDE theory, they have during recent years been studied in diverse areas such as control theory, optimization, real algebraic geometry, probability theory, computer science and combinatorics, see [22, 23, 24, 25] and the references therein. To each hyperbolic polynomial is associated a closed convex (hyperbolicity) cone. Over the past 20 years methods have been developed to do optimization over hyperbolicity cones, which generalize semidefinite programming. A problem that has received considerable interest is the generalized Lax conjecture which asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices (of some size). Hence if the generalized Lax conjecture is true then hyperbolic programming is the same as semidefinite programming.

Choe *et al.* [7] and Gurvits [13] proved that hyperbolic polynomials give rise to a class of matroids. This class of matroids, called hyperbolic matroids or matroids with the weak half-plane property, properly contains the class of matroids which are representable over the complex numbers, see [7]. The second author used this fact to construct counterexamples to algebraic (stronger) versions of the generalized Lax conjecture. To better understand, and to identify potential counterexamples to the generalized Lax conjecture, it is of interest to study hyperbolic matroids which are not representable over $\mathbb{C}$, or even better not representable over any (skew) field. However previous to this work essentially just two such matroids were known: The Vámos matroid V_8 [26] and a generalization V_{10} [6]. In this paper we first show that the Non-Pappus and Non-Desargues matroids are hyperbolic (but not representable over any field) by considering a well known connection between hyperbolic polynomials and Euclidean Jordan algebras. Then we construct a family of hyperbolic matroids which are parametrized by uniform hypergraphs, and prove that many of these matroids fail to be representable over any field, and more generally over any modular lattice. The proof of the main result uses several ingredients. In order to prove that the polynomials coming from our family of matroids are hyperbolic we need to prove that certain symmetric polynomials are nonnegative. The results obtained generalize and strengthen several inequalities in the literature, such as the Laguerre–Turán inequality and Jensen's inequality. We finally explore some consequences to algebraic versions of the generalized Lax conjecture. We will only sketch some proofs in this extended abstract. We refer to [1] for full proofs.

2 Hyperbolic and stable polynomials

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e} - \mathbf{x})$ has only real zeros, see [23, 24]. Note that if h is a hyperbolic polynomial of degree d then we may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})),$$

where

$$\lambda_{\max}(\mathbf{x}) = \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) = \lambda_{\min}(\mathbf{x})$$

are called the *eigenvalues* of $\mathbf{x}$ (with respect to $\mathbf{e}$). The *hyperbolicity cone* of h with respect to $\mathbf{e}$ is the set $\Lambda_+(h, \mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \lambda_d(\mathbf{x}) \geq 0\}$. We usually abbreviate and write Λ_+ if there is no risk for confusion. We denote by Λ_{++} the interior Λ_+.

Example 2.1. An important example of a hyperbolic polynomial is $\det(X)$ where $X = (x_{ij})_{i,j=1}^n$ is a matrix of variables where we impose $x_{ij} = x_{ji}$. Note that $t \mapsto \det(tI - X)$ where $I = \text{diag}(1, \dots, 1)$, is the characteristic polynomial of a symmetric matrix so it has only real zeros. Hence $\det(X)$ is a hyperbolic polynomial with respect to I.

The next theorem which follows (see [19]) from a theorem of Helton and Vinnikov [14] proved the Lax conjecture (after Peter Lax [20]).

Theorem 2.1. *Suppose $h(x, y, z) \in \mathbb{R}[x, y, z]$ is of degree d and hyperbolic with respect to $e = (e_1, e_2, e_3)^T$. Suppose further that h is normalized such that $h(e) = 1$. Then there are symmetric $d \times d$ matrices A, B, C such that $e_1 A + e_2 B + e_3 C = I$ and*

$$h(x, y, z) = \det(xA + yB + zC).$$

Remark 2.2. The exact analogue of the Helton–Vinnikov theorem fails for $n > 3$ variables. This may be seen by a simple count of parameters, see [24].

A convex cone in $\mathbb{R}^n$ is *spectrahedral* if it is of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \text{ is positive semidefinite} \right\},$$

where A_i, $i = 1, \dots, n$ are symmetric matrices such that there exists a vector $(y_1, \dots, y_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n y_i A_i$ positive definite.

Conjecture 2.3 (Generalized Lax conjecture [24])**.** *All hyperbolicity cones are spectrahedral.*

We may reformulate Conjecture 2.3 as follows, see [24]. The hyperbolicity cone of $h(\mathbf{x})$ with respect to $\mathbf{e} = (e_1, \dots, e_n)$ is spectrahedral if there is a homogeneous polynomial $q(\mathbf{x})$ and real symmetric matrices $A_1, \dots, A_n$ of the same size such that

$$q(\mathbf{x})h(\mathbf{x}) = \det\left(\sum_{i=1}^n x_i A_i\right) \tag{2.1}$$

where $\Lambda_{++}(h, \mathbf{e}) \subseteq \Lambda_{++}(q, \mathbf{e})$ and $\sum_{i=1}^n e_i A_i$ is positive definite.

3 Hyperbolic polymatroids

We refer to [21] for undefined matroid terminology. The connection between hyperbolic/stable polynomials and matroids was first realized in [7]. A polynomial $P(\mathbf{x}) \in \mathbb{C}[x_1, \dots, x_n]$ is *stable* if $P(z_1, \dots, z_n) \neq 0$ whenever $\text{Im}(z_j) > 0$ for all $1 \leq j \leq n$. Choe *et al.* proved that if

$$P(\mathbf{x}) = \sum_{B \subseteq [m]} a(B) \prod_{i \in B} x_i \tag{3.1}$$

is a homogeneous, multiaffine and stable polynomial, then its *support*, $\mathcal{B} = \{B : a(B) \neq 0\}$, is the set of bases of a matroid, $\mathcal{M}$, on $[m]$. Such matroids are called *weak half-plane property matroids* (abbreviated WHPP–matroids). If further $P(\mathbf{x})$ can be chosen so that $a(B) \in \{0, 1\}$, then $\mathcal{M}$ is called a *half-plane property matroid* (abbreviated HPP–matroid). Then $P(\mathbf{x})$ is the *bases generating polynomial* of $\mathcal{M}$.

- All matroids representable over $\mathbb{C}$ are WHPP, [7].
- A binary matroid is WHPP if and only if it is HPP if and only if it is regular, [5, 7].
- No finite projective geometry $\mathrm{PG(r,n)}$ is WHPP, [5].
- The Vámos matroid V_8 is HPP (but not representable over any field), [26].

We shall now see how weak half-plane property matroids may conveniently be described in terms of hyperbolic polynomials.

For a positive integer m, let $[m] := \{1, 2, \dots, m\}$. A *polymatroid* is a function $r : 2^{[m]} \to \mathbb{N}$ satisfying

1. $r(\emptyset) = 0$,
2. $r(S) \leq r(T)$ whenever $S \subseteq T \subseteq [m]$,
3. r is *semimodular*, i.e.,
$$r(S) + r(T) \geq r(S \cap T) + r(S \cup T),$$
for all $S, T \subseteq [m]$.

Recall that rank functions of matroids on $[m]$ coincide with polymatroids r on $[m]$ with $r(\{i\}) \leq 1$ for all $i \in [m]$. Let $\mathcal{V} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$ be a tuple of vectors in $\Lambda_+ = \Lambda_+(h, \mathbf{e})$, where $\mathbf{e} \in \mathbb{R}^n$. The *(hyperbolic) rank*, $\mathrm{rk}(\mathbf{x})$, of $\mathbf{x} \in \mathbb{R}^n$ is defined to be the number of non-zero eigenvalues of $\mathbf{x}$, i.e., $\mathrm{rk}(\mathbf{x}) = \deg h(\mathbf{e} + t\mathbf{x})$. Define a function $r_{\mathcal{V}} : 2^{[m]} \to \mathbb{N}$ by

$$r_{\mathcal{V}}(S) = \mathrm{rk}\left(\sum_{i \in S} \mathbf{v}_i\right).$$

It follows from [13] (see also [4]) that $r_{\mathcal{V}}$ is a polymatroid. We call such polymatroids *hyperbolic polymatroids*. Hence if the vectors in $\mathcal{V}$ have hyperbolic rank at most one, then we obtain the hyperbolic rank function of a *hyperbolic matroid*. The following proposition is implicit in [4, 13].

Proposition 3.1. *A matroid is hyperbolic if and only if it has the weak half-plane property.*

4 Modularly represented hyperbolic matroids

Suppose L is a lattice with a smallest element $\hat{0}$, and $f : L \to \mathbb{N}$ is a function satisfying

1. $f(\hat{0}) = 0$,
2. if $x \leq y$, then $f(x) \leq f(y)$,
3. for any $x, y \in L$, $\quad f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$.

If $x_1, \dots, x_m \in L$, then the function $r : 2^{[m]} \to \mathbb{N}$ defined by

$$r(S) = f\left(\bigvee_{i \in S} x_i\right)$$

defines a polymatroid. Indeed all polymatroids arise in this manner. However if f is modular we say that r is *modularly represented*. Hence all linear matroids as well as all projective geometries are modularly represented. Although finite projective geometries are not hyperbolic (cf [5]) we shall now see that certain infinite projective geometries coming from real Euclidean Jordan algebras give rise to modularly represented (but non-linear) hyperbolic matroids.

Let C be a closed convex cone in $\mathbb{R}^n$. Recall that a *face* F of C is a convex subcone of C with the property that $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{y} - \mathbf{x} \in C$ and $\mathbf{y} \in F$ implies $\mathbf{x} \in F$. The collection of all faces of C is a lattice, $L(C)$, under containment with smallest element $\{0\}$ and largest element C. Clearly $F \wedge G = F \cap G$ and $F \vee G = \bigcap_H H$, where H ranges over all faces containing F and G. The collection of all relative interiors of faces of C partitions C. If $F_{\mathbf{x}}$ is the unique face that contains $\mathbf{x} \in C$ in its relative interior, then $F_{\mathbf{x}} \vee F_{\mathbf{y}} = F_{\mathbf{x}+\mathbf{y}}$. The *rank* of a face F of the hyperbolicity cone Λ_+ is defined by

$$\mathrm{rk}(F) = \max_{\mathbf{x} \in F} \mathrm{rk}(\mathbf{x}).$$

Note that if $L(\Lambda_+)$ is a graded lattice, then the above hyperbolic rank function is not necessarily the rank function of $L(\Lambda_+)$.

An algebra $(A, \circ)$ over a field $\mathbb{K}$ is said to be a *Jordan algebra* if for all $a, b \in A$

$$a \circ b = b \circ a \qquad \text{and} \qquad a \circ (a^2 \circ b) = a^2 \circ (a \circ b).$$

A Jordan algebra is *Euclidean* if

$$a_1^2 + \cdots + a_k^2 = 0 \implies a_1 = \cdots = a_k = 0$$

for all $a_1, \ldots, a_k \in A$. We refer to [9] for the facts about Euclidean Jordan algebras mentioned below. By a theorem of Jordan, von Neumann and Wigner [17] the simple finite dimensional real Euclidean Jordan algebras classify into four infinite families and one exceptional algebra (the Albert algebra) as follows:

1. $H_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) - the algebra of Hermitian $n \times n$ matrices over $\mathbb{K}$ with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.
2. $\mathbb{R}^n \oplus \mathbb{R}$ - the real inner product space with inner product $(u \oplus \lambda, v \oplus \mu) = (u, v)_{\mathbb{R}^n} + \lambda\mu$ and Jordan product $(u \oplus \lambda) \circ (v \oplus \mu) = (\mu u + \lambda v) \oplus ((u, v)_{\mathbb{R}^n} + \lambda\mu)$.
3. $H_3(\mathbb{O})$ - the algebra of octonionic Hermitian 3×3 matrices with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

A characteristic property of finite dimensional real Euclidean Jordan algebras is the existence of a spectral theorem. A finite dimensional real Euclidean Jordan algebra A is also equipped with a hyperbolic determinant polynomial $\det : A \to \mathbb{R}$. The hyperbolicity cone of $\det$ is $\{a^2 : a \in A\}$. Each projective geometry is a (simple) modular geometric lattice, and each modular geometric lattice is a direct product of a Boolean algebra with projective geometries, see [2, p. 93]. The following proposition is a famous connection between Jordan algebras and projective geometries (see e.g [1]).

Proposition 4.1. *Let A be a finite dimensional real Euclidean Jordan algebra and let $\Lambda_+ = \{a^2 : a \in A\}$ denote the hyperbolicity cone of $\det : A \to \mathbb{R}$. Then $L(\Lambda_+)$ is a modular geometric lattice. In particular if A is simple, then $L(\Lambda_+)$ is a projective geometry.*

The Non-Pappus and Non-Desargues configurations are depicted in Fig. 1. The configurations give rise to rank 3 matroids where three points are dependent if and only if they are collinear. The Non-Pappus and Non-Desargues matroids are not linear but may be represented over the projective geometries associated to the Euclidean Jordan algebras $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$, respectively. This may be deduced from the coordinatizations in [11, Theorem 6.21] and [12]. Hence, since the determinants associated to $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$ are hyperbolic (Proposition 4.1):

Theorem 4.2. *The Non-Pappus and Non-Desargues matroids are hyperbolic.*

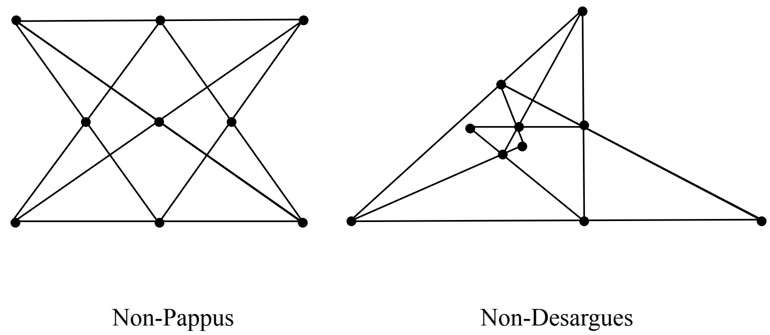


Fig. 1: The Non-Pappus and Non-Desargues configurations.

5 Non-modularly represented hyperbolic matroids

In this section we provide an infinite family of hyperbolic matroids that do not arise from modular geometric lattices. Although Ingleton's proof [15] of the next lemma only concerns linear matroids it extends verbatim to modularly represented matroids.

Lemma 5.1 (Ingleton's Inequality, [15]). *Suppose* $r : 2^{[m]} \to \mathbb{N}$ *is a modularly represented polymatroid and* $A, B, C, D \subseteq [m]$. *Then*

$$r(A \cup B) + r(A \cup C \cup D) + r(C) + r(D) + r(B \cup C \cup D) \leq$$
$$r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) + r(C \cup D).$$

The *Vámos matroid* V_8 is the rank-four matroid on $E = [8]$ having set of bases

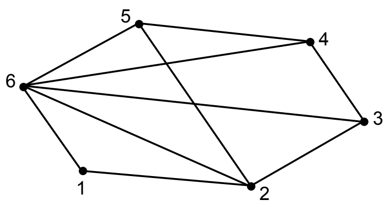
$$\mathcal{B}(V_8) = \binom{E}{4} \setminus \{\{1,2,3,4\}, \{1,2,5,6\}, \{1,2,7,8\}, \{3,4,5,6\}, \{5,6,7,8\}\}.$$

The rank function of the Vámos matroid fails to satisfy Ingleton's inequality (see [15]), and hence it is not modularly represented. Nevertheless Wagner and Wei [26] proved that V_8 has the half-plane property, and hence V_8 is hyperbolic. This was used in [4] to provide counterexamples to stronger algebraic versions of the generalized Lax conjecture.

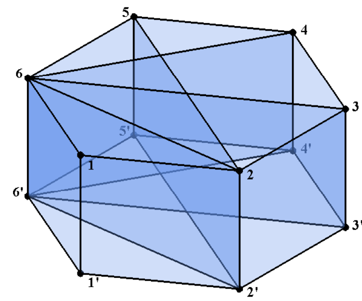
Burton, Vinzant and Youm [6] studied an infinite family of generalized Vámos matroids, $\{V_{2n}\}_{n \geq 4}$, and conjectured that all members of the family have the half-plane property. They proved their conjecture for $n = 5$. Below we generalize their construction and construct a family of matroids; one matroid for each uniform hypergraph. We prove that all matroids corresponding to simple graphs are HPP, and that all matroids corresponding to uniform hypergraphs are WHPP. In particular this proves the conjecture of Burton *et al.*

Recall that a *paving matroid* is a rank r matroid such that all its circuits have size at least r. A paving matroid of rank r is *sparse* if its hyperplanes all have size $r - 1$ or r.

Recall that a *hypergraph* H consists of a set $V(H)$ of *vertices* together with a set $E(H) \subseteq 2^{V(H)}$ of *hyperedges*. We say that a hypergraph H is r-*uniform* if all hyperedges have size r.



(a) A simple graph G



(b) The matroid V_G

Theorem 5.2. *Let H be an r-uniform hypergraph on $[n]$, and let $E = \{1, 1', \dots, n, n'\}$. Then*

$$\mathcal{B}(V_H) = \binom{E}{2r} \setminus \{e \cup e' : e \in E(H)\},$$

where $e' := \{i' : i \in e\}$, is the set of bases of a sparse paving matroid V_H of rank $2r$.

Theorem 5.3. *For each uniform hypergraph H, V_H is hyperbolic i.e., has the weak half-plane property.*

Theorem 5.4. *For each simple graph G, V_G has the half-plane property.*

Note that if remove one edge from the complete graph K_4, then we obtain a graph G such that V_G is the Vámos matroid V_8. If V_H contains a minor isomorphic to V_8, then V_H fails to be modularly represented (and fails to satisfy Ingleton's inequality). Hence Theorem 5.3 provides a large family of hyperbolic matroids which are not modularly represented.

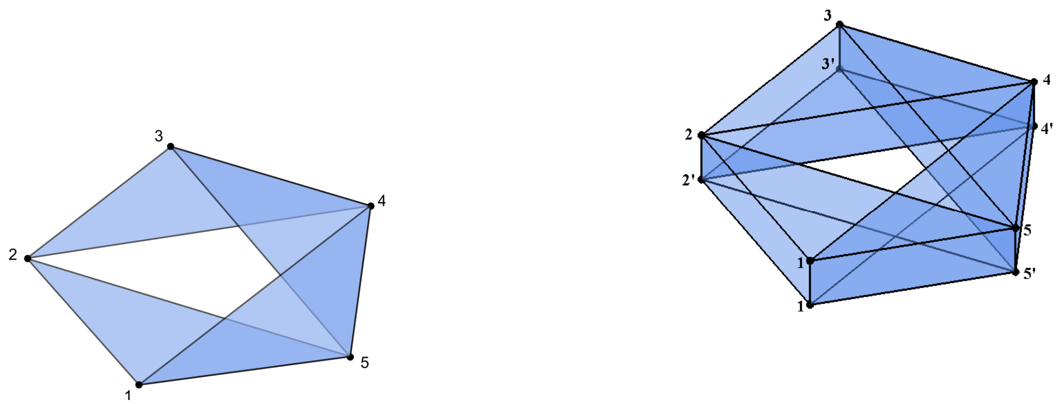
6 Consequences for the generalized Lax conjecture

Helton and Vinnikov [14] conjectured that for every hyperbolic polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ with respect to $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{R}^n$ there exists a positive integer N such that

$$h(\mathbf{x})^N = \det\left(\sum_{i=1}^{n} x_i A_i\right)$$

for some symmetric matrices $A_1, \dots, A_n$ such that $e_1 A_1 + \dots + e_n A_n$ is positive definite. In [4] the second author used the bases generating polynomial h_{V_8} of the Vámos matroid to prove that there is no linear polynomial $l(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ which is nonnegative on the hyperbolicity cone of h_{V_8} and positive integers N, M such that

$$l(\mathbf{x})^M h_{V_8}(\mathbf{x})^N = \det\left(\sum_{i=1}^{8} x_i A_i\right)$$



(a) A 3-uniform hypergraph H (b) The matroid V_H

for some symmetric matrices $A_1, \ldots, A_8$ with $e_1 A_1 + \cdots + e_8 A_8$ positive definite. We here provide further restrictions on the factor $q(\mathbf{x})$ in (2.1). Given a positive integers n and k, consider the k-uniform hypergraph $H_{n,k}$ on $[n+2]$ containing all hyperedges $e \in \binom{[n+2]}{k}$ except those for which $\{n+1, n+2\} \subseteq e$. By Theorem 5.3 the matroid $V_{H_{n,k}}$ is hyperbolic and therefore has a stable weighted bases generating polynomial $h_{V_{H_{n,k}}}$ by Proposition 3.1. By Grace–Walsh–Szegő theorem the polynomial $h_{n,k} \in \mathbb{R}[x_1, \ldots, x_{n+2}]$ obtained from the multiaffine polynomial $h_{V_{H_{n,k}}}$ by identifying the variables x_i and $x_{i'}$ pairwise for all $i \in [n+2]$ is stable.

Theorem 6.1. *Let n and k be positive integers. Suppose there exists a positive integer N and a hyperbolic polynomial $q(\mathbf{x})$ such that*

$$q(\mathbf{x}) h_{n,k}(\mathbf{x})^N = \det\left(\sum_{i=1}^{n+2} x_i A_i\right)$$

with $\Lambda_+(h_{n,k}) \subseteq \Lambda_+(q)$ for some symmetric matrices $A_1, \ldots, A_{n+2}$ such that $A_1 + \cdots + A_{n+2}$ is positive definite and

$$q(\mathbf{x}) = \prod_{i=1}^{s} p_j(\mathbf{x})^{\alpha_i}$$

for some irreducible hyperbolic polynomials $p_1, \ldots, p_s \in \mathbb{R}[x_1, \ldots, x_{n+2}]$ of degree at most $k-1$ where $\alpha_1, \ldots, \alpha_s$ are positive integers. Then $n < (2s+1)k - 1$.

Hence, for n sufficiently large, q either has a irreducible factor of large degree or is the product of many factors of low degree.

Consider

$$h_{2,2} = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4).$$

The polynomial $h_{2,2}$ comes from the bases generating polynomial of the Vámos matroid under the restriction $x_i = x_{i'}$ for $i = 1, \ldots, 4$. Kummer [18] found real symmetric matrices A_i, $i = 1, \ldots, 4$ with $A_1 + A_2 + A_3 + A_4$ positive definite and a hyperbolic polynomial q of degree 3 with $\Lambda_+(h_{2,2}) \subseteq \Lambda_+(q)$ such that

$$q(\mathbf{x})h_{2,2}(\mathbf{x}) = \det\left(x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4\right),$$

where

$$q(\mathbf{x}) = 32(2x_1 + 3x_2 + 3x_3 + 4x_4)(x_1x_2 + x_1x_3 + 2x_1x_4 + x_2x_4 + x_3x_4).$$

If $s = 2$ and $k = 3$ in Theorem 6.1 it follows that there exists no linear and quadratic hyperbolic polynomials $l(\mathbf{x}), q(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_{16}]$ respectively such that $h_{14,3}(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_{16}]$ has a positive definite representation of the form

$$l(\mathbf{x})q(\mathbf{x})h_{14,3}(\mathbf{x}) = \det\left(\sum_{i=1}^{16} x_iA_i\right)$$

with $\Lambda_+(h_{14,3}) \subseteq \Lambda_+(lq)$.

7 Sketch of proof of Theorem 5.4

The *elementary symmetric polynomial* $e_d \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d is defined by

$$e_d(\mathbf{x}) = \sum_{\substack{S\subseteq[n]\\|S|=d}} \prod_{i\in S} x_i.$$

An important ingredient in the proof of Theorem 5.4 is to show that certain symmetric polynomials are nonnegative. The results are interesting in their own right since they generalize several well-known inequalities in the literature. The Laguerre-Turán inequalities (see [8]) state that

$$re_r(\mathbf{x})^2 - (r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^m, m \geq 1.$$

The following inequalities are due to Jensen [16]:

$$\sum_{k=0}^{2r} (-1)^{r+j} \binom{2r}{k} f^{(k)}(t) f^{(2r-k)}(t) \geq 0, \quad t \in \mathbb{R}, \tag{7.1}$$

for all real–rooted polynomials f. Our next lemma is a refinement of the Laguerre–Turán inequalities and may be formulated as *the Laguerre–Turán inequalities beat Jensen's inequalities*. Lemma 7.1 is also a generalization of [10, Theorem 3], where the case $r = 2$ was proved. If $P, Q \in \mathbb{R}[\mathbf{x}]$, we write $P \leq Q$ if $Q - P$ is a nonnegative polynomial.

Lemma 7.1. *If* $r \geq 1$*, then*

$$e_r(x_1^2, \ldots, x_n^2) \leq re_r(\mathbf{x})^2 - (r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^n$.

Lemma 7.2. *If $r \geq 2$ is an integer, then*

$$(a_r e_{r-1}(\mathbf{x}) e_r(\mathbf{x}) - e_{r-2}(\mathbf{x}) e_{r+1}(\mathbf{x}))^2 \geq C_r e_{r-2}(\mathbf{x}) e_r(\mathbf{x}) e_r(x_1^2, \dots, x_n^2), \tag{7.2}$$

where

$$a_r = 3\frac{r-1}{r+1} \quad \text{and} \quad C_r = 9\frac{r-1}{(r+1)^2}.$$

The following theorem provides families of stable polynomials which are closed under convex sums.

Theorem 7.3. *Let $r \geq 2$ be an integer, and let*

$$M(\mathbf{x}) = \sum_{|S|=r} a(S) \prod_{i \in S} x_i^2 \in \mathbb{R}[x_1, \dots, x_n],$$

where $0 \leq a(S) \leq 1$ for all $S \subseteq [n]$, where $|S| = r$. Then the polynomial

$$4e_{r+1}(\mathbf{x}) e_{r-1}(\mathbf{x}) + \frac{3}{r+1} M(\mathbf{x})$$

is stable.

The next theorem is a version of the Grace–Walsh–Szegő coincidence theorem, see e.g. [3, Prop. 3.4].

Theorem 7.4 (Grace–Walsh–Szegő)**.** *Suppose $P(x_1, \dots, x_n) \in \mathbb{C}[\mathbf{x}]$ is a polynomial of degree at most d in the variable x_1:*

$$P(x_1, \dots, x_n) = \sum_{k=0}^{d} P_k(x_2, \dots, x_n) x_1^k.$$

Let Q be the polynomial in the variables $x_2, \dots, x_n, y_1, \dots, y_n$

$$Q = \sum_{k=0}^{d} P_k(x_2, \dots, x_n) \frac{e_k(y_1, \dots, y_d)}{\binom{d}{k}}.$$

Then P is stable if and only if Q is stable.

Lemma 7.5. *Let $r \geq 2$. Then*

$$\begin{aligned} &e_{2r}(x_1, x_1, \dots, x_n, x_n) - e_r(x_1^2, \dots, x_n^2) = \\ &4(e_{r-1}(\mathbf{x}) e_{r+1}(\mathbf{x}) + e_{r-3}(\mathbf{x}) e_{r+3}(\mathbf{x}) + e_{r-5}(\mathbf{x}) e_{r+5}(\mathbf{x}) + \cdots). \end{aligned}$$

Proof of Theorem 5.3. By definition the bases generating polynomial of $V_H \in \mathcal{V}$ is given by

$$h_{V_H} = \sum_{B \in \mathcal{B}(V_H)} \prod_{i \in B} x_i = e_{2r}(x_1, x_{1'}, \dots, x_n, x_{n'}) - e_r(x_1 x_{1'}, \dots, x_n x_{n'}) + N(\mathbf{x}).$$

where

$$N(\mathbf{x}) = \sum_{\{i_1, \dots, i_r\} \notin E(H)} \prod_{j=1}^{r} x_{i_j} x_{i_j'}.$$

The polynomial h_{V_H} is clearly multiaffine and symmetric pairwise in $x_i, x_{i'}$ for all $i \in [n]$. Set $x_{i'} = x_i$ for all $1 \leq i \leq n$ and obtain the polynomial

$$f_{V_H} = e_{2r}(x_1, x_1, \ldots, x_n, x_n) - e_r(x_1^2, \ldots, x_n^2) + N(x_1, x_1, \ldots, x_n, x_n).$$

By Lemma 7.5

$$f_{V_H} = 4 \sum_{j=0}^{\lceil r/2 \rceil - 1} e_{r+2j+1}(\mathbf{x}) e_{r-2j-1}(\mathbf{x}) + N(x_1, x_1, \ldots, x_n, x_n).$$

The support of $e_{r+j}(\mathbf{x})e_{r-j}(\mathbf{x})$ is contained in the support of $e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x})$ for each $1 \leq j \leq r$. Hence f_{V_H} has the same support as the polynomial

$$W_{V_H} = 4e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x}) + \frac{3}{r+1} N(x_1, x_1, \ldots, x_n, x_n)$$

which in turn is stable by Theorem 7.3. Hence if we replace x_i^k, $k = 0, 1, 2$, in W_{V_H} with $e_k(x_i, x_{i'})/\binom{2}{k}$, we obtain a polynomial which is stable by the Grace–Walsh–Szegő theorem, and has the same support as h_{V_H}. Thus V_H is a WHPP-matroid so V_H is hyperbolic by Proposition 3.1. □

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